Final exam

- This is a closed book exam. No calculators are allowed.
- Unless stated otherwise, justify all your steps.
- You may use lemmas and theorems that were proven in class and on assignments unless stated otherwise.
- Four appendices are attached at the end of this exam. Appendix 1 contains a list of rules of inference. Appendix 2 contains two tables of propositional equivalences. Appendix 3 contains a list of definitions and theorems. Appendix 4 contains a glossary of symbols.
- All graphs are simple graphs unless stated otherwise. All graphs have no loops.

Appendix 1: Rules of inference

Rule		Name
$\dfrac{P \wedge Q}{P}$	$\dfrac{P \wedge Q}{Q}$	$\wedge\mathcal{E}$
$\dfrac{\begin{array}{c}P\\ Q\end{array}}{P \wedge Q}$	$\dfrac{\begin{array}{c}P\\ Q\end{array}}{Q \wedge P}$	$\wedge\mathcal{I}$
$\dfrac{\begin{array}{c}\cancel{P}\\ \vdots\\ Q\end{array}}{P \to Q}$		$\to\mathcal{I}$
$\dfrac{\begin{array}{c}P\\ P \to Q\end{array}}{Q}$		$\to\mathcal{E}$
$\dfrac{P}{P \vee Q}$	$\dfrac{P}{Q \vee P}$	$\vee\mathcal{I}$
$\dfrac{\begin{array}{c}P \vee Q\\ P \to R\\ Q \to R\end{array}}{R}$	$\dfrac{\begin{array}{c}Q \vee R\\ P \to R\\ Q \to R\end{array}}{R}$	$\vee\mathcal{E}$
$\dfrac{P \to \mathbf{F}}{\neg P}$		$\neg\mathcal{I}$
$\dfrac{\begin{array}{c}P\\ \neg P\end{array}}{\mathbf{F}}$		$\neg\mathcal{E}$
$\dfrac{\neg\neg P}{P}$		$\neg\neg\mathcal{E}$
$\dfrac{\mathbf{F}}{P}$		$\mathbf{F}\mathcal{E}$

Appendix 2: Table of equivalences

For propositional logic.

$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
$p \vee p \equiv p$ $p \wedge p \equiv p$
$\neg(\neg p) \equiv p$
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$
$p \rightarrow q \equiv \neg p \vee q$

For first order logic. The above as well as the following.

$\neg \exists x \; p(x) \equiv \forall x \; \neg p(x)$ $\neg \forall x \; p(x) \equiv \exists x \; \neg p(x)$

Appendix 3: Definitions and theorems

Definition 1. We call $P_1, \ldots, P_k \vdash Q$ an **argument**. An argument is **valid** if we can infer the conclusion Q given the hypotheses $P_1, \ldots, P_k$ and **invalid** otherwise.

Definition 2. A **graph** G is an ordered pair (V, E) where V is a set of vertices and E is a (multi) set of edges: 2-element subsets of V.

Lemma 3. (Handshaking lemma) *Let $G = (V, E)$ be a graph. Then*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Theorem 4. *Let G be a graph. The number of odd degree vertices in G is even.*

Definition 5. A **walk** consists of an alternating sequence of vertices and edges consecutive elements of which are incident, that begins and ends with a vertex. A **trail** is a walk without repeated edges. A **path** is a walk without repeated vertices.

If a walk (resp. trail, path) begins at x and ends at y then it is an $x - y$ walk (resp. $x - y$ trail, resp. $x - y$ path).

A walk (trail) is **closed** if it begins and ends at the same vertex. A closed trail whose origin and internal vertices are distinct is a **cycle**.

Definition 6. A **circuit** is a trail that begins and ends at the same vertex.

Some equivalent definitions of paths and cycles.

Definition 7. A **path** in a graph G is a sequence of vertices $p_1, \ldots, p_k$ such that for all $1 \leq i \leq k-1$, (p_i, p_{i+1}) is an edge in G.

A **cycle** in a graph G is a path $p_1, \ldots, p_k$ such that (p_k, p_1) is an edge of G.

Definition 8. A **subgraph** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq \{(u, v) | (u, v) \in E(G), u \in V(H), v \in V(H)\}$.

An **induced subgraph** H of G is a subgraph of G where $E(H) = \{(u, v) | (u, v) \in E(G), u \in V(H), v \in V(H)\}$ (i.e., we have all edges between vertices of H).

