1 Rules of Inference

[1]

Rules of inference are used to deduce true formulas from other true formulas.

Notation. Upper case letters P, Q, R, S will be used to denote propositional formulas (rather than single variables).

The rules of inference are stated using the following notation

$$\frac{\text{Hypothesis}}{\text{Conclusion}}$$

which mean given the Hypothesis, we can infer (or deduce) the Conclusion. If there is more than one hypothesis, it is written

$$\frac{\begin{array}{c}\text{Hypothesis}_1\\ \text{Hypothesis}_2\\ \vdots\\ \text{Hypothesis}_k\end{array}}{\text{Conclusion}}$$

Rule	Name	Book name
$\dfrac{P \wedge Q}{P}$	$\wedge\mathcal{E}$	Simplification
$\dfrac{\begin{array}{c}P\\ Q\end{array}}{P \wedge Q}$	$\wedge\mathcal{I}$	Conjunction
$\dfrac{\begin{array}{c}\not{P}\\ \vdots\\ Q\end{array}}{P \to Q}$	$\to\mathcal{I}$	The rules given in the book avoids this rule by using the equivalence of $p \to q$ and $\neg p \vee q$
$\dfrac{\begin{array}{c}P\\ P \to Q\end{array}}{Q}$	$\to\mathcal{E}$	Modus Ponens
$\dfrac{P}{P \vee Q}$	$\vee\mathcal{I}$	Addition
$\dfrac{\begin{array}{c}P \vee Q\\ P \to R\\ Q \to R\end{array}}{R}$	$\vee\mathcal{E}$	Disjunctive syllogism (not exactly the same)
$\dfrac{P \to \mathbf{F}}{\neg P}$	$\neg\mathcal{I}$	Modus Tollens (not exactly the same)
$\dfrac{\begin{array}{c}P\\ \neg P\end{array}}{\mathbf{F}}$	$\neg\mathcal{E}$	
$\dfrac{\neg\neg P}{P}$	$\neg\neg\mathcal{E}$	
$\dfrac{\mathbf{F}}{P}$	$\mathbf{F}\mathcal{E}$	

These are the inference rules of a natural deduction system. We will be using these rules but there are other set of rules that we could have used.

[1] Made using Paul Taylor's boxuser LaTeX macros

Notation. The "$\mathcal{I}$" in the table means "introduction" and the "$\mathcal{E}$" in the table means "elimination". So, for example, the name of the first rule is "and elimination".

The first rule can be read as "From $P \wedge Q$, we can infer P".

The rule $\rightarrow\mathcal{I}$ states that if taking P as an assumption, after a number of steps we arrive at Q then we can infer $P \rightarrow Q$ while losing our assumption P (and any formula derived from it).

Notation. We could have also written a rule

$$\frac{\text{Hypothesis}}{\text{Conclusion}}$$

in a "linear form" as

$$\text{Hypothesis} \vdash \text{Conclusion}$$

and the case with multiple hypotheses as

$$\text{Hypothesis}_1, \text{Hypothesis}_2, \ldots, \text{Hypothesis}_k \vdash \text{Conclusion}$$

This can be read as "Given Hypotheses$_1$,...,Hypotheses$_k$, we can infer the Conclusion".

Example 1. We will now prove $p \wedge q \rightarrow p$ (using no premise, thus showing that it is a tautology)

1	$p \wedge q$	assumption
2	p	$1, \wedge\mathcal{E}$
3	$p \wedge q \rightarrow p$	$1-2, \rightarrow\mathcal{I}$

Note that we have proven $p \wedge q \rightarrow p$ without any hypothesis. Thus we can write $\vdash p \wedge q \rightarrow p$.

Definition 1. Any formula which can be infered using no premise is called a *theorem*.

Example 2. We will now prove what is called "Modus Tollens" in the book. The statement is $\neg q, p \rightarrow q \vdash \neg p$.

