1. (b) The propositional variables are

p: the box contains a blue ball
q: the box contains a green ball
r: the box contains a red ball

so we have to prove

$$\frac{r \vee (\neg p \wedge \neg q)}{(p \vee q) \to r}$$

Proof.

1	$r \vee (\neg p \wedge \neg q)$	premise
2	r	assumption
3	$p \vee q$	assumption
4	r	2
5	$(p \vee q) \to r$	$3-4, \to\mathcal{I}$
6	$r \to ((p \vee q) \to r)$	$2, 5, \to\mathcal{I}$
7	$\neg p \wedge \neg q$	assumption
8	$p \vee q$	assumption
9	p	assumption
10	$\neg p$	$7, \wedge\mathcal{E}$
11	$\mathbf{F}$	$9, 10, \neg\mathcal{E}$
12	$p \to \mathbf{F}$	$9-11, \to\mathcal{I}$
13	q	assumption
14	$\neg q$	$7, \wedge\mathcal{E}$
15	$\mathbf{F}$	$13, 14, \neg\mathcal{E}$
16	$q \to \mathbf{F}$	$13-15, \to\mathcal{I}$
17	$\mathbf{F}$	$8, 12, 16, \vee\mathcal{E}$
18	r	$17, \mathbf{F}\mathcal{E}$
19	$(p \vee q) \to r$	$8-18, \to\mathcal{I}$
20	$(\neg p \wedge \neg q) \to ((p \vee q) \to r)$	$7-19, \to\mathcal{I}$
21	$(p \vee q) \to r$	$1, 6, 20, \vee\mathcal{E}$

$\square$

(a)

$$
\begin{aligned}
& \neg\forall a((\exists b\ p(a,b) \to \exists c\ p(a,c)) \wedge \neg\forall d\ q(a,d)) \\
\equiv\ & \exists a(\neg((\exists b\ p(a,b) \to \exists c\ p(a,c)) \wedge \neg\forall d\ q(a,d))) \\
\equiv\ & \exists a(\neg(\exists b\ p(a,b) \to \exists c\ p(a,c)) \vee \neg\neg\forall d\ q(a,d)) \\
\equiv\ & \exists a(\neg(\exists b\ p(a,b) \to \exists c\ p(a,c)) \vee \forall d\ q(a,d)) \\
\equiv\ & \exists a(\neg(\neg\exists b\ p(a,b) \vee \exists c\ p(a,c)) \vee \forall d\ q(a,d)) \\
\equiv\ & \exists a(\neg\neg\exists b\ p(a,b) \wedge \neg\exists c\ p(a,c)) \vee \forall d\ q(a,d)) \\
\equiv\ & \exists a(\exists b\ p(a,b) \wedge \neg\exists c\ p(a,c)) \vee \forall d\ q(a,d)) \\
\equiv\ & \exists a(\exists b\ p(a,b) \wedge \forall c\ \neg p(a,c)) \vee \forall d\ q(a,d))
\end{aligned}
$$

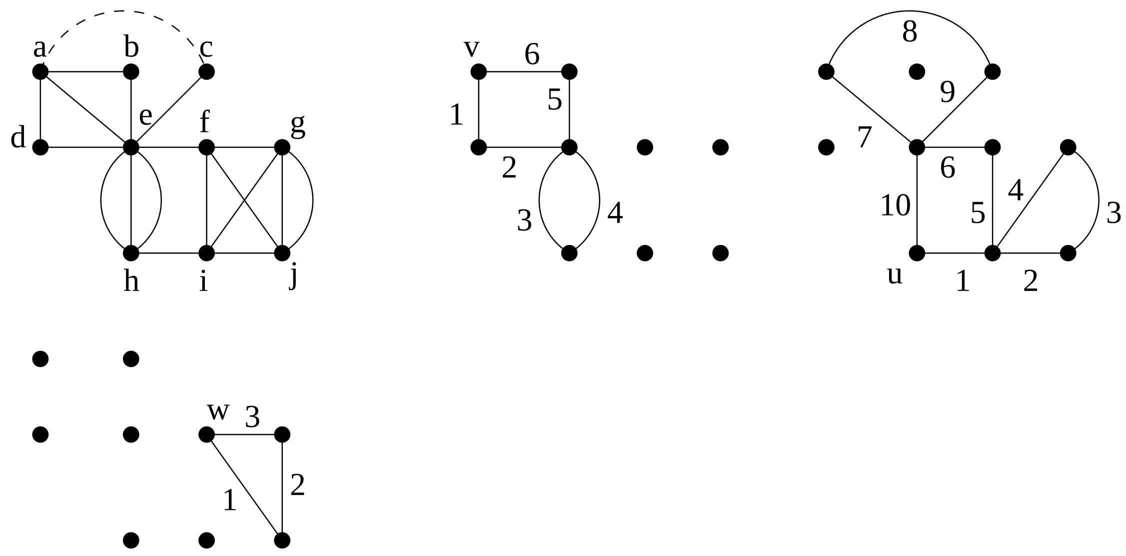
2. (a) This, of course, depends on the parity of the sum of the digits of your student number.