Definition 9. P_n is the graph on n vertices $v_1, \ldots, v_n$ and edges (v_i, v_{i+1}) for each i from 1 to $n-1$.

C_n is the graph on n vertices $v_1, \ldots, v_n$ and edges (v_1, v_n) and (v_i, v_{i+1}) for each i from 1 to $n-1$.

K_n, the complete graph, is the graph on n vertices $v_1, \ldots, v_n$ and all edges (i.e., (v_i, v_j) for all $1 \leq i < j \leq n$).

Q_n, the hypercube graph, is the graph on 2^n vertices with each vertex labelled by a different binary string of length n and two vertices are adjacent if and only if their labels in exactly one bit.

Definition 10. A **path** in a graph G is a subgraph of G that is a copy of P_k for some k

A **cycle** in a graph G is a subgraph of G that is a copy of C_k for some k

Definition 11. The **length** of a path P is the number of vertices in it and is denote $|P|$ or $|V(P)|$. The **length** of a cycle is the number of vertices in it.

Definition 12. An **Eulerian circuit** in a graph G is a circuit which contains every edge of G.

An **Eulerian trail** in a graph G is a trail which contains every edge of G.

Definition 13. A graph G is **connected** if there is a path between every pair of vertices. G is **disconnected** otherwise.

A graph G is **k-connected** if there does not exist a set of at most $k-1$ vertices of G whose removal yield a disconnected graph.

A **connected component** of a graph G is a maximal connected subgraph (meaning we cannot add more edges and vertices while preserving connectivity).

Theorem 14. *Let G be a multigraph. G is a connected and all vertices of G have even degree if and only if G has an Eulerian circuit and G has no zero degree vertex.*

Definition 15. An **Hamiltonian cycle** in a graph G is a cycle which contains every vertex of G.
An **Hamiltonian path** in a graph G is a path which contains every vertex of G.

Theorem 16. *There exists an ordering (or sequence) containing all n-bit binary strings exactly once where every consecutive string differ in exactly one bit and the first and last string differ in exactly one bit.*

Lemma 17. *If a graph G has a Hamiltonian cycle then G is 2-connected.*

Theorem 18. (Dirac's theorem) *If a graph G has at least 3 vertices and the degree of every vertex of G is at least $\frac{|V(G)|}{2}$ then G has a Hamiltonian cycle.*

Definition 19. A **tree** is a connected graph with no cycles.
A **forest** is a graph with no cycles (which is not necessarily connected).

Definition 20. A **rooted tree** is digraph obtained from a tree T and a special vertex $r \in V(T)$ called the **root** by directing every edge "towards" the root (e.g., from the vertex farthest from the root to the vertex closest to the root).

Lemma 21. *If T is a tree with at least 2 vertices then T has at least 2 vertices of degree 1.*

Theorem 22. *Every tree on n vertices has exactly $n-1$ edges.*

Problem 23. Minimum spanning tree
Input: A connected graph $G = (V, E)$ and weights $w_e \geq 0$ for each edge $e \in E$
Output: A subset F of E such that (V, F) is connected and given these restrictions, $\sum_{e \in F} w_e$ is minimized.

Algorithm 24. Kruskal's algorithm
Initialize F to the empty set.
Sort the edges in ascending order of weights
For each edge e in this ordering.
If $(V, F \cup \{e\})$ does not contain a cycle then add e to F
Return F

Theorem 25. *Kruskal's algorithm returns a minimum spanning tree.*

Problem 26. Shortest path
Input: A connected graph $G = (V, E)$, weights $w_e > 0$ for each edge $e \in E$ and two vertices $s, t \in V$.
Output: A minimum weight path from s to t in G.

Algorithm 27. (Simplified) Dijkstra's algorithm
Initialize an array d indexed by V to ∞
$d[s] \leftarrow 0$
$S \leftarrow \{s\}$
Initialize an array prev indexed by V to null.
While $t \notin S$
Find $e = (u, v) \in E$ with $u \in S, v \in V \setminus S$ minimizing $d[u] + w_{(u,v)}$.
$d[v] \leftarrow d[u] + w_{(u,v)}$
$\text{prev}[v] \leftarrow u$
$S \leftarrow S \cup \{v\}$
Return d and prev

To obtain the path from the output, repeatedly follow the prev pointers, starting from t.