1	$p \rightarrow q$	premise
2	$\neg q$	premise
3	p	assumption
4	q	$1, 3, \rightarrow\mathcal{E}$
5	$\mathbf{F}$	$2, 4, \neg\mathcal{E}$
6	$p \rightarrow \mathbf{F}$	$3-5, \rightarrow\mathcal{I}$
7	$\neg p$	$6, \neg\mathcal{I}$

Note. Each line of a proof should either be a premise, an assumption or a formula which is true given all formulas in previous lines are true.

There should not be any assumptions left at the end of a proof (only premises).

Example 3. Here is an example of a mathematical proof which has been formatted differently than usual so that it ressembles a proof in logic.

Theorem 1. *If* $\sqrt{2} > \frac{3}{2}$ *then* $2 > \frac{9}{4}$.

Proof.

1	$\sqrt{2} > \frac{3}{2}$	assumption
2	$\sqrt{2}^2 > \sqrt{2}\frac{3}{2}$	1, multiply both sides by $\sqrt{2}$
3	$\sqrt{2}\frac{3}{2} > \left(\frac{3}{2}\right)^2$	1, multiply both sides by $\frac{3}{2}$
4	$\sqrt{2}^2 > \left(\frac{3}{2}\right)^2$	2, 3, transitivity of $>$
5	$2 > \frac{9}{4}$	evaluation of the left-hand side and right-hand side

□

Of course, the premise $\sqrt{2} > \frac{3}{2}$ is false. But this does not make the implication and thus our theorem false.

Here is what the proof may look like if it were normally formatted.

Proof. Suppose $\sqrt{2} > \frac{3}{2}$. Then $\sqrt{2}^2 > \sqrt{2}\frac{3}{2} > \left(\frac{3}{2}\right)^2$. Therefore, $2 > \frac{9}{4}$. □

Definition 2. We call Hypothesis $\vdash$ Conclusion an *argument*. An argument is *valid* if we can infer the Conclusion given Hypotheses$_1$,...,Hypotheses$_k$ and *invalid* otherwise.

Example 4. In this example, we prove $\vdash (p \to q) \to ((q \to r) \to (p \to r)$ (and thus, show that it is a valid argument).

1	$p \to q$	assumption
2	$q \to r$	assumption
3	p	assumption
4	q	$1, 3, \to\mathcal{E}$
5	r	$2, 4, \to\mathcal{E}$
6	$p \to r$	$3 - 5, \to\mathcal{I}$
7	$(q \to r) \to (p \to r)$	$2 - 6, \to\mathcal{I}$
8	$(p \to q) \to ((q \to r) \to (p \to r))$	$1 - 7, \to\mathcal{I}$

Exercise 1. Show that $(p \wedge q) \to r \vdash p \to (q \to r)$.

At first, there is no reason to believe that if we can prove P from no premise using rules of inference that P should be a tautology. And perhaps even less believable is the fact that *all* tautologies can be proven using these rules. These concepts are refered to as *soundness* and *completeness* respectively. We formalise these ideas with the $\models$ symbol.

Definition 3. Hypothesis$_1$,...,Hypothesis$_k$ $\models$ Conclusion if for any value of the variables which makes Hypothesis$_1$,...,Hypothesis$_k$ true, the Conclusion is true.

$P \models Q$ is read as "P models Q".

Example 5. The following table shows that $\neg q, p \to q \models \neg p$.

p	q	$\neg q$	$p \to q$	$\neg p$
T	T	F	T	F
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

The hypothesis are all true only when p and q are both false and in that case, $\neg p$ is indeed true. Thus, $\neg q, p \rightarrow q \models \neg p$.

We can now write what we would like to be true.

Theorem 2 (Soundness)**.**

$$P_1, \ldots, P_k \vdash Q \Rightarrow P_1, \ldots, P_k \models Q$$

Theorem 3 (Completeness)**.**

$$P_1, \ldots, P_k \models Q \Rightarrow P_1, \ldots, P_k \vdash Q$$