   If the parity is odd, by the Handshaking lemma, such a multigraph does not exist.

   If the parity is even, then we can calculate how many edges the multgraph would have if it existed. It is half the sum of the degrees. If the degree of some vertex is greater than the number of edges, again the multigraph does not exist.

   Otherwise, such a multigraph exist. We can start with an empty graph and repeatedly add an edge between two vertices with the highest "remaining" degree.

   (b) Here is a run of the (algorithmic) proof seen in class.



We first add an imaginary edge between the two odd degree vertices of the graph. Our first starting vertex v is chosen randomly. The circuit we took to get back to it is drawn. Then, we picked a second starting vertex u on our first circuit which is incident to some unvisited edge. The circuit starting from u is drawn. Finally, we visit the 3 remaining edges.

Thus, the vertices of the first circut (in the order they are visited) are $adeheba$. Then we add the second circuit between visiting edges 3 and 4 to get $adehijgifeaceheba$. Finally, we add the last 3 edges between visiting edges 5 and 6 of the second circuit to get $adehijgifjgfeaceheba$. Now we shift this so that we visit our imaginary edge (a, c) last. This gives the Eulerian trail $cehebadehijgifjgfea$.

(c) As in a), the answer depends on your student number.

Claim 1. *If the sum of the digits is not a multiple of 5 then it is not possible that each student answered exactly 5 question.*

The proof follows the proof of the Handshaking lemma.

Proof. We prove the contrapositive.

Suppose each student answered exactly 5 questions.

If for each question, we count the students that answered that question then we have counted each student exactly 5 times. But we have counted each question exactly the number of times it was answer (which is the sum of the digits of your student number). Thus, this sum is a multiple of 5. □

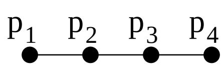
If the sum is a multiple of 5 then we know how many students there were by dividing the sum of the digits by 5. If the number of students is greater than the number of times a question was answered, again it is impossible the question i was answered d_i times for each i.

Otherwise, it is possible. To construct an example, we proceed as follows. We say the first student answered the 5 most answered questions (breaking ties arbitrarily). Then the second student answered the 5 most answered question that remained. And so on.

3. (a) This statement is true.

Proof. Let C be a cycle and $1 \leq j \leq k$. All we need to do is check that there are edges between every pair of consecutive vertices in $c_j, c_{j+1}, \ldots, c_{k-1}, c_k, c_1, c_2, \ldots, c_{j-2}, c_{j-1}$. Since C is a cycle and therefore a path, there is an edge c_jc_{j+1}, $c_{j+1}c_{j+2}$ up to $c_{k-1}c_k$. Since C is a cycle, there is an edge between its last vertex and its first vertex. That is, the edge c_kc_1 exists. Again, since C is a cycle, there is an edge c_1c_2, c_2c_3 up to $c_{j-1}c_j$. So all required edges are present (including $c_{j-1}c_j$ from the last vertex to the first vertex). □

(b) This statement is false. For example, if the whole graph is just a path and $j = 2$ as in the following example.



(c) The proof of this theorem is essentially the same as the proof of Dirac's theorem.

Proof. Suppose the theorem is false. Then there exists a graph G with no Hamiltonian cycle, at least 3 vertices and every vertex of G has degree at least $\frac{n}{2}$ where $n = |V(G)|$.

Let $P = p_1, \ldots, p_k$ be a longest path in G. We claim that G contains a cycle of length k (that is, a cycle with k vertices).

Proof. If p_1 is adjacent to p_k, this extra edge makes P a cycle. So from now on, we may assume p_1 is not adjacent to p_k.

Suppose p_1 is adjacent to a vertex u of G not on P. Then $u, p_1, \ldots, p_k$ is a longer path than G. This is a contradiction to our choice of P.

Thus, p_1 is only adjacent to vertices of P. By symmetry, p_k is only adjacent to vertices of P.

Suppose p_1 is adjacent to p_i and p_k is adjacent to p_{i-1} then $p_1, p_i, p_{i+1}, \ldots, p_{k-1}, p_k, p_{i-1}, \ldots, p_2$ is a cycle of length k.

So we will try to find such an i. If $S = \{i | p_i \in N(p_1)\}$ and $T = \{i+1 | p_i \in N(p_k)\}$ contain a common element then we have found the i we needed. Otherwise, the intersection of S and T is empty and their union is contained in $\{2, 3, \ldots, k\}$. So $k - 1 = |\{2, \ldots, k\}| \geq |S \cup T| = |S| + |T| = \deg(p_1) + \deg(p_k) \geq n$ since p_1 and p_k are non-adjacent. This is a contradiction since $k \leq n$ (there cannot be more vertices in the path P than there are vertices in the whole graph G). □

Let $C = c_1, \ldots, c_k$ be a cycle of length k in G. If there is any edge $c_i u$ between a vertex in C and a vertex not in C then $u, c_i, c_{i+1}, \ldots, c_{k-1}, c_k, c_1, c_2, \ldots, c_{i-1}$ is a path (exercise: check that this is indeed a path). But this path has length $k + 1$.

