On the Efficiency of Markets with Two-Sided Proportional Allocation Mechanisms

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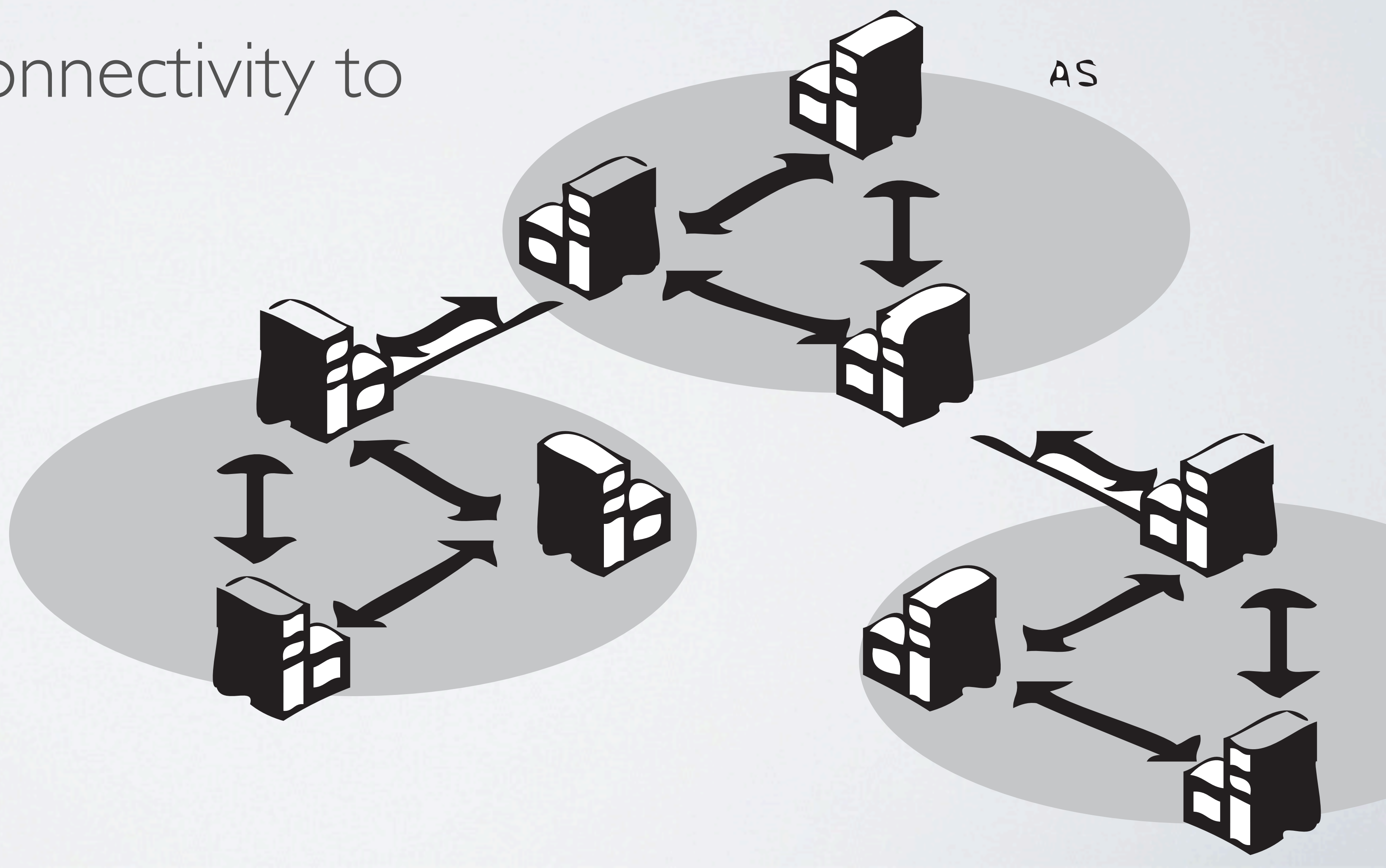
What is the most fair way
to share a good between people,
given their competing interests?


Bandwidth sharing
Electricity production

INTERNET AUTONOMOUS SYSTEMS

- The Internet is made up of smaller independent networks.
- They wish to have connectivity to each other.



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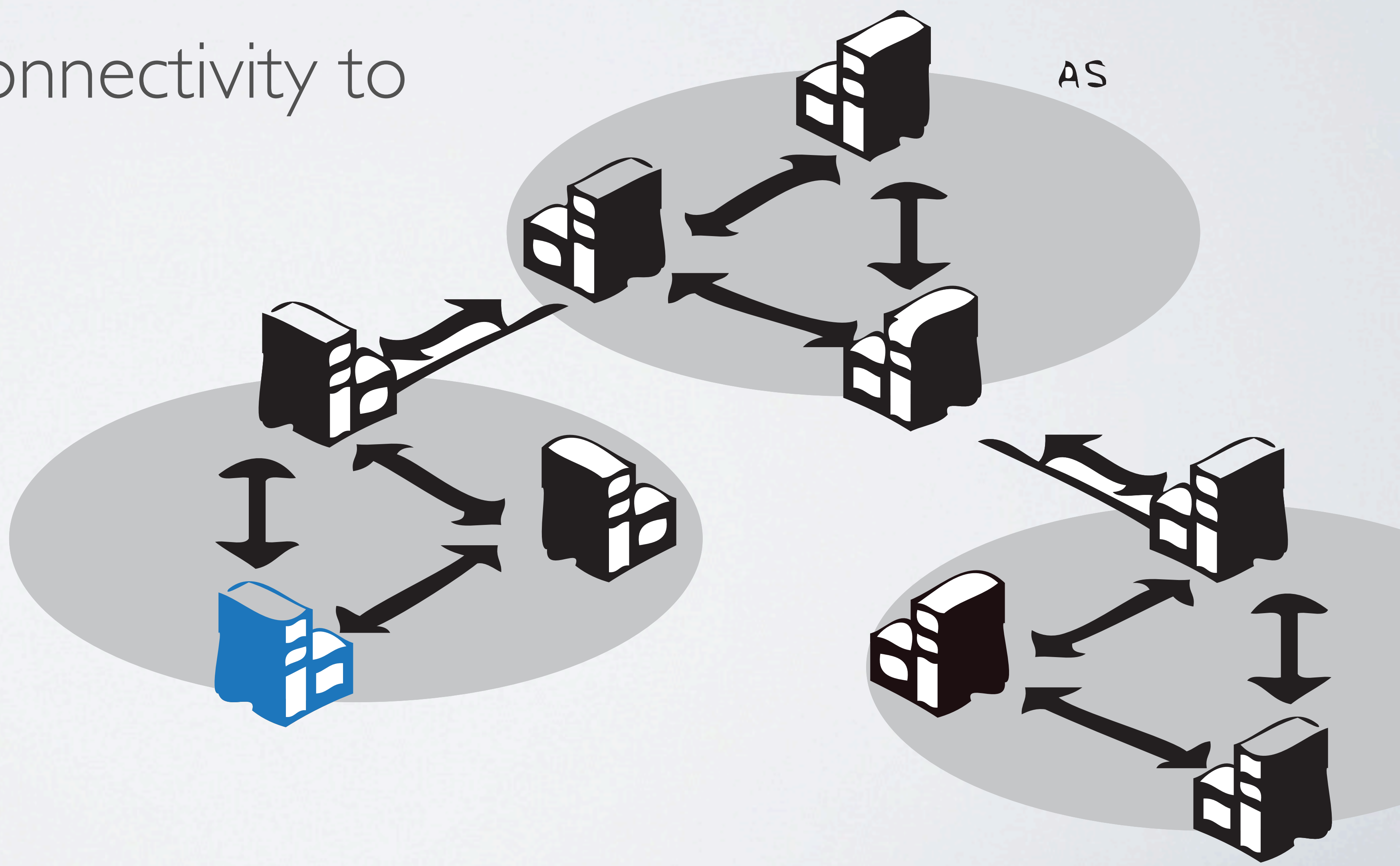

