McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 363
DISCRETE MATHEMATICS FOR ENGINEERS

Examiner: Zhentao Li

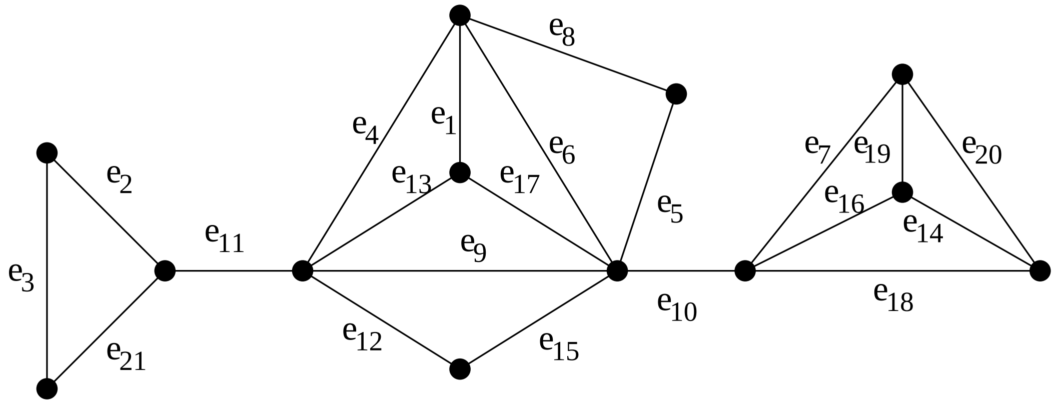
Date: Wednesday April 28, 2010

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. This is a closed book exam.
3. Calculators are not permitted.
4. Use of a translation dictionaries is permitted.
5. This exam contains 21 questions. You must answer question 1.
6. Unless stated otherwise, justify all your steps.
7. You may use lemmas and theorems that were proven in class and on assignments unless stated otherwise.
8. Four appendices are attached at the end of this exam. Appendix 1 contains a list of rules of inference. Appendix 2 contains two tables of propositional equivalences. Appendix 3 contains a list of definitions and theorems. Appendix 4 contains a glossary of symbols.
9. All graphs are simple graphs unless stated otherwise. All graphs have no loops.
10. The exam will be marked out of 74 but it is possible and likely to obtain more points.

This exam comprises of x pages, including this cover page.

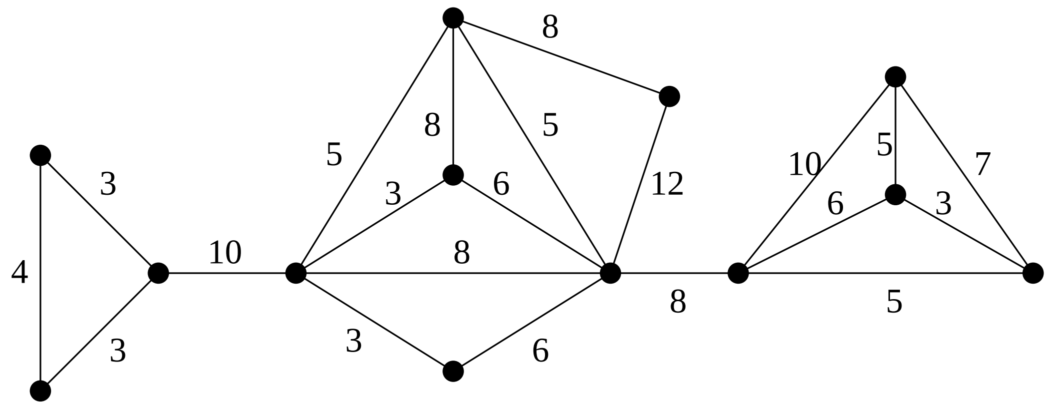
1. (8 points) Let $G^* = (V^*, E^*)$ be the following graph.



Choose a subset F^* of E^* such that (V, F^*) contains no cycles and write down your choice as the answer to this question.

Answer question i of this midterm only if you chose edge e_i. **You must answer this question** (i.e., you must choose e_1). Questions you did not choose will not be marked. Questions corresponding to the last edge forming a cycle (in your choice of F^*) will not be marked.

For your convenience, the above graph with weight corresponding to point values of each question is drawn below.



2. (3 points)

   There are 4 cards on the table. We see "7", "M", "A" and "10" respectively on these cards.

   We want to flip the minimum number of cards to verify if the statement "Every cards with an even number has a vowel on the other side" is true. Which cards to we need to flip?

3. (4 points)

   Prove the following statement without using the Pigeonhole principle (Theorem 57).

   If we put more than n objects into n boxes then there is a box containing at least 2 objects.

4. (5 points)

   A staircase has n steps. You walk up taking one or two steps at a time. How many ways can you go up. Given a closed form formula as a function of n.

