A proof of the Tutte-Berge formula

Youri Tamitegama

September 25, 2018

1 Definitions and statement

A few bits of notation that will be used throughout:

- $def(G)$: the deficiency of the graph G, i.e. the number of unmatched vertices in a maximum matching of G
- $\nu(G)$: the size of a maximum matching of the graph G
- $odd(G)$: the number of connected components with and odd number of vertices in G
- Alternating path for a matching M
- Augmenting path for a matching M : an M-alternating path with endpoints not covered by M

The statement of the Tutte-Berge formula:

Theorem 1.1 (Tutte-Berge formula). *In any graph $G = (V, E)$, the following holds:*

$$def(G) = \max_{X \subset V} \left\{ odd(G \backslash X) - |X| \right\}$$

One can rewrite this equation to express it in terms of the size of the matching:

$$\nu(G) = \frac{1}{2} \left(n - \max_{X \subset V} \left\{ odd(G \backslash X) - |X| \right\} \right)$$

2 A first proof

Here is a first proof by contradiction that roughly amounts to obtaining a contradiction by reducing the problem in a big graph to a smaller graph.

Proof.

Lemma 2.1. *If the formula holds for a graph G, then if we let X_G be a set satisfying the equality, then every vertex of X_G must be contained in every maximum matching of G.*

proof of lemma 2.1. There are $def(G) + |X_G|$ odd components in $G \backslash X_G$. As each odd component leaves at least one vertex unmatched, it means that at keast $def(G) + |X_G|$ vertices cannot be matched in these components alone. Hence, since each component sees only X_G, at most $|X_G|$ of the unmatched vertices from odd components can be matched. Suppose now for contradiction that $\exists u \in X_G$ and a maximum matching M such that $u \notin M$. Then from the odd components alone we have at least $def(G) + 1$ unmatched vertices, which contradicts the maximality of M. □

Back to the main proof. Suppose for contradiction that the formula does not always hold; let G be the smallest (with respect to $|V(G)|$) counterexample.

Lemma 2.2. *There is no vertex in $V(G)$ that is in every maximum matching of G.*

Proof. 2.2 Suppose there is such a v. Then if we define $H:=G\backslash\{v\}$, we know H satisfies the formula (G is the smallest counterexample). Thus there is some set $X_H \subset V(H)$ such that

$$def(H) = odd(X_H) + |X_H|$$

Letting $X_G = X_H \cup \{v\}$, we see $|X_H| = |X_G| - 1$. Further, we know $def(H) = def(G) + 1$ as v is in every maximum matching of G. Plugging in these values in the formula for H, we get that G satisfies the formula with X_G, which is a contradiction. □

Now for the last big lemma.

Lemma 2.3. *For every edge $uv \in E(G)$, $\exists$ an odd cycle C through uv such that there are two maximum matchings M, N such that $E(C)\backslash\{uv\} \subset M \cup N$.*

Proof. 2.3 We are in the following situation: PICTURE!!!!!! Say M covers u, N covers v.

Now if we start at uv and follow the edges in M and N, we get three cases:

- we obtain a path
- we obtain an even cycle
- we obtain an odd cycle

In the last case we are happy, so we'll focus on why the first two cannot happen.

If we have a path, then it means that after repeating the procedure, we came to a stop, i.e. there was an edge wx of (WLOG) M and then no neighbour of x was covered by N. But in particular, x is not covered by N. Thus if we consider the path starting at the vertex u and ending at x, we get an N-augmenting path. Contradiction.

If we have an even cycle, then since there is already a vertex in the cycle that is not covered by (WLOG) M, there must be a second unmatched vertex in the cycle. But this contradicts the way we constructed the cycle.

So the only case left is that we have an odd cycle, so we're done. □

This means that $\nu(G\backslash C) = \nu(G) - \frac{|V(C)|-1}{2}$.

Note that G does have at least one edge, otherwise taking X_G empty would yield a contradiction. So take a cycle C like in the last lemma and define GxC from G by "contracting" the cycle C. That is to say

- $V(GxC) = V(G) \cup$ a new vertex c
- $E(GxC) = E(G\backslash C) \cup \left\{xc \mid x \in V(G)\backslash C, \text{ and } \exists y \in C \text{ s.t. } xy \in E(G)\right\}$

For any maximum matching M' in GxC there is a maximum matching M in G with $M \subset M' \cup E(C)$. This can easily be seen by considering two cases: M' covers and does not cover c.

Since $|V(GxC)| < |V(G)|$ we can apply the Tutte-Berge formula to GxC and get a contradiction from $\nu(GxC) = \nu(G) - \frac{V(G)-1}{2}$

□