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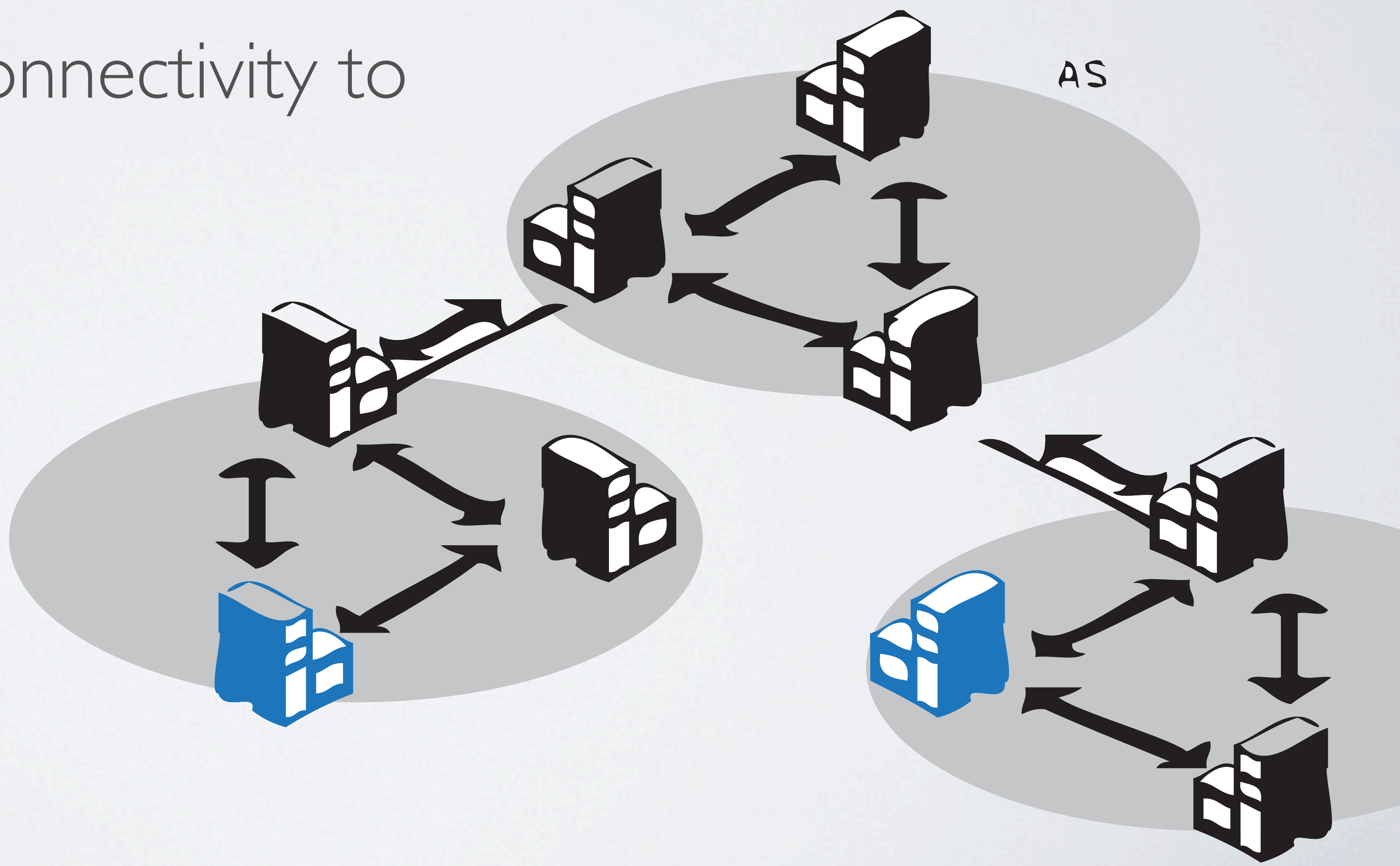

INTERNET AUTONOMOUS SYSTEMS

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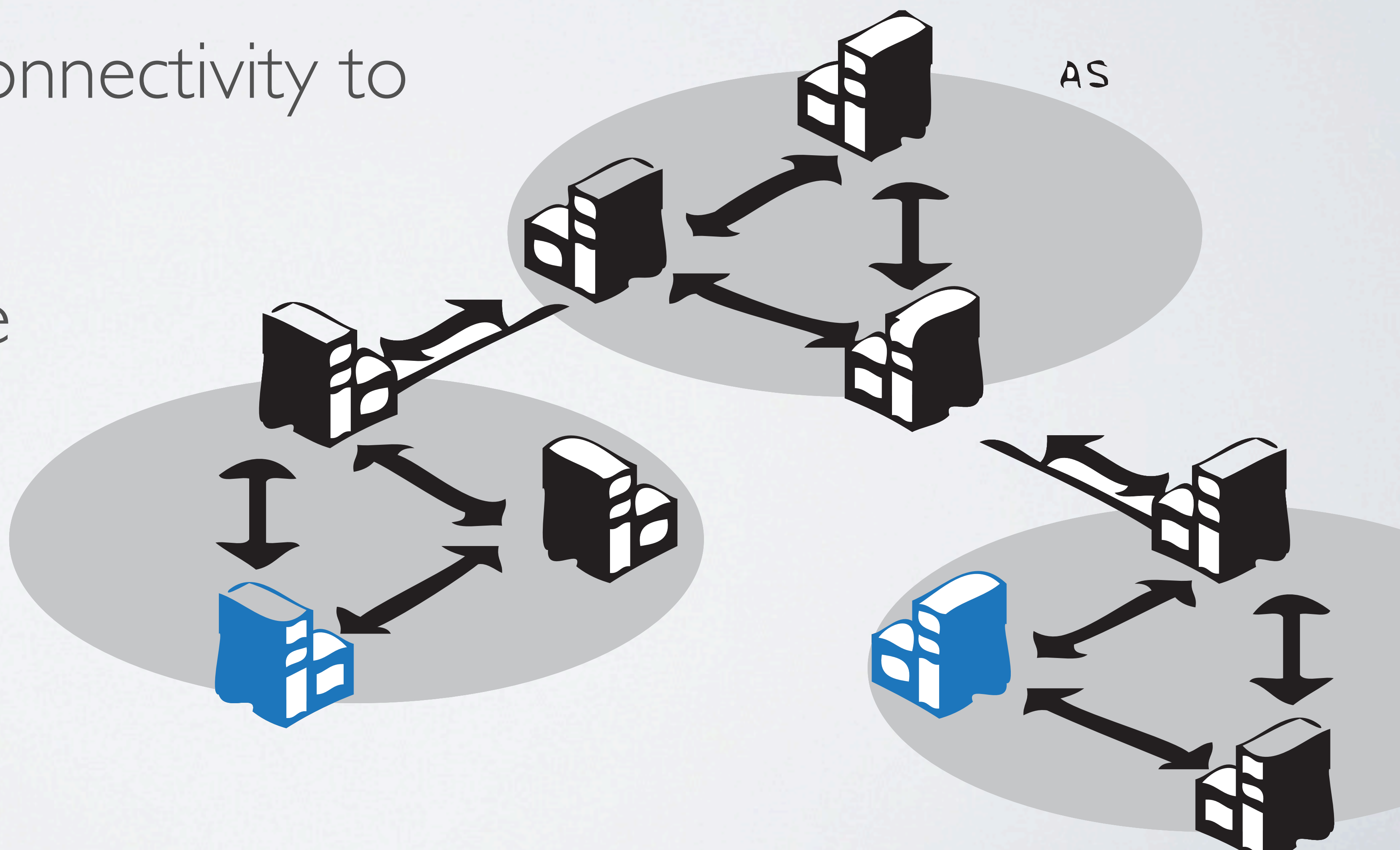
- They wish to have connectivity to each other.
- Network owners are willing to sell transit



INTERNET AUTONOMOUS SYSTEMS

- How can we efficiently organize supply and demand?

