Eigenfunctions of the Laplacian on the Torus

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1 Introduction

Given a compact Riemannian manifold (M, g), we define the Laplacian to be the following differential operator $C^{\infty}(M) \longrightarrow C^{\infty}(M)$: $\Delta u = -div\nabla u$. In this note we investigate its eigenvalues and eigenfunctions on the n dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$ for some lattice Γ. First we treat a simple one dimensional case, then we go over the more general case.

2 The one dimensional case

In dimension one, a torus is simply a circle of some given length: $\mathbb{T} = \mathbb{R}/l\mathbb{Z}$ for some $l \in \mathbb{R}$. So any function defined on it can be thought of as a periodic function on $\mathbb{R}$ with period l.

In particular, any solution of the eigenvalue problem

$$\frac{d^2}{dx^2}\phi + \lambda\phi = 0$$

will satisfy this periodicity condition.

The general solution to this eigenvalue problem is of the form

$$\phi(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

This can be seen somewhat easily by solving the ODE:

$$\phi(x) = Ce^{i\sqrt{\lambda}x} + De^{i\sqrt{\lambda}x}$$

Then using Euler's formula and setting $A = C + D$, $-iB = C - D$. Now to find the eigenvalues we use the periodicity condition. It gives us in particular that $\phi(-l/2) = \phi(l/2)$.

The first constraint forces that

$$Acos(\sqrt{\lambda}l/2) + Bsin(\sqrt{\lambda}l/2) = Acos(-\sqrt{\lambda}l/2) + Bsin(-\sqrt{\lambda}l/2)$$
$$2Bsin(\sqrt{\lambda}l/2) = 0$$

We see that this can only be the case if $\lambda = (\frac{2\pi k}{l})^2$ for $k \in \mathbb{Z}$. With the correct coefficients, this set of eigenfunctions is a complete basis for the L^2 Hilbert space on the torus.

3 The n dimensional case

The general setting is the following: we are given a lattice Γ generated by a basis of vectors $S = \{v_1, \ldots, v_n\}$ and define $\mathbb{T} = \mathbb{R}^n/\Gamma$. The periodicity condition in this setting becomes: $\forall k_1, \ldots, k_n \in \mathbb{Z}$, $\forall x \in \mathbb{R}^n$, $f(x + \sum_{i=1}^n k_i v_i) = f(x)$. In other words, the function is invariant under translations by generators of the lattice.

Here we can do the same sort of reasoning as in the 1D case, with the subtlety that we're no longer working with an ODE but a PDE. It is possible however to use the separation of variables technique, which yields a solution of the form

$$\phi(x) = C \prod_{s=1}^{n} e^{ic_s x_s}$$

where $c_s \in \mathbb{R}$ such that $\sum c_s^2 = \lambda$. This equation can be rewritten using Euler's formula:

$$\begin{aligned}\phi(x) &= \prod_{s=1}^{n} e^{ic_s x_s} \\ &= \exp(i \sum_{s=1}^{n} c_s x_s) \\ &= \cos(\sum_{s=1}^{n} c_s x_s) + i\sin(\sum_{s=1}^{n} c_s x_s)\end{aligned}$$

To get a real valued solution we can consider the following solution to the equation (for the same eigenvalue):

$$\begin{aligned}\psi(x) &= \prod_{s=1}^{n} e^{-ic_s x_s} \\ &= \cos(\sum_{s=1}^{n} c_s x_s) - i\sin(\sum_{s=1}^{n} c_s x_s)\end{aligned}$$

So we get two solutions to the PDE, one by adding ψ and ϕ, one by subtracting them. To conclude, we can write our eigenfunctions in the form

$$\varphi(x) = A\cos(\sum_{s=1}^{n} c_s x_s) + B\sin(\sum_{s=1}^{n} c_s x_s)$$

Now let's see what the periodicity conditions translates to. For simplicity, let's fix an arbitrary generator v of Γ and denote by $< \cdot, \cdot >$ the standard inner product of vectors. The periodicity condition tells us in particular that

$$\varphi(-v/2) = \varphi(v/2)$$

$$Acos(\frac{1}{2} < c, v >) - Bsin(\frac{1}{2} < c, v >) = Acos(\frac{1}{2} < c, v >) + Bsin(\frac{1}{2} < c, v >)$$

$$sin(\frac{1}{2} < c, v >) = 0$$

We conclude that for φ to be an eigenfunction on the torus, c needs to satisfy $\forall v \in S$, $< c, v >= 2\pi k$ for some $k \in \mathbb{Z}$. It is also easy to see that if c satisfies the above, then the corresponding function is an eigenfunction of the Laplacian on the torus with eigenvalue $4\pi^2 k^2 ||c||$.

So each element in the set $\Gamma^* = \{x \in \mathbb{R}\colon < x, v >= 2\pi k \text{ s.t. } k \in \mathbb{Z}\}$ corresponds uniquely to an eigenfunction of the Laplacian on the torus. This set is also commonly referred to as the dual lattice.

