Assignment 4

This assignment is worth 4% of your final grade. Answer all questions from 1 to 4. You may also answer the bonus questions for extra marks but you cannot receive partial marks for the bonus questions.

Please show steps of your reasoning. You may use any theorems (or lemmas or claims) that we proved in class if you clearly state which theorem you are using. Note that if a theorem you want to use is similar to (but not exactly) one seen in class, you should prove it first. Do not simply state that the proof is similar (write down the whole proof).

1. (a) (3 points) Prove or disprove the following statement.

   Let $S_1, S_2, \ldots, S_k$ be k sets. If for any i and j between 1 and k, $S_i \cap S_j$ is non-empty then there exists an element x such that for any i between 1 and k, $x \in S_i$ (i.e., the sets contain a common element).

   (b) (7 points)

   Definition 1. A *subtree* of a tree T is a subgraph of T which is also a tree.

   Note that all subgraphs of a tree are forests but not necessarily trees.

   Prove the following statement.

   Let T be a tree. Let $T_1, \ldots, T_k$ be a set of subtrees of T. If for all i, j, $V(T_i) \cap V(T_j)$ is non-empty then

   $$\bigcap_{i=1}^{k} V(T_i)$$

   is non-empty.

   [Hint: Try to prove this by induction on the number of vertices in T rather than k.]

2. (a) (5 points)

   Prove or disprove the following statement.

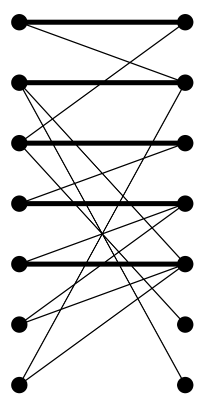
   Let $G = (V, E)$ be a graph with positive weights $w_e > 0$ for all edges $e \in E$. For all minimum weight paths $P = p_1, \ldots, p_k$ (from 1 to k) in G and all $1 \leq i < j \leq k$, the subpath $p_i, p_{i+1}, \ldots, p_j$ is a minimum weight path between p_i and p_j.

   (b) (5 points)

   Let F be the set of chosen edges during a run of (simplified) Dijkstra's algorithm on a graph $G = (V, E)$ (with single source s and no target vertex). Prove that (V, F) is a tree.

3. (a) (3 points)

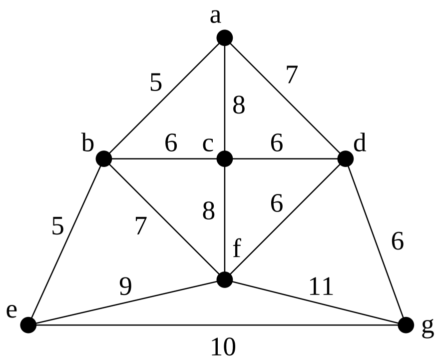
   Let G be the following graph and M be a matching consisting of the highlighted edges.



Find a perfect matching in G by starting from M and repeatedly finding a M-augmenting path and modifying M. You may use either algorithm (you are told that G contains a perfect matching). Do not forget to show your steps.

(b) (3 points)

Solve the Chinese postman problem where the input is the following graph.



(c) (4 points)

Recall that for a set S of vertices in a graph, $N(S) = \{v | \exists s \in S, (v, s) \in E\}$.

Prove or disprove the following statement.

Let $G = (V, E)$ be a graph. If G has an even number of vertices and for all $S \subseteq V$, $|S| \leq |N(S)|$ then G contains a perfect matching.

4. (a) (5 points)

Definition 2. Let $S_1, S_2, \ldots, S_k$ be sets. $T = \{(s_1, s_2, \ldots, s_k) | s_1 \in S_1, s_2 \in S_2, \ldots, s_k \in S_k\}$ is called the *Cartesian product* of $S_1, S_2, \ldots, S_k$ and is denoted by $S_1 \times S_2 \times \ldots \times S_k$.

Prove the following statement by explicitly defining a bijection.

For any k and any k sets $S_1, S_2, \ldots, S_k$, $|S_1 \times S_2 \times \ldots \times S_k| = |S_1||S_2| \ldots |S_k|$ (i.e., the size of the Cartesian product of these sets is the product of the sizes of these sets).

(b) (5 points)

Prove the following statement by explicitly defining a bijection.

The number of functions $f : A \to B$ where $|A| = x$ and $|B| = y$ is y^x.

5. **Bonus** (10 points)

Definition 3. A *vertex cover* in a graph $G = (V, E)$ is a subset X of V such that all edges are incident to at least one vertex of X (i.e., $\forall (u, v) \in E, u \in X$ or $v \in X$).

Let $G = (V, E)$ be a bipartite graph. Prove that the size of a maximum matching in G is equal to the size of a minimum vertex cover in G.