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Economic efficiency

Leave the users well-off.

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Computational efficiency
Scale to the size of the Internet

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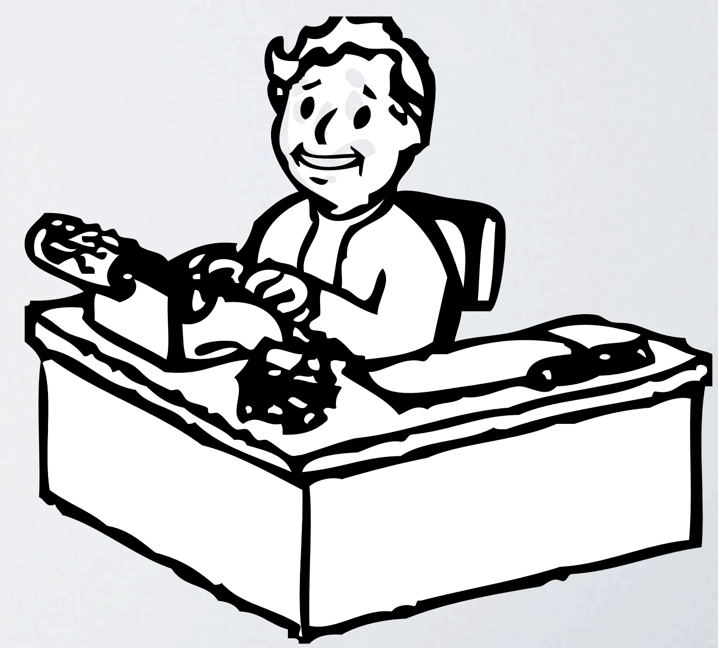

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Economic efficiency
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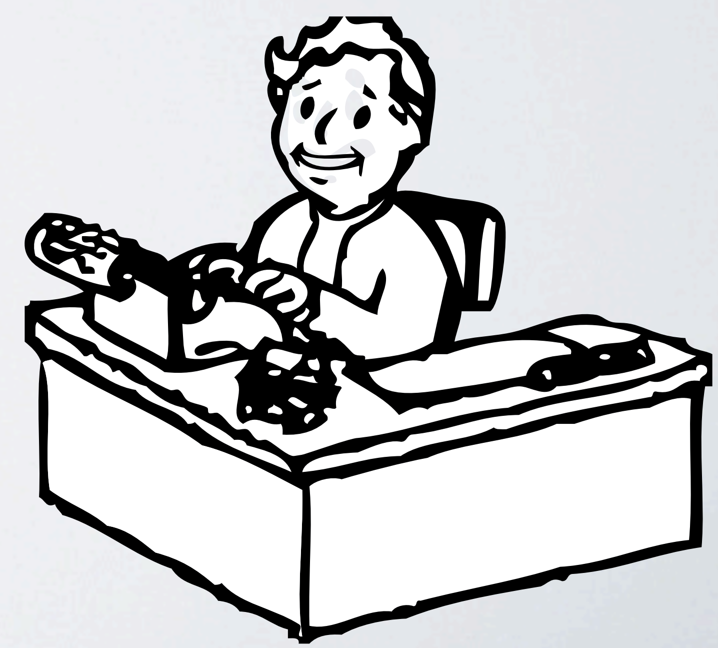
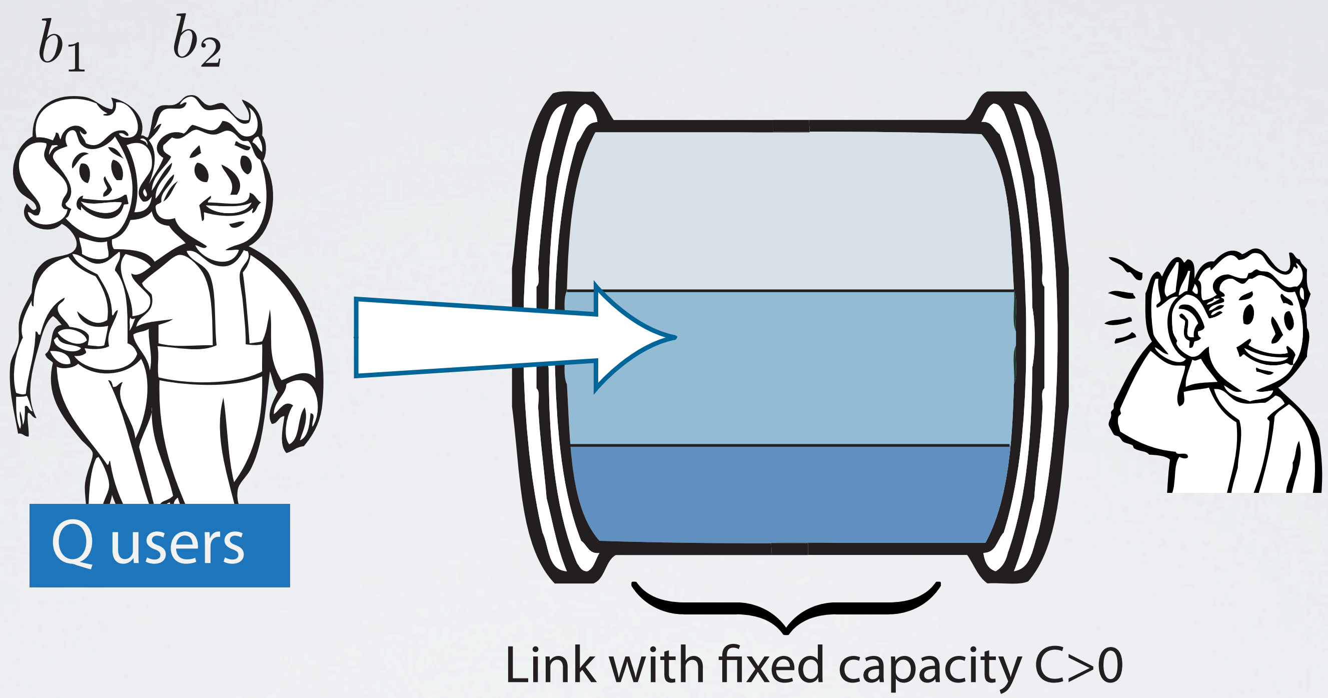


There is a fundamental tradeoff between them.

THE PROPORTIONAL ALLOCATION MECHANISM



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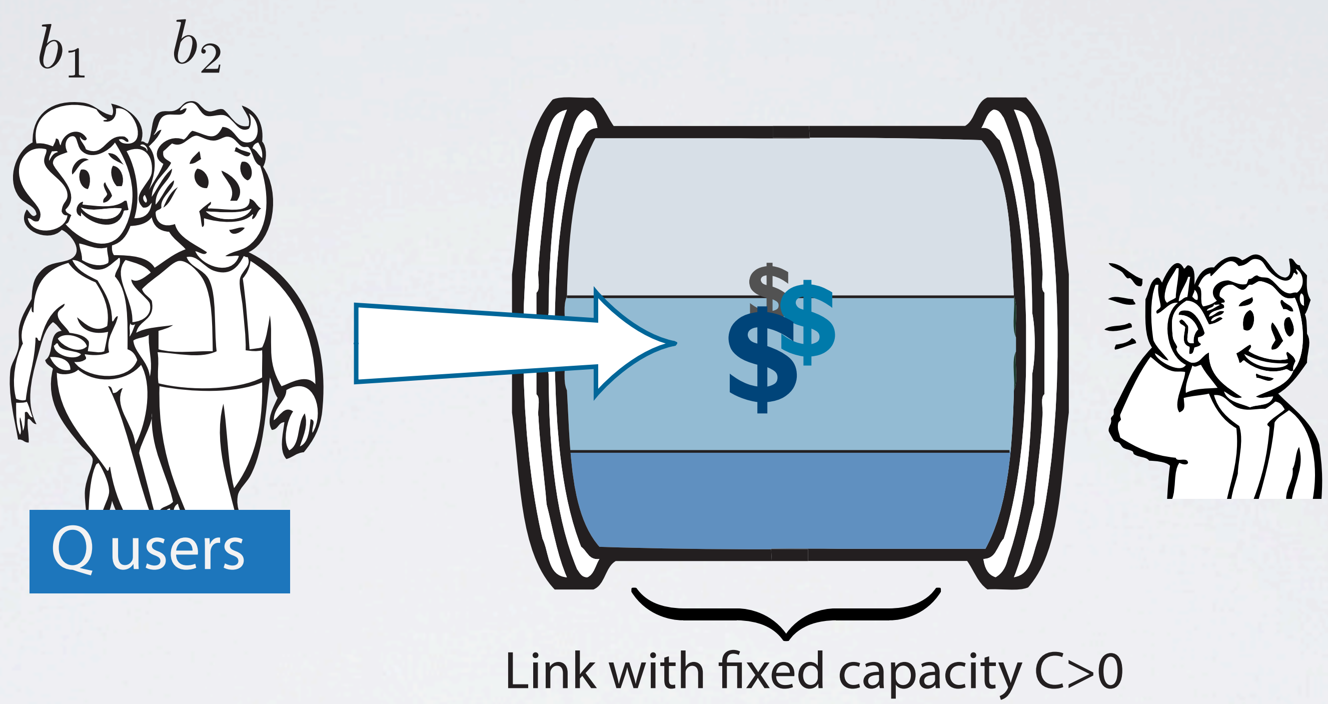