   [Hint: Write the number of ways as a linear recurrence.]

5. (12 points)

   Prove the following statement without using Hall's theorem (Theorem 33).

   Let G be a bipartite graph with parts A and B. G contains a perfect matching if and only if $|A| = |B|$ and for all $S \subseteq A$, $|S| \leq |N(S)|$.

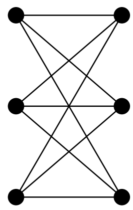
6. (5 points)

   Prove the following statement without using Lemma 21.

   Every tree with at least 2 vertices contains a vertex of degree 1.

7. (10 points)

   Prove that $K_{3,3}$ is a not a planar graph (without using Kuratowski's theorem).



$K_{3,3}$

8. (8 points)

   Let $\Delta(G)$ denote the maximum degree in G (i.e., $\Delta(G) = \max_{v \in V(G)} \deg(v)$). Prove that any graph G is $\Delta + 1$-colourable.

9. (8 points)

   Prove the following statement without using Theorem 68 or Corollary 69.

   Every planar graph with a vertex has a vertex of degree at most 5.

10. (8 points)

    Definition 1. A graph G is *d-regular* if all vertices of G have degree d.

    Prove the following statement.

    For any $d > 0$, any bipartite d-regular graph G has a perfect matching.

11. Consider the following argument.

    $$\frac{\text{It is not true that there is a red ball or green ball (or both) in the box.}}{\text{There is no red ball and there is no green ball in the box.}}$$

    (a) (2 points) Rewrite the argument using propositional logic.

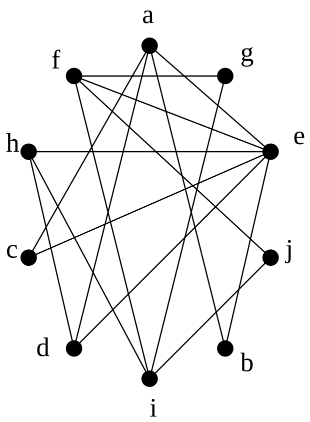
    (b) (3 points) Write down a single truth table containing both the premise and the conclusion.

    (c) (5 points) Prove the argument is valid using rules of inference given in Appendix 1.
    Do not use any other rules. Do not use equivalences. Do not refer to the truth table in (b).

12. (3 points) Prove $\vdash p \wedge q \rightarrow p$ using rules of inference given in Appendix 1.

    Do not use any other rules. Do not use equivalence. Do not use truth table.

13. (3 points) Determine if this multigraph contains an <u>Eulerian trail</u>. If it does, write down the **vertices** visited by your circuit in the order they are visited (no justification needed in this case). If it does not, give a reason why.



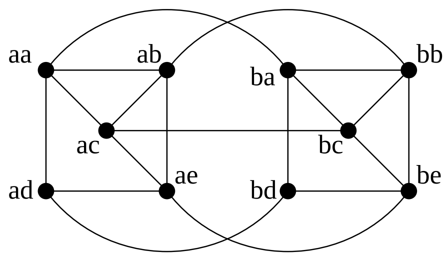
14. (3 points)

    How many different hands of 9 cards can you draw from a standard deck of 52 cards if only suits matter. e.g., $\{J\heartsuit, K\heartsuit, 9\heartsuit, 7\heartsuit, 10\diamondsuit, A\diamondsuit, 2\spadesuit, 4\clubsuit, 7\clubsuit, 9\clubsuit, 10\clubsuit\}$ and $\{9\heartsuit, 3\heartsuit, 2\diamondsuit, 8\spadesuit, Q\clubsuit, 5\clubsuit, 6\heartsuit, 10\clubsuit, 3\diamondsuit, 9\clubsuit, 5\heartsuit\}$ are considered the same hand.

15. (6 points)

    (Inspired by discrete mathematics elementary and beyond, p.239-240)

    We would like to seat guests around a (round) dinner table so that only friends sit next to each other. In the following graph, vertices represent guests and there is an edge between two vertices if and only if the corresponding guests are friends.



    Give such a seating if it is possible. If not, give a reason why.

16. (6 points)

    Prove the following statement without using Theorem 42. Give an explicit bijection.

    The number of subsets of a set of size n is 2^n.

17. (6 points)

    Prove the following statement.

    If G is a graph with at least 2 vertices then G contains two vertices of the same degree.

18. (5 points)

    Prove or disprove the following statement.

    Every bipartite graph is 2-colourable.

19. (5 points)

    Prove the following statement without using Theorem 60.

    A graph with 6 vertices contains either a clique of size 3 or a stable set of size 3 (or both).

20. (7 points) Prove the following statement.

    For two events A and B (in a sample space Ω with probability mass function p),

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

21. (3 points)

    Suppose we randomly order the integers 1,2,3,4 where each ordering is equally likely.

    What is the probability that 4 precedes 3 and 2 precedes 1 in the ordering?