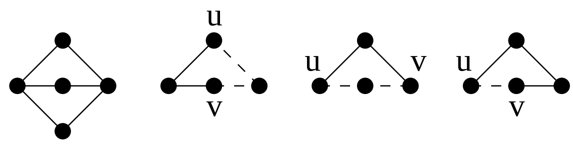
So there are no edges between vertices in C and vertices not in C. So the degrees of a (any) vertex x in C and a (any) vertex y not in C must satisfy $\deg(x)+\deg(y) \geq n$. But $\deg(x)$ is at most the number of vertices of C minus 1 (because x is not its own neighbour) which is $k - 1$ and $\deg(y)$ is at most the number of vertices not in C minus 1 which is $n - k - 1$. But $k - 1 + n - k - 1 = n - 2$ which is less than n. Contradiction. □

4. (a) This statement is true.

Proof. We prove the contrapositive.

If G is not $k - 1$ connected then there exists a set X of at most $k - 2$ vertices whose removal disconnects G. But a set of at most $k - 2$ vertices is a set of at most $k - 1$ vertices (that is, $|X| \leq k - 2 \leq k - 1$). Therefore, G is not k connected (we have exhibited as set, namely X, whose removal disconnects G). □

(b) This statement is false. For example the following graph has 2 internally vertex disjoint paths between every pair of vertices but no Hamiltonian cycle.



By symmetry, we only need to check that the pairs of vertices show above have 2 internally vertex disjoint paths. One path is shown with dashed lines and the other with full lines.

In a (Hamiltonian) cycle, each of the three degree 2 vertices would need one of its two neighbours right before it and one of its two neighbours right after it. But any degree 3 vertex can only have one vertex before it in the cycle and one vertex after it.

(c) This statement is true.

Proof. We prove this by contradiction.

Suppose every pair of vertices $u, v \in V(G)$ has two internally vertex disjoint paths between them and G is not 2-connected.

Since G is not 2-connect, there exists a set X of size at most 1 whose removal disconnects G. By definition of disconnectivity, there exists two vertices u and v in $G - X$ such that there is no path from u to v.

But we know that there are 2 internally vertex disjoint paths P_1, P_2 between u and v in G. Since $u, v \in G - X$, neither u nor v is in X. Since X contains at most one vertex, there is a path (either P_1 or P_2) which does not contain any vertex of X. This is a contradiction to u and v having no paths between them in $G - X$. □

5. This statement combined with 4c) is what is known as Menger's theorem for the special case of 2-connectivity (instead of k-connectivity).

Proof. We prove the statement for each value of n where n is the number vertices in the graph G. Suppose the statement does not hold for all graphs on n vertices. Then there is a graph on n vertices that is 2-connected but there is a pair of vertices $u, v \in V(G)$ which do not have two internally vertex disjoint paths between them. Pick G, u and v such that $\deg(v)$ is maximized. Note that we can do this since n is fixed so $\deg(v)$ is bounded by n.

If u is adjacent to v then there is a path which does not use the edge (u, v) as otherwise, removing either u or v disconnects G (e.g., if we remove u then there is no path from v to any neighbour of u). This is a contradiction to G being 2-connected.

Thus, u is not adjacent to v. Since G is 2-connected, G is connected and there exists some path $P = u, p_2, \ldots, v$ from u to v in G. Since u and v are non-adjacent, the second vertex p_2 of P exists and is neither u nor v. Let Q be a path from u to v in $G - p_2$ (Q exists since G is 2-connected).

The "middle" (non-endpoint) of Q and P intersect as otherwise, they are two internally vertex disjoint paths between u and v which we assumed do not exist. Let p_i be the last (highest indexed) "middle" vertex of P in which they intersect (so, $p_i = q_\ell$, say).

So the subpath P' of P from p_i to v and the subpath Q' of Q from p_i to v do not intersect except at their endpoints.

Let H be the graph obtain from G by adding all (missing) edges from v to vertices of P' and Q'. By maximality of the degree of v, there are two internally vertex disjoint paths from u to v in H. We can pick these path so that neither of them contains more than one neighbour of v in H (e.g., by picking the shortest possible paths).

Now either these paths are already paths of G or we can replace their last edge by subpaths of P' and Q' (possibly in reversed order). For example, if the two neighbours of v used are p_j and p_k with $i \leq j < k$ (i.e., p_j comes before p_k in P) then we can replace the (possibly missing) edge (p_k, v) by the subpath $p_k, p_{k+1}, \ldots, v$ of P and the (possibly missing) edge (p_j, v) by $p_j, p_{j-1}, \ldots, p_i = q_\ell, q_{\ell+1}, \ldots, v$.

Thus, in either case, we have found 2 internally vertex disjoint paths between u and v. Contradiction. $\square$