1. User q submits a payment of b_q

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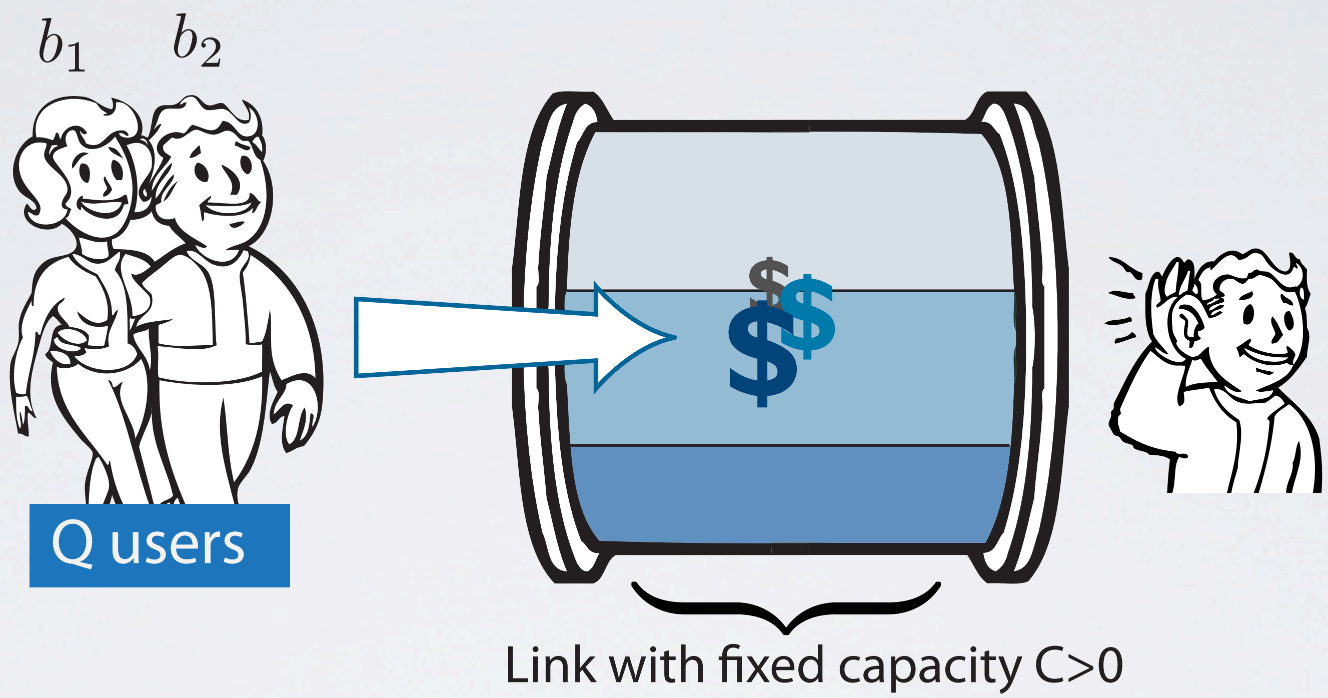
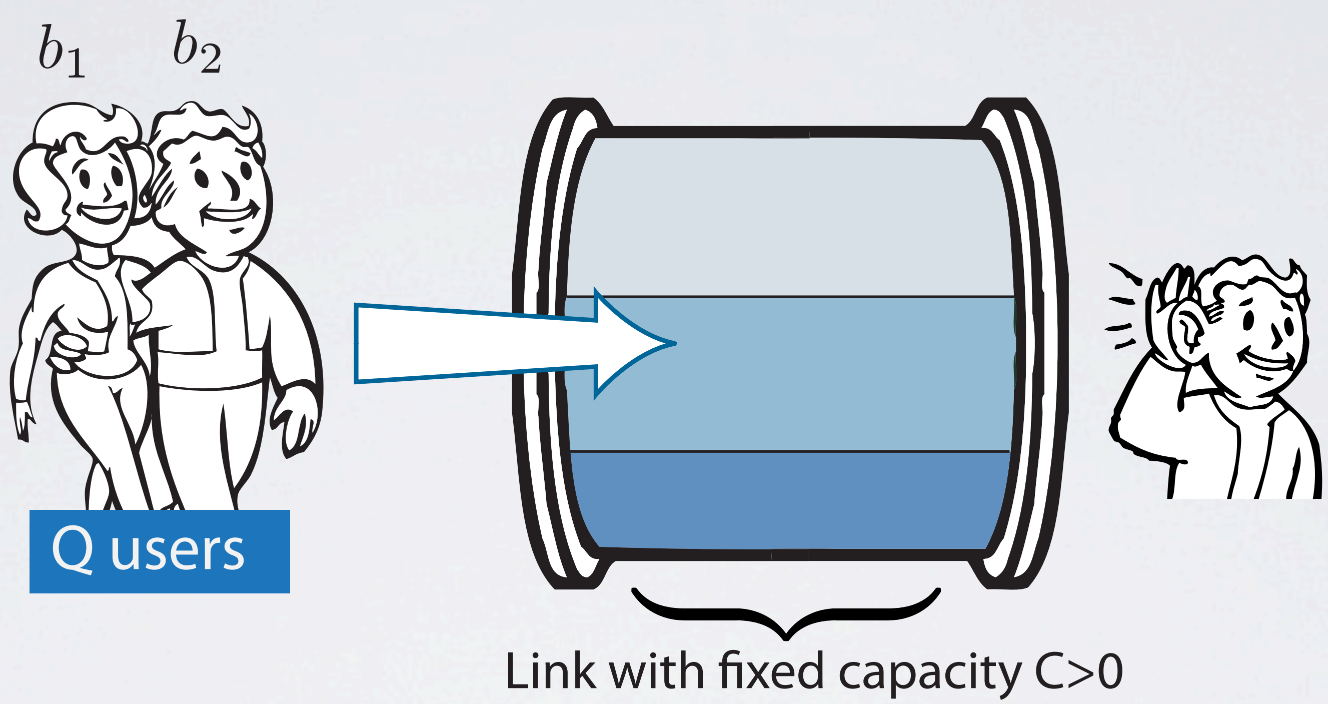


