A proof of the Tutte-Berge formula

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1 Definitions and statement

A few bits of notation that will be used throughout:

- $\text{def}(G)$: the deficiency of the graph $G$, i.e. the number of unmatched vertices in a maximum matching of $G$
- $\nu(G)$: the size of a maximum matching of the graph $G$
- $\text{odd}(G)$: the number of connected components with and odd number of vertices in $G$
- Alternating path for a matching $M$
- Augmenting path for a matching $M$: an $M$-alternating path with endpoints not covered by $M$

The statement of the Tutte-Berge formula:

**Theorem 1.1** (Tutte-Berge formula). In any graph $G = (V,E)$, the following holds:

$$\text{def}(G) = \max_{X \subseteq V} \left\{ \text{odd}(G \setminus X) - |X| \right\}$$

One can rewrite this equation to express it in terms of the size of the matching:

$$\nu(G) = \frac{1}{2} \left( n - \max_{X \subseteq V} \left\{ \text{odd}(G \setminus X) - |X| \right\} \right)$$

2 A first proof

Here is a first proof by contradiction that roughly amounts to obtaining a contradiction by reducing the problem in a big graph to a smaller graph.

 prova. 

**Lemma 2.1.** If the formula holds for a graph $G$, then if we let $X_G$ be a set satisfying the equality, then every vertex of $X_G$ must be contained in every maximum matching of $G$.

**proof of lemma 2.1.** There are $\text{def}(G) + |X_G|$ odd components in $G \setminus X_G$. As each odd component leaves at least one vertex unmatched, it means that at least $\text{def}(G) + |X_G|$ vertices cannot be matched in these components alone. Hence, since each component sees only $X_G$, at most $|X_G|$ of the unmatched vertices from odd components can be matched. Suppose now for contradiction that $\exists u \in X_G$ and a maximum matching $M$ such that $u \notin M$. Then from the odd components alone we have at least $\text{def}(G) + 1$ unmatched vertices, which contradicts the maximality of $M$.

Back to the main proof. Suppose for contradiction that the formula does not always hold; let $G$ be the smallest (with respect to $|V(G)|$) counterexample.

**Lemma 2.2.** There is no vertex in $V(G)$ that is in every maximum matching of $G$. 

Proof. 2.2 Suppose there is such a $v$. Then if we define $H = G\setminus \{v\}$, we know $H$ satisfies the formula ($G$ is the smallest counterexample). Thus there is some set $X_H \subset V(H)$ such that
\[ \text{def}(H) = \text{odd}(X_H) + |X_H| \]
Letting $X_G = X_H \cup \{v\}$, we see $|X_H| = |X_G| - 1$. Further, we know $\text{def}(H) = \text{def}(G) + 1$ as $v$ is in every maximum matching of $G$. Plugging in these values in the formula for $H$, we get that $G$ satisfies the formula with $X_G$, which is a contradiction.

Now for the last big lemma.

Lemma 2.3. For every edge $uv \in E(G)$, $\exists$ an odd cycle $C$ through $uv$ such that there are two maximum matchings $M, N$ such that $E(C)\setminus \{uv\} \subset M \cup N$.

Proof. 2.3 We are in the following situation: PICTURE!!!!!! Say $M$ covers $u$, $N$ covers $v$.

Now if we start at $uv$ and follow the edges in $M$ and $N$, we get three cases:

- we obtain a path
- we obtain an even cycle
- we obtain an odd cycle

In the last case we are happy, so we’ll focus on why the first two cannot happen.

If we have a path, then it means that after repeating the procedure, we came to a stop, i.e. there was an edge $wx$ of (WLOG) $M$ and then no neighbour of $x$ was covered by $N$. But in particular, $x$ is not covered by $N$. Thus if we consider the path starting at the vertex $u$ and ending at $x$, we get an $N$-augmenting path. Contradiction.

If we have an even cycle, then since there is already a vertex in the cycle that is not covered by (WLOG) $M$, there must be a second unmatched vertex in the cycle. But this contradicts the way we constructed the cycle.

So the only case left is that we have an odd cycle, so we’re done.

This means that $\nu(G\setminus C) = \nu(G) - \frac{|V(G)|-1}{2}$.

Note that $G$ does have at least one edge, otherwise taking $X_G$ empty would yield a contradiction.

So take a cycle $C$ like in the last lemma and define $GxC$ from $G$ by "contracting" the cycle $C$.

That is to say

- $V(GxC) = V(G) \cup \text{a new vertex } c$
- $E(GxC) = E(G\setminus C) \cup \{xc \mid x \in V(G) \setminus C, \text{ and } \exists y \in C \text{ s.t. } xy \in E(G)\}$

For any maximum matching $M'$ in $GxC$ there is a maximum matching $M$ in $G$ with $M \subset M' \cup E(C)$. This can easily be seen by considering two cases: $M'$ covers and does not cover $c$.

Since $|V(GxC)| < |V(G)|$ we can apply the Tutte-Berge formula to $GxC$ and get a contradiction from $\nu(GxC) = \nu(G) - \frac{|V(G)|-1}{2}$

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2