Note on the bipartite matchings

We start with a well known theorem characterizing bipartite graphs with perfect matching.

Definition 1. A *perfect matching* in a graph G is a matching M where all vertices of G are incident to some edge of M.

Theorem 1. Hall's theorem *Let G be a bipartite graph with parts A and B. G contains a perfect matching if and only if $|A| = |B|$ and for all $S \subseteq A$, $|S| \leq |N(S)|$.*

Proof. First, we see that the two conditions are necessary for G to contain a perfect matching.

Since G is bipartite, every edge of a matching matches a vertex in A to a vertex in B. Therefore, $|M| \leq \min(|A|, |B|)$ for all matchings M.

Similarly, if $|S| > |N(S)|$ for some subset S of A then not all of S can be matched since each edge of a matching with one end in S has the other end in $N(S)$. But each vertex of $N(S)$ can be incident to at most one edge of any matching.

Now for the more difficult direction. That is, the two conditions are sufficient.

Claim 1. *For all bipartite graphs G, if $|A| = |B|$ and for all $S \subseteq A$, $|S| \leq |N(S)|$ then G contains a perfect matching.*

Proof. Suppose the claim is false. Let G be a counter-example which minimizes $|V(G)|$. Since G is a counter-example, G is bipartite with parts A and B, $|A| = |B|$, for all $S \subseteq A$, $|S| \leq |N(S)|$ and G does not contain a perfect matching.

G contains at least one vertex as otherwise, the empty matching is a perfect matching in G.

G contains at least one edge since $|N(v)| = 0$ for any vertex $v \in A$ but $|\{v\}| = 1 > 0$.

Let $e = (u, v) \in G$ with (up to relabelling u and v) $u \in A$ and $v \in B$.

Let H be the graph obtain from G by deleting u and v. That is, $H = G - \{u, v\}$. H is bipartite with parts $A \cap V(H)$ and $B \cap V(H)$. Note that these parts have the same size since $|A \cap V(H)| = |A \setminus \{u\}| = |A| - 1 = |B| - 1 = |B \setminus \{v\}| = |B \cap V(H)|$.

We will now use the minimality of G. The graph we apply minimality to depends on if H satisfies the hypothesis in the claim.

If H satisfies the hypothesis in the claim. That is, for all $S \subseteq A \cap V(H)$, $|S| \leq |N_H(S)|$. Here, we write N_H instead of N to emphasize the fact that we are looking at the neighbours of S in H rather than G. Then, by minimality, H contains a perfect matching M_1. Adding $e = (u, v)$ to M_1 gives a perfect matching M in G. Contradiction to G being a counter-example.

If H does not satisfy the hypothesis, there exists $S \subseteq A \cap V(H)$ with $|S| > |N_H(S)|$. Since G satisfies the hypothesis, for the same set S, $|S| \leq |N_G(S)| \leq |N_H(S)| - 1$ (G only has one more vertex than H in B, namely, v). So $|S| = |N_G(S)|$.

Let H_1 be the subgraph with vertex set $S \cup N_G(S)$ with all edges of G between S and $N_G(S)$ (i.e., H_1 is the subgraph of G induced by $S \cup N_G(S)$). Let H_2 be the subgraph of G with vertex set $V(G) \setminus (S \cup N_G(S))$ (i.e., H_1 is the subgraph of G induced by $V(G) \setminus (S \cup N_G(S))$).

We claim that H_1 satisfies the hypothesis of the claim. Indeed, if not, there is a subset $S' \subseteq S$ with $|S'| > |N_{H_1}(S')|$. But $S' \subseteq S \subseteq A$ and $N_{H_1}(S') = N_G(S')$ (since $N(S') \subseteq N(S)$). So $|S'| > |N_G(S')|$. This contradicts the fact that G satisfies the hypothesis of the claim.

We claim that H_2 also satisfies the hypothesis of the claim with the two parts switched. Indeed, if not, there is a subset $S' \subseteq B \setminus N(S)$ with $|S'| > |N_{H_2}(S')|$. But $S' \subseteq B \setminus N(S) \subseteq B$ and $N_{H_2}(S') = N_G(S')$ (since there are no edges from $B \setminus N(S)$ to S by definition of $N(S)$). So $|A \setminus N(S')| > |N_G(A \setminus N(S'))|$ (since $|A| = |B|$). This contradicts the fact that G satisfies the hypothesis of the claim.

Thus, both H_1 and H_2 satisfy the hypothesis in the claim. Furthermore, both H_1 and H_2 have fewer vertices than G (for example, u is not in H_1 and v is not in H_2). Therefore, by minimality of G, H_1 contains a perfect matching in M_1 and H_2 contains a perfect matching in M_2. But the union of M_1 and M_2 is a perfect matching in G. Contradiction to G being a counter-example.

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Based, on Hall's theorem, we can design an algorithm for finding augmenting paths in graphs with perfect matchings.

Algorithm 1. **Input:** A bipartite graph $G = (V, E)$ with parts A and B, a matching M in G, the set of unmatched vertices U of A and the set of unmatched vertex W of B.
Output: Either

1. An M-augmenting path in G, or
2. A subset S of A with $|S| > |N(S)|$.

Initialize an array prev of pointers
$S \leftarrow U$
$T \leftarrow \emptyset$
For $s \in S$, set prev$[s]$ $\leftarrow$null
While true
  If $\exists e = (u, v) \in E$ with $u \in S$ and $v \notin T$ then prev$[v] \leftarrow u$
    If $v \in W$ then
      return the path from v following prev pointers. $T \leftarrow T \cup \{v\}$
    $w \leftarrow$ the vertex matched to v in M
    $S \leftarrow S \cup \{w\}$
    prev$[w] \leftarrow v$
  Else
    return S

Remark 1. At the beginning of every iteration,

- $|T| < |S|$
- $T \subseteq N(S)$
- No edge (u, v) with $u \in S, v \notin T$ exists only if $T = N(S)$.

A similar algorithm for finding an M-augmenting path regardless of whether G contains a perfect matching.

Algorithm 2. **Input:** A bipartite graph $G = (V, E)$ with parts A and B, a matching M in G, the set of unmatched vertices U of A and the set of unmatched vertex W of B.
Output: Either

1. An M-augmenting path in G, or
2. "An M-augmenting path does not exist in G."

Build a digraph H with vertex set A and directed edges $\{(u, v) | \exists v \in B, (u, v) \notin M, (v, w) \in M\}$.
Run a graph search algorithm (e.g., DFS or BFS) in H starting from U and see if we can reach a vertex in $N(W)$.