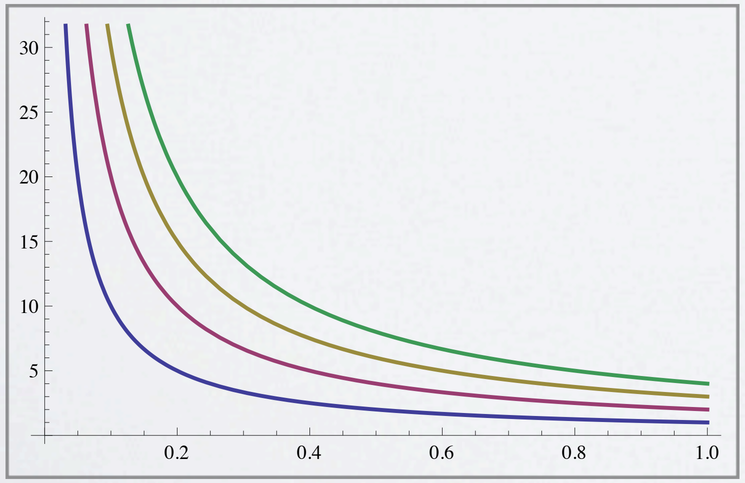
1. User q submits a payment of b_q

2. Capacity is allocated proportionally to the bids. If you pay $50 out of $100, you receive one half.

THE PROPORTIONAL ALLOCATION MECHANISM



1. Let $\mathcal{D} = \{D(p,b) = b/p \,|\, b > 0\}$ be a set of demand functions.



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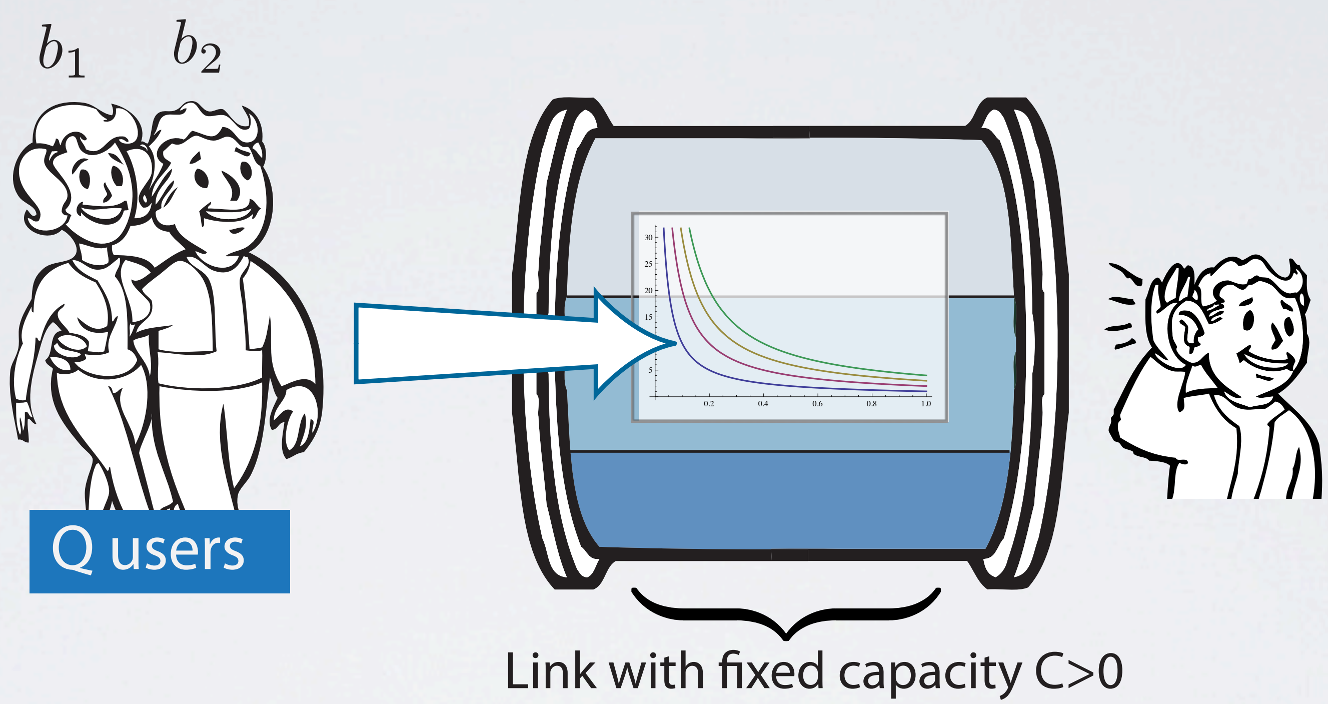
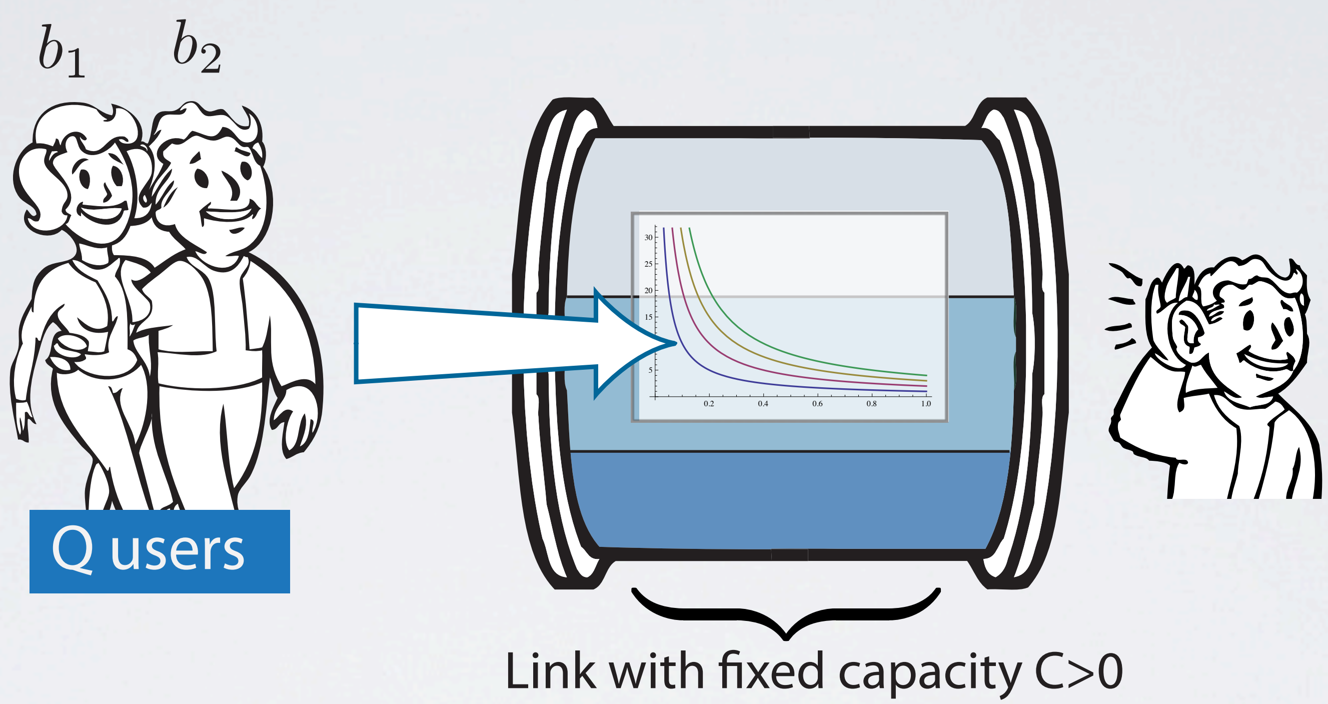