Again, with the proper coefficients, these eigenfunctions can be made into an orthonormal basis of $L^2(\mathbb{T})$.

In general, one can find a complete orthonormal basis for the L^2 space of a compact Riemannian manifold M composed of the eigenfunctions of the Laplacian. This is commonly referred to as the Sturm-Liouville decomposition.

4 Sturm-Liouville decomposition

The theorem we want to prove in this section is the following:

Theorem 4.1 (Sturm-Liouville decomposition)**.** *Let (M, g) be a compact Riemannian manifold. There is a complete orthonormal basis $\{\phi_0, \phi_1, \ldots\}$ of $L^2(M)$ consisting of eigenfunctions of Δ_g with ϕ_j having eigenvalue λ_j satisfying*

$$\lambda_0 \leq \lambda_1 \leq \ldots \longrightarrow \infty$$

Further, for every j we have $\phi_j \in \mathcal{C}^\infty(M)$ and that the heat kernel can be expressed from the eigenfunctions as a series:

$$p(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y)$$

First, let's recall some crucial definitions:

Definition 4.2. *We define the* ***heat operator*** *to be $L = \Delta_g + \partial_t$*

Definition 4.3. *A* ***fundamental solution*** *of the heat equation is a continuous function $p\colon M \times M \times (0, \infty) \longrightarrow \mathbb{R}$ which is $\mathcal{C}^2$ on $M \times M$, $\mathcal{C}^1$ on $(0, \infty)$ and such that*

$$\begin{aligned} L_y p &= 0 \\ \lim_{t \to 0} p(\cdot, y, t) &= \delta_y \text{ (in the sense of distributions)} \end{aligned}$$

Fundamental solutions can be shown to be unique and symmetric in the first two variables.

Definition 4.4. *For $t > 0$, the* ***heat propagator*** *operator, $e^{-t\Delta_g}\colon L^2(M) \to L^2(M)$ is defined as:*

$$e^{-t\Delta_g} f(x) = \int_M p(x, y, t) f(y) d\omega_g(y)$$

It is essentially the solution of the heat diffusion equation with initial condition $f(x)$. We list a few of its essential properties:

1. $e^{-t\Delta_g} \circ e^{-s\Delta_g} = e^{-(t+s)\Delta_g}$
2. $(e^{-\Delta_g})^t = e^{-t\Delta_g}$
3. $e^{-t\Delta_g}$ is self-adjoint and positive
4. $e^{-t\Delta_g}$ is a compact operator

We're now ready to prove the main theorem of the section.

Proof. Since $e^{-\Delta_g}$ is a compact self adjoint operator, it admits eigenvalues $\beta_0 \geq \beta_1 \geq \ldots$ such that $\beta_n \to 0$ as $n \to \infty$ with corresponding eigenfunctions $\phi_0, \phi_1, \ldots$ forming a complete orthonormal basis of $L^2(M)$.

We will show that in fact these correspond to eigenfunctions of the Laplacian, with eigenvalues $\lambda_i = -\ln\beta_i$. We'll use this definition from now on.

As the natural logarithm preserves ordering and $\ln(1) = 0$, to show the eigenvalues of the Laplacian are all non-negative it is enough to show that $\beta_0 \leq 1$. But from the properties of the heat propagator,

$$e^{-t\Delta_g}\phi_0 = (e^{-\Delta_g})^t\phi_0 = \beta_0^t\phi_0$$

But as we know $e^{-t\Delta_g}$ is in fact the harmonic solution to the BVP $u(\cdot, 0) = \phi_0$ and that such solutions need to be decreasing (in L^2 norm) with time, we see that $\beta_0 \leq 1$.

Now we show that the λ's we defined are indeed eigenvalues of the Laplacian. We get from their definition that $e^{-t\Delta_g}\phi_k = \beta_k^t\phi_k = e^{-t\lambda_k}\phi_k$.

Since $e^{-t\lambda_k}\phi_k$ solves the heat equation we can write

$$\begin{aligned}
0 &= L(e^{-t\Delta_g}\phi_k) \\
&= L(e^{-t\lambda_k}\phi_k) \\
&= \Delta_g e^{-t\lambda_k}\phi_k + \partial_t e^{-t\lambda_k}\phi_k \\
&= e^{-t\lambda_k}(\Delta_g\phi_k - \lambda_k\phi_k)
\end{aligned}$$

So as the exponential is strictly positive, $\Delta_g\phi_k = \lambda_k\phi_k$ as desired.

Now as the ϕ_k form a complete orthonormal basis, we can express the fundamental solution in terms of that base:

$$p(x, y, t) = \sum_{k=0}^{\infty} < p(x, \cdot, t), \phi_k > \phi_k(y)$$

But this expression can be reduced by using the definition of the L^2 inner product:

$$\begin{aligned}
< p(x, y, t), \phi_k > &= \int_M p(x, y, t)\phi(y)d\omega_g(y) \\
&= e^{-t\Delta_g}\phi_k(x) \\
&= e^{-t\lambda_k}\phi_k(x)
\end{aligned}$$

From this we conclude:

$$p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y)$$

$\square$