# 1 Rules of Inference

[1]

Rules of inference are used to deduce true formulas from other true formulas.

**Notation.** Upper case letters $P, Q, R, S$ will be used to denote propositional formulas (rather than single variables).

The rules of inference are stated using the following notation

$$\frac{\text{Hypothesis}}{\text{Conclusion}}$$

which mean given the Hypothesis, we can infer (or deduce) the Conclusion. If there is more than one hypothesis, it is written

$$\frac{\begin{array}{c}\text{Hypothesis}_1\\ \text{Hypothesis}_2\\ \vdots\\ \text{Hypothesis}_k\end{array}}{\text{Conclusion}}$$

| Rule | Name | Book name |
|---|---|---|
| $\dfrac{P \land Q}{P}$ | $\land\mathcal{E}$ | Simplification |
| $\dfrac{\begin{array}{c}P\\ Q\end{array}}{P \land Q}$ | $\land\mathcal{I}$ | Conjunction |
| $\dfrac{\begin{array}{c}\cancel{P}\\ \vdots\\ Q\end{array}}{P \to Q}$ | $\to\mathcal{I}$ | The rules given in the book avoids this rule by using the equivalence of $p \to q$ and $\neg p \lor q$ |
| $\dfrac{\begin{array}{c}P\\ P \to Q\end{array}}{Q}$ | $\to\mathcal{E}$ | Modus Ponens |
| $\dfrac{P}{P \lor Q}$ | $\lor\mathcal{I}$ | Addition |
| $\dfrac{\begin{array}{c}P \lor Q\\ P \to R\\ Q \to R\end{array}}{R}$ | $\lor\mathcal{E}$ | Disjunctive syllogism (not exactly the same) |
| $\dfrac{P \to \mathbf{F}}{\neg P}$ | $\neg\mathcal{I}$ | Modus Tollens (not exactly the same) |
| $\dfrac{\begin{array}{c}P\\ \neg P\end{array}}{\mathbf{F}}$ | $\neg\mathcal{E}$ | |
| $\dfrac{\neg\neg P}{P}$ | $\neg\neg\mathcal{E}$ | |
| $\dfrac{\mathbf{F}}{P}$ | $\mathbf{F}\mathcal{E}$ | |

These are the inference rules of a natural deduction system. We will be using these rules but there are other set of rules that we could have used.

[1] Made using Paul Taylor's boxuser LaTeX macros

**Notation.** The "$\mathcal{I}$" in the table means "introduction" and the "$\mathcal{E}$" in the table means "elimination". So, for example, the name of the first rule is "and elimination".

The first rule can be read as "From $P \wedge Q$, we can infer $P$".
The rule $\rightarrow\mathcal{I}$ states that if taking $P$ as an assumption, after a number of steps we arrive at $Q$ then we can infer $P \rightarrow Q$ while losing our assumption $P$ (and any formula derived from it).

**Notation.** We could have also written a rule

$$\frac{\text{Hypothesis}}{\text{Conclusion}}$$

in a "linear form" as

$$\text{Hypothesis} \vdash \text{Conclusion}$$

and the case with multiple hypotheses as

$$\text{Hypothesis}_1, \text{Hypothesis}_2, \ldots, \text{Hypothesis}_k \vdash \text{Conclusion}$$

This can be read as "Given $\text{Hypotheses}_1$,. . .,$\text{Hypotheses}_k$, we can infer the Conclusion".

**Example 1.** We will now prove $p \wedge q \rightarrow p$ (using no premise, thus showing that it is a tautology)

| | | |
|---|---|---|
| 1 | $p \wedge q$ | assumption |
| 2 | $p$ | $1, \wedge\mathcal{E}$ |
| 3 | $p \wedge q \rightarrow p$ | $1-2, \rightarrow\mathcal{I}$ |

Note that we have proven $p \wedge q \rightarrow p$ without any hypothesis. Thus we can write $\vdash p \wedge q \rightarrow p$.