1. Let $\mathcal{D} = \{D(p,b) = b/p \,|\, b > 0\}$ be a set of demand functions.

2. User q chooses a demand function $D_q(p) = D(p, b_q) \in \mathcal{D}$

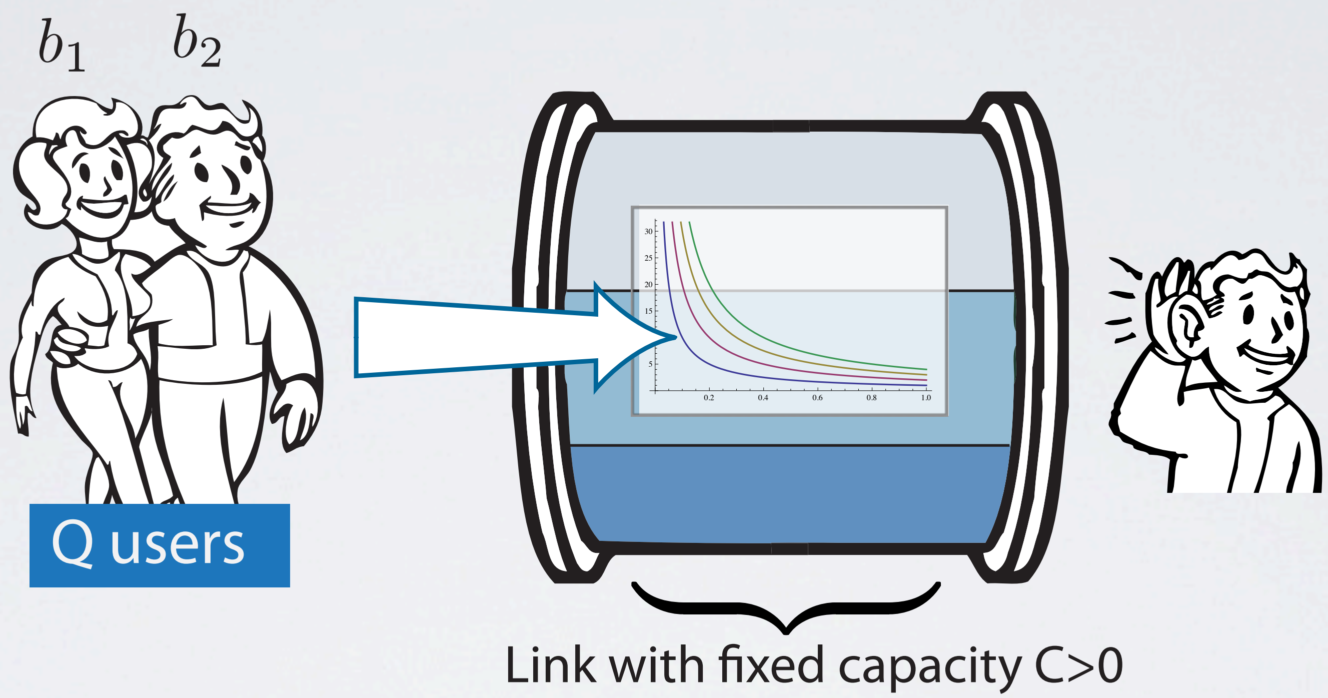
THE PROPORTIONAL ALLOCATION MECHANISM



3. The mechanism chooses a price p so that $\sum_q D_q(p) = C$

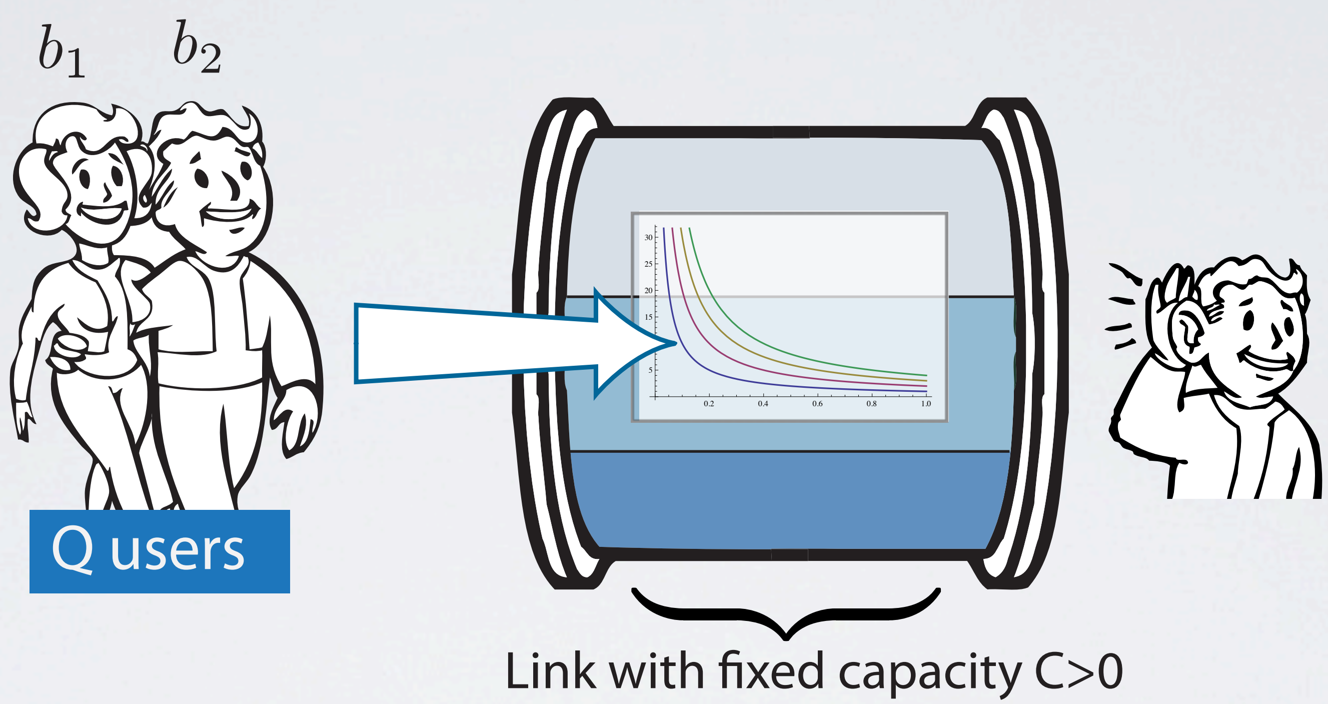
4. User q buys $D_q(p)$ at price p

THE PROPORTIONAL ALLOCATION MECHANISM



$$\sum_q D_q(p) = \sum_q \frac{b_q}{p} = C \implies p = \frac{\sum_q b_q}{p}$$

THE PROPORTIONAL ALLOCATION MECHANISM



$$\sum_q D_q(p) = \sum_q \frac{b_q}{p} = C \implies p = \frac{\sum_q b_q}{p}$$

$$\implies D_q(p) = \frac{b_q}{p} = \frac{b_q}{\sum_q b_q} C$$

THAT WAS AN EXAMPLE OF A PRICING MECHANISM

- We focus on pricing mechanisms.
- A single price minimizes communication with the users.
- Pricing is standard tool for sharing resources, e.g. road tolls, electricity pricing.

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

- User q has utility:

$$U_q(d_q) = \underbrace{V_q(d_q)}_{\text{value}} - \underbrace{pd_q}_{\text{money}}$$

- Every user makes his best bid given the others' bids:

$$b_q \in \arg\max_b U_q(b, \mathbf{b}_{-q})$$

- We measure welfare loss using the price of anarchy.

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

$$U_q(d_q) = V_q(d_q) - pd_q$$

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

$$U_q(d_q) = V_q(d_q) - p \underbrace{d_q(b_q)}_{\text{allocation}}$$

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

$$U_q(d_q) = V_q(d_q) - \underbrace{p(b_q)}_{\text{price}} \; \underbrace{d_q(b_q)}_{\text{allocation}}$$

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

$$U_q(d_q) = V_q(d_q) - \underbrace{p(b_q)}_{\text{price}} \; \underbrace{d_q(b_q)}_{\text{allocation}}$$

Theorem. (Kelly, 1997) *When users do not exercise their market power, the Kelly mechanism is optimal. It maximizes*

$$\sum_{q \in Q} V_q(d_q)$$

ECONOMIC EFFICIENCY OF THE PROP. ALLOC. MECH.

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Theorem. (Johari and Tsitsiklis, 2004) *Given some natural assumptions on the utility functions, the price of anarchy in Kelly's mechanism is* $\mathbf{3/4}$.

