Assignment 2

This assignment is worth 4% of your final grade. Recall that the marking scheme has been changed to max(20% Assignments + 20% Midterm + 60% Final, 20% Assignments + 80% Final).

Answer all questions from 1 to 4. You may also answer the bonus question for extra marks but you cannot receive partial marks for the bonus question.

Please show steps of your reasoning. You may use any theorems (or lemmas or claims) that we proved in class if you clearly state which theorem you are using. Note that if a theorem you want to use is similar to (but not exactly) one seen in class, you should prove it first. Do not simply state that the proof is similar (write down the whole proof).

1. (a) (3 points) Write a first order logic formula which is equivalent to the following but all $\neg$ symbols appear immediately in front of a predicate.

$$\neg\forall a((\exists b\ p(a,b) \to \exists c\ p(a,c)) \land \neg\forall d\ q(a,d))$$

   Here p and q are predicates.

   Note that all equivalences that we have seen for propositional logic also apply when the formulas in the equivalences are first order logic formulas. This happens to be the case for propositional logic and first order logic but it needs not be true in general.

   For this question, you are allowed to make *substitutions*. That is, you are allowed to replace a part of a formula by a different formula which is equivalent to that part (e.g., $P \to (Q \to R) \equiv P \to (\neg Q \lor R)$). But in this case, indicate which formula is substituted with which one (in that step).

   (b) (7 points) For the following arguments, rewrite it using propositional logic (not first order logic). Determine if the argument is valid. If it is valid, give a proof using rules of inference. If it is invalid, give a set of values for the propositions you have defined which makes all premises true but the conclusion false and write what these propositions with these values mean in English.

   Make no assumption about the truth value of individual proposition variable (e.g., do not assume that $1 + 1 = 2$).

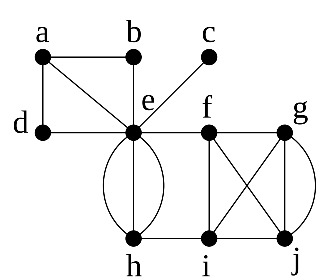
   Note that you cannot use the completeness theorem (namely because we haven't proven it).

   Either the box contains a red ball or the box contains neither a blue ball nor a green ball.

   If the box contains a blue ball or a green ball then the box also contains a red ball.

2. (a) (3 points) Does there exist a multigraph on 9 vertices whose ith vertex has degree equal to the ith digit of your student number? If it exists, draw one such multigraph and label the vertices from v_1 to v_9 so that $\deg(v_i)$ is the i digit of your student number. If it does not exist, show why with a proof.

   (b) (3 points) Find an Eulerian trail in the following graph (you should show your steps, not just the final answer). Write down the **vertices** visited by your circuit in the order they are visited (i.e., there is no need to write down which edge were used). The vertices are labelled a to j.



(c) (4 points) Let d_i be the ith digit of your student number. In an assignment containing 9 questions, d_i students answered the ith question. Is it possible that each student answered exactly 5 questions? If it is possible, give an example where this could happen (e.g., student A and B answered questions $1, 2, 4, 5, 8$, etc). If it is impossible, show why with a proof.

3. Prove or disprove each of the following statements. Depending on the negation of the statement, it may be sufficient to produce a counter-example (e.g., a graph) and verify that it is indeed a counter-example in case the statement is false.

   (a) (2 points) Let G be a graph. If $C = c_1, c_2, \ldots, c_{k-1}, c_k$ is a cycle in G then for any j (between 1 and k), $c_j, c_{j+1}, \ldots, c_{k-1}, c_k, c_1, c_2, \ldots, c_{j-2}, c_{j-1}$ is also a cycle in G.

   (b) (3 points) Let G be a graph. If $P = p_1, p_2, \ldots, p_{k-1}, p_k$ is a path in G then for any j (between 1 and k), $p_j, p_{j+1}, \ldots, p_{k-1}, p_k, p_1, p_2, \ldots, p_{j-2}, p_{j-1}$ is also a path in G.

   (c) (5 points) Let G be a graph. If G has at least 3 vertices and for every pair of non-adjacent vertices $u, v \in V(G)$, $\deg(u) + \deg(v) \geq |V(G)|$ then G has a Hamiltonian cycle.

4. Prove or disprove each of the following statements. Depending on the negation of the statement, it may be sufficient to produce a counter-example (e.g., a graph) and verify that it is indeed a counter-example in case the statement is false.

   (a) (3 points) Let G be a graph. For any $k > 2$, if G is k-connected then G is $k-1$ connected.

   (b) (3 points) A set of path $P_1, \ldots, P_k$ with the same starting and ending vertex is said to be *internally vertex disjoint* if no two paths have a vertex in common except for their endpoints. That is, if $P_i = u, p_{i,1}, p_{i,2}, \ldots, v$ then there does not exist i, j, k, ℓ with $i \neq k$ such that $p_{i,j} = p_{k,\ell}$.

   Let G be a graph. If every pair of (distinct) vertices $u, v \in V(G)$, there are two internally vertex disjoint paths P_1, P_2 starting at u and ending at v then G has a Hamiltonian cycle.

   (c) (4 points) Let G be a graph. If every pair of (distinct) vertices $u, v \in V(G)$, there are two vertex disjoint paths P_1, P_2 starting at u and ending at v then G is 2-connected.

5. **Bonus** (10 points) Let G be a graph. Prove that if G is 2-connected then for every pair of (distinct) vertices $u, v \in V(G)$, there exists two internally vertex disjoint paths P_1, P_2 from u to v.