**Definition 1.** Any formula which can be infered using no premise is called a *theorem*.

**Example 2.** We will now prove what is called "Modus Tollens" in the book. The statement is $\neg q, p \rightarrow q \vdash \neg p$.

| | | |
|---|---|---|
| 1 | $p \rightarrow q$ | premise |
| 2 | $\neg q$ | premise |
| 3 | $p$ | assumption |
| 4 | $q$ | $1, 3, \rightarrow\mathcal{E}$ |
| 5 | $\mathbf{F}$ | $2, 4, \neg\mathcal{E}$ |
| 6 | $p \rightarrow \mathbf{F}$ | $3-5, \rightarrow\mathcal{I}$ |
| 7 | $\neg p$ | $6, \neg\mathcal{I}$ |

**Note.** Each line of a proof should either be a premise, an assumption or a formula which is true given all formulas in previous lines are true.
There should not be any assumptions left at the end of a proof (only premises).

**Example 3.** Here is an example of a mathematical proof which has been formatted differently than usual so that it ressembles a proof in logic.

**Theorem 1.** *If* $\sqrt{2} > \frac{3}{2}$ *then* $2 > \frac{9}{4}$.

*Proof.*

| | | |
|---|---|---|
| 1 | $\sqrt{2} > \frac{3}{2}$ | assumption |
| 2 | $\sqrt{2}^2 > \sqrt{2}\frac{3}{2}$ | 1, multiply both sides by $\sqrt{2}$ |
| 3 | $\sqrt{2}\frac{3}{2} > \left(\frac{3}{2}\right)^2$ | 1, multiply both sides by $\frac{3}{2}$ |
| 4 | $\sqrt{2}^2 > \left(\frac{3}{2}\right)^2$ | 2, 3, transitivity of $>$ |
| 5 | $2 > \frac{9}{4}$ | evaluation of the left-hand side and right-hand side |

$\square$

Of course, the premise $\sqrt{2} > \frac{3}{2}$ is false. But this does not make the implication and thus our theorem false.

Here is what the proof may look like if it were normally formatted.

*Proof.* Suppose $\sqrt{2} > \frac{3}{2}$. Then $\sqrt{2}^2 > \sqrt{2}\frac{3}{2} > \left(\frac{3}{2}\right)^2$. Therefore, $2 > \frac{9}{4}$. $\square$

**Definition 2.** We call Hypothesis $\vdash$ Conclusion an *argument*. An argument is *valid* if we can infer the Conclusion given $\text{Hypotheses}_1, \ldots, \text{Hypotheses}_k$ and *invalid* otherwise.

**Example 4.** In this example, we prove $\vdash (p \to q) \to ((q \to r) \to (p \to r)$ (and thus, show that it is a valid argument).

| | | |
|---|---|---|
| 1 | $p \to q$ | assumption |
| 2 | $q \to r$ | assumption |
| 3 | $p$ | assumption |
| 4 | $q$ | $1, 3, \to\mathcal{E}$ |
| 5 | $r$ | $2, 4, \to\mathcal{E}$ |
| 6 | $p \to r$ | $3-5, \to\mathcal{I}$ |
| 7 | $(q \to r) \to (p \to r)$ | $2-6, \to\mathcal{I}$ |
| 8 | $(p \to q) \to ((q \to r) \to (p \to r))$ | $1-7, \to\mathcal{I}$ |

**Exercise 1.** Show that $(p \wedge q) \to r \vdash p \to (q \to r)$.

At first, there is no reason to believe that if we can prove $P$ from no premise using rules of inference that $P$ should be a tautology. And perhaps even less believable is the fact that *all* tautologies can be proven using these rules. These concepts are refered to as *soundness* and *completeness* respectively. We formalise these ideas with the $\models$ symbol.

**Definition 3.** $\text{Hypothesis}_1, \ldots, \text{Hypothesis}_k \models$ Conclusion if for any value of the variables which makes $\text{Hypothesis}_1, \ldots, \text{Hypothesis}_k$ true, the Conclusion is true.

$P \models Q$ is read as "$P$ models $Q$".

**Example 5.** The following table shows that $\neg q, p \to q \models \neg p$.

| $p$ | $q$ | $\neg q$ | $p \to q$ | $\neg p$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

The hypothesis are all true only when $p$ and $q$ are both false and in that case, $\neg p$ is indeed true. Thus, $\neg q, p \to q \models \neg p$.

We can now write what we would like to be true.

**Theorem 2** (Soundness)**.**

$$P_1, \ldots, P_k \vdash Q \Rightarrow P_1, \ldots, P_k \models Q$$

**Theorem 3** (Completeness)**.**

$$P_1, \ldots, P_k \models Q \Rightarrow P_1, \ldots, P_k \vdash Q$$