SUPPLY-SIDE PROPORTIONAL ALLOCATION MECHANISM

SUPPLY-SIDE PROPORTIONAL ALLOCATION MECHANISM

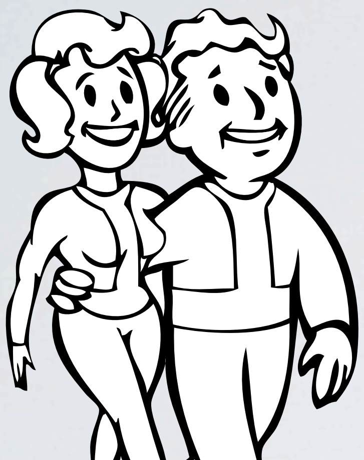
Theorem. (Johari, 2004) *Given some natural assumptions on the cost functions, the price of anarchy in Kelly's supply-side mechanism is* $\mathbf{1/2}$.

But in reality, competition occurs on both sides of the market.

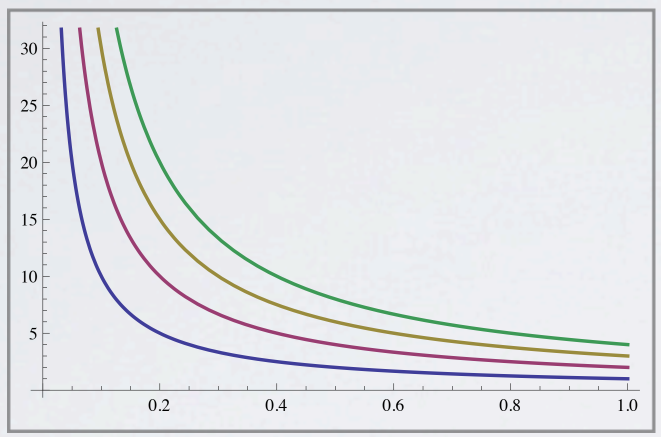
WHY STUDY TWO-SIDED PRICING MECHANISMS?

- Real-world markets are two-sided.
- Current pricing mechanisms apply only to one-sided markets.
- VCG mechanisms cannot be used in the two-sided setting.

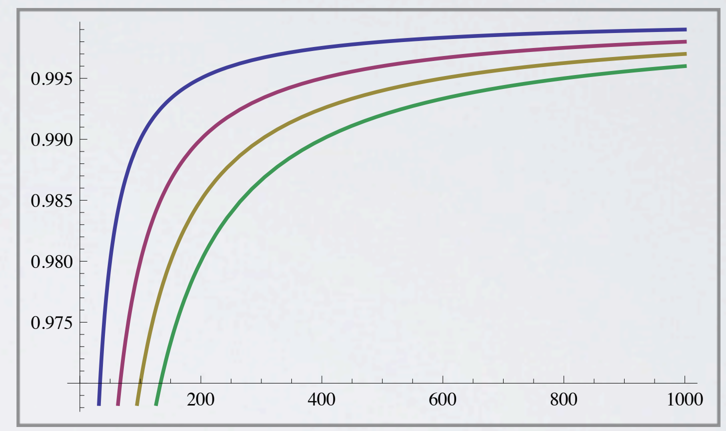
TWO-SIDED PROPORTIONAL ALLOCATION MECHANISM



Q users

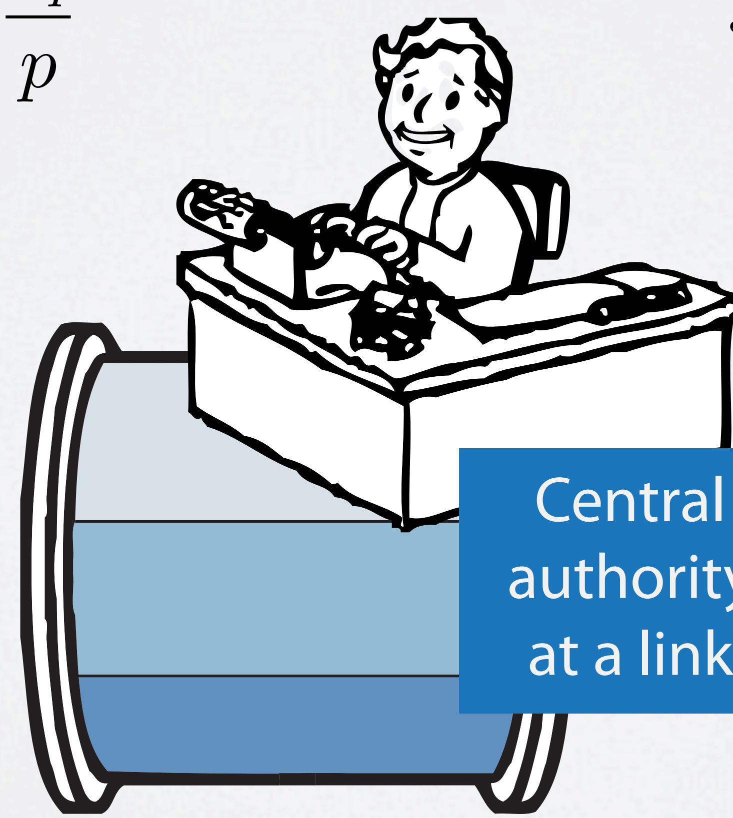


$$d_q = \frac{b_q}{p}$$



$$s_r = 1 - \frac{b_r}{p}$$

R providers



Central authority at a link

A TWO-SIDED MARKET

- Users' utilities are:

$$U_q(d_q) = \underbrace{V_q(d_q)}_{\text{value}} - \underbrace{p(d_q)d_q}_{\text{money}}$$

(Valuations are concave.)

$$U_r(s_r) = \underbrace{p(s_r)s_r}_{\text{money}} - \underbrace{C_r(s_r)}_{\text{costs}}$$

(Marginal costs are convex.)

- The optimal solution is:

$$\text{maximize} \ \sum_{q=1}^{Q} V_q(d_q) - \sum_{r=1}^{R} C_r(s_r)$$

$$\text{such that} \ \ \text{supply equals demand}$$

MAIN RESULT

Theorem. *The price of anarchy of the two-sided market involving* $R > 1$ *suppliers equals*

$$\frac{s^2(S^2 + 4Ss + 2s^2)}{S(S + 2s)}$$

where $S = R - 1$, *and* s *is the unique positive root of the polynomial*

$$\gamma(s) = 16s^4 + 10S^2s(3s - 2) + S^3(5s - 4) + Ss^2(49s - 24)$$

Furthermore, this bound is tight.

MAIN RESULT

Corollary. *The worst inefficiency occurs when $R = 2$. It equals approximately* **0.588727**.

Theorem. *The price of anarchy of the two-sided market involving $R > 1$ suppliers equals*

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where $S = R - 1$, and s is the unique positive root of the polynomial

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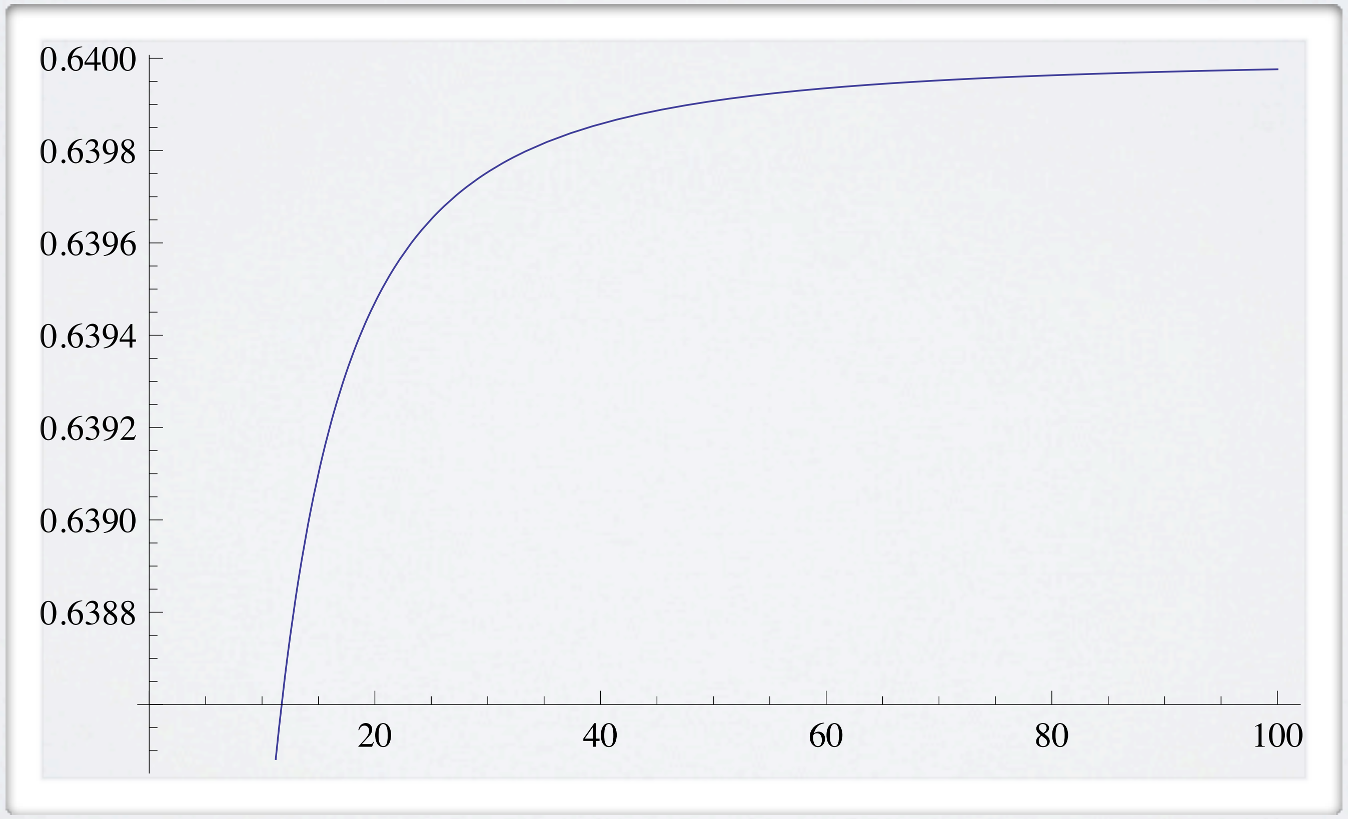
OBSERVATIONS

- Supply-side competition improves the price of anarchy.
- In a fully competitive market, the price of anarchy equals 0.64.



OBSERVATIONS

- Demand-side competition worsens the price of anarchy!
- The best price of anarchy occurs in a monopsony market. It equals 0.72.

PROOF TECHNIQUE

- We formulate the price of anarchy as an optimization problem and analytically compute its solution.

$$\begin{aligned} \underset{U,d,s}{\text{minimize}} \quad & \frac{\sum_q U_q(d_q^{\text{NE}}) + \sum_r U_r(s_r^{\text{NE}})}{\sum_q U_q(d_q^{\text{OPT}}) + \sum_r U_r(s_r^{\text{OPT}})} \\ \text{such that} \quad & d_q^{\text{NE}}, s_r^{\text{NE}} \text{ form a Nash equilibrium allocation} \\ & d_q^{\text{OPT}}, s_r^{\text{OPT}} \text{ form an optimal allocation} \end{aligned}$$

PROOF TECHNIQUE

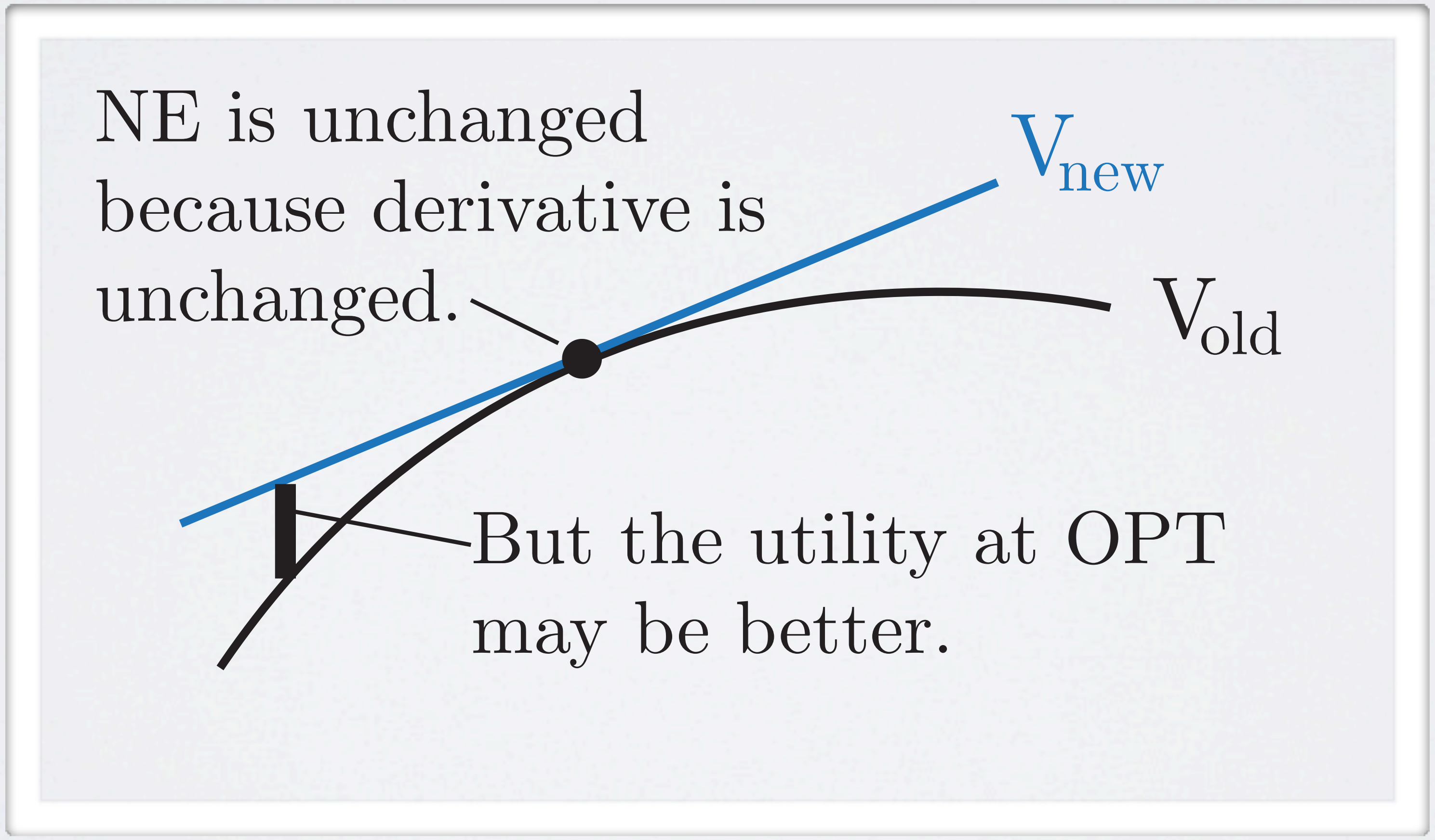
1. Derive necessary and sufficient conditions for an allocation to be Nash equilibrium:

$$\frac{dU_q(d_q^{NE})}{d\theta_q} = 0 \quad \begin{cases} U_q'(d_q)\left(1 - \frac{d_q}{R}\right) & \geq \quad p \; if \; d_q > 0 \\ U_q'(d_q)\left(1 - \frac{d_q}{R}\right) & \leq \quad p \end{cases}$$

$$\frac{dU_r(s_r^{NE})}{d\theta_r} = 0 \quad \begin{cases} C_r'(s_r)\left(1 + \frac{s_r}{R-1}\right) & \leq \quad p \; if \; 0 < s_r \leq 1 \\ C_r'(s_r)\left(1 + \frac{s_r}