Lemma 28. *Dijkstra's algorithm assigns d values in a non-decreasing order.*

Lemma 29. *A subpath of a minimum weight path is a minimum weight path (between different endpoints).*

Theorem 30. *The d values returned by Dijkstra's algorithm corresponds to minimum weight distance from s.*

Definition 31. A **matching** in a graph $G = (V, E)$ is a subset of the edges $M \subseteq E$ where all vertices of (V, M) have degree at most 1.

Definition 32. A **perfect matching** in a graph G is a matching M where all vertices of G are incident to some edge of M.

Theorem 33. (Hall's theorem) *Let G be a bipartite graph with parts A and B. G contains a perfect matching if and only if $|A| = |B|$ and for all $S \subseteq A$, $|S| \leq |N(S)|$.*

Algorithm 34. Input: A bipartite graph $G = (V, E)$ with parts A and B, a matching M in G, the set of unmatched vertices U of A and the set of unmatched vertex W of B.

Output: Either

1. An M-augmenting path in G, or
2. A subset S of A with $|S| > |N(S)|$.

```
Initialize an array prev of pointers
S ← U
T ← ∅
For s ∈ S, set prev[s] ←null
While true
  If ∃e = (u, v) ∈ E with u ∈ S and v ∉ T then
    prev[v] ← u
    If v ∈ W then
      return the path from v following prev pointers.
    T ← T ∪ {v}
    w ← the vertex matched to v in M
    S ← S ∪ {w}
    prev[w] ← v
  Else
    return S
```

Algorithm 35. Input: A bipartite graph $G = (V, E)$ with parts A and B, a matching M in G, the set of unmatched vertices U of A and the set of unmatched vertex W of B.

Output: Either

1. An M-augmenting path in G, or
2. "An M-augmenting path does not exist in G."

Build a digraph H with vertex set A and directed edges $\{(u, v) | \exists v \in B, (u, v) \notin M, (v, w) \in M\}$.
Run a graph search algorithm (e.g., DFS or BFS) in H starting from U and see if we can reach a vertex in $N(W)$.

Problem 36. Chinese postman problem
Input: A connected graph $G = (V, E)$, weights $w_e \geq 0$ for each edge $e \in E$.
Output: A minimum weight set of edge of G that we need to "double" to make the graph Eulerian.

Algorithm 37. for solving the Chinese postman problem
Compute the degrees of all vertices in G
Let S be the set of odd degree vertices in G
Build H, the weighted complete graph with vertex set S and weights
$w_{u,v}$ =shortest path distance from u to v in G
Find a minimum weight maximum matching M in H.
Let F be the union of all edges of G on paths corresponding to edges of M.
Return F.

Definition 38. A **set** is a (unordered) collection of distinct elements.

Definition 39. A **function** f from a set A to a set B, denoted $f : A \to B$, is an assignment of one element of B to each element of A.

$f(a)$ is the element of B assigned to $a \in A$.

Definition 40. A function $f : A \to B$ is said to be **injective** (or **one-to-one**) if $\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ (i.e., no two elements of A get assigned the same element of B).

A function $f : A \to B$ is said to be **surjective** (or **onto**) if $\forall b \in B \exists a \in A$, such that $f(a) = b$ (i.e., all elements of B are assigned some element of A).

A function is **bijective** if it is both injective and surjective.

A bijective function is called a **bijection**.

Theorem 41. *If there is a bijection between A and B then $|A| = |B|$..*

Theorem 42. *The number of subsets of a set S of size n is 2^n.*

Theorem 43. *The complete graph on n vertices (K_n) has $\frac{n(n-1)}{2}$ edges.*

Theorem 44. *Let $G = (V, E)$ be a bipartite graph with parts A and B. Then*

$$\sum_{v \in A} \deg(v) = |E| = \sum_{v \in B} \deg(v)$$

Corollary 45. Let $f : A \to B$ be a function such that every element of B is a assigned exactly k elements of A. Then $k|B| = |A|$.

Lemma 46. *For any two sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$.*

Lemma 47. *For any three sets A_1, A_2 and B, $(A_1 \cap B) \cap (A_2 \cap B) = A_1 \cap A_2 \cap B$.*

Lemma 48. *For any $k+1$ sets $A_1, \ldots, A_k$ and B, $(A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_k \cap B) = (A_1 \cup A_2 \cup \ldots \cup A_k) \cap B$.*

Theorem 49. (Inclusion-exclusion principle) *For any n sets $S_1, S_2, \ldots, S_n$,*

$$|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1 \le i < j \le n} |S_i \cap S_j| + \sum_{1 \le i < j < k \le n} |S_i \cap S_j \cap S_k| - \ldots + (-1)^n |S_1 \cap S_2 \cap \ldots \cap S_n|$$

Lemma 50. *The number of subsets of size k of a set of size n is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Definition 51. A **fixed point** of a function $f : A \to A$ is an element $a \in A$ such that $f(a) = a$.

Definition 52. A **derangement** is a function $f : A \to A$ with no fixed points.

Theorem 53. *The number of derangements from A to A where $|A| = n$ is*

$$\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

Theorem 54. *The number of ways of choosing k elements out of n elements when repetition is allowed is* $\binom{n+k-1}{n-1}$.

Theorem 55. *If $f : A \to B$ and $g : B \to A$ are functions and $\forall a \in A, g(f(a)) = a$ and $\forall b \in B, f(g(b)) = b$ then f and g are bijections.*

We write f^{-1} for g in this case.

Theorem 56. *Every graph with at least 2 vertices has two vertices of the same degree.*

Theorem 57. (Pigeonhole principle) *If we put more than n objects into n boxes then there is a box with at least 2 objects in it.*

Theorem 58. (Generalized pigeonhole principle) *If we put n objects into k boxes then there is a box with at least $\lceil \frac{n}{k} \rceil$ objects in it.*

Definition 59. Let $G = (V, E)$ be a graph.

A **clique** in G is a subset U of V such that there is an edge between every pair of vertices in U.

A **stable set** in G is a subset U of V such that there no edge between any pair of vertices in U.

Theorem 60. *A graph with 6 vertices contains either a clique of size 3 or a stable set of size 3 (or both).*

Definition 61. The Ramsey number $R(s,t)$ is the minimum number such that every graph with (at least) $R(s,t)$ vertices contains either a clique of size s or a stable set of size t.

Lemma 62. *If $s \geq 3, t \geq 3$ then $R(s,t) \leq R(s-1,t) + R(s,t-1)$.*

Theorem 63. *If $s \geq 2, t \geq 2$ then $R(s,t) \leq 2^{s+t}$.*

Theorem 64. *For every $s > 2$, there exists a graph with $2^{s/2}$ vertices with no clique of size s and no stable set of size s.*

Definition 65. A **colouring** with k colours of a graph $G = (V, E)$ is assignment $c : V \to \{1, 2, \ldots, k\}$ such that adjacent vertices are assigned different values.

If such an assignment exists, we say that G is k-colourable.

Definition 66. A graph is **planar** if it can be drawn in the plane in such a way that no two edges cross.

Theorem 67. *If a planar graph has v vertices, e edges and f faces then $f + v = e + 2$.*

Theorem 68. *If a planar graph G has n vertices then G has at most $3n - 6$ edges.*

Corollary 69. Every planar graph has a vertex of degree less or equal to 5.

Theorem 70. *Every planar graph is 5-colourable.*

Definition 71. A **partially ordered set (or poset)** is a set S with a "less than" relation $<$ such that if $a < b$ and $b < c$ then $a < c$.

Definition 72. A **chain** in a poset $(A, <)$ is a set of elements $a_1, a_2, \ldots, a_k$ in A such that $a_1 < a_2 < \ldots < a_k$.

An **anti-chain** in a poset $(A, <)$ is a set of elements $a_1, a_2, \ldots, a_k$ in A such that a_i and a_j are incomparable for all $i \neq j$.

Theorem 73. *Let G be a bipartite graph. The minimum size of a vertex cover in G is equal to the size of a maximum matching in G.*

Theorem 74. Dilworth's theorem *Let $(A, <)$ be a poset. The maximum number of elements in an anti-chain of A is the minimum size of a partition of A into chains.*

Some more lemmas and theorems from the assignments.

Lemma 75. *Let G be a graph. If $C = c_1, c_2, \ldots, c_{k-1}, c_k$ is a cycle in G then for any j (between 1 and k), $c_j, c_{j+1}, \ldots, c_{k-1}, c_k, c_1, c_2, \ldots, c_{j-2}, c_{j-1}$ is also a cycle in G.*

Theorem 76. (Ore's theorem)
Let G be a graph. If G has at least 3 vertices and for every pair of non-adjacent vertices $u, v \in V(G)$, $\deg(u) + \deg(v) \geq |V(G)|$ then G has a Hamiltonian cycle.

Lemma 77. *Let G be a graph. For any $k > 2$, if G is k-connected then G is $k-1$ connected.*

Definition 78. A set of path $P_1, \ldots, P_k$ with the same starting and ending vertex is said to be **internally vertex disjoint** if no two paths have a vertex in common except for their endpoints. That is, if $P_i = u, p_{i,1}, p_{i,2}, \ldots, v$ then there does not exist i, j, k, ℓ with $i \neq k$ such that $p_{i,j} = p_{k,\ell}$.

Theorem 79. (Part of Menger's theorem)
Let G be a graph. If every pair of (distinct) vertices $u, v \in V(G)$, there are two vertex disjoint paths P_1, P_2 starting at u and ending at v then G is 2-connected.

Definition 80. The **Cartesian product** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted $G_1 \times G_2$, is a graph with vertex set V and edge set E defined as follows. V consists of all pair (v_1, v_2) for each vertex v_1 in V_1, and each vertex v_2 in V_2 (i.e., $V = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$). Two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if either

- $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2, or
- $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

In other words, $E = \{((u_1, u_2), (v_1, v_2)) | u_1 = v_1, (u_2, v_2) \in E(G_2)\} \cup \{((u_1, v_1), (u_2, v_2)) | u_2 = v_2, (u_1, v_1) \in E(G_1)\}$.

Theorem 81. *If G has a Hamiltonian cycle then $G \times P_2$ has a Hamiltonian cycle where P_2 is the graph on two vertices with a single edge*

Theorem 82. *If G_1 and G_2 both have Hamiltonian cycles and $|V(G_1)| = |V(G_2)|$ then $G_1 \times G_2$ has a Hamiltonian cycle*

Theorem 83. *If $T = (V, E)$ is a tree then for any $e \in E$, $(V, E \setminus \{e\})$ is disconnected.*

Definition 84. A **subtree** of a tree T is a subgraph of T which is also a tree.

Definition 85. Let $S_1, S_2, \ldots, S_k$ be sets. $T = \{(s_1, s_2, \ldots, s_k) | s_1 \in S_1, s_2 \in S_2, \ldots, s_k \in S_k\}$ is called the **Cartesian product** of $S_1, S_2, \ldots, S_k$ and is denoted by $S_1 \times S_2 \times \ldots \times S_k$.

Theorem 86. *For any k and any k sets $S_1, S_2, \ldots, S_k$, $|S_1 \times S_2 \times \ldots \times S_k| = |S_1||S_2| \ldots |S_k|$ (i.e., the size of the Cartesian product of these sets is the product of the sizes of these sets).*

Theorem 87. *The number of functions $f : A \to B$ where $|A| = x$ and $|B| = y$ is y^x.*

Appendix 4: Glossary of symbols

Symbol	Name	Example or definition	Example read as
$\vee$	Logical or	$p \vee q$	p or q.
$\wedge$	Logical and	$p \wedge q$	p and q.
$\neg$	Logical not	$\neg p$	not p.
$\rightarrow$	Implication	$p \rightarrow q$	p implies q. If p then q. q whenever p.
$\leftrightarrow$	Bi-implication	$p \leftrightarrow q$	p if and only if q.
$\equiv$	Equivalence	$p \equiv q$	p is equivalent to q.
$\mathbf{F}$	Contradiction	$\mathbf{F} \rightarrow p$	False implies p.
$\mathbf{T}$	Tautology	$\mathbf{T} \rightarrow \mathbf{F}$	True implies false.
$\vdash$	Infer	$P_1, \ldots, P_k \vdash Q$	We can infer Q from $P_1, \ldots, P_k$.
$\models$	Models	$P_1, \ldots, P_k \models Q$	$P_1, \ldots, P_k$ models Q.
$\in$	Containment	$x \in S$	x is in S. x is an element of S.
$\subseteq$	Subset	$S \subseteq T$	S is a subset of T.
$\cap$	Intersection	$S \cap T = \{x \mid x \in S, x \in T\}$	S intersect T. The elements in both S and T.
$\cup$	Union	$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$	S union T. The elements in either S or T.
$\setminus$	Set difference	$S \setminus T = \{x \mid x \in S, x \notin T\}$	S minus T. The elements in S but not T.
$\mid\ \mid$	Cardinality	$\|S\|$	The size of S.
$\forall$	Universal quantifier	$\forall x \in \mathbb{Z}, x^2 \geq 0$	For all integers x, x^2 is greater or equal to zero.
$\exists$	Existential quantifier	$\exists x \in \mathbb{Z}, x + 5 = 0$	There exists an integer x such that $x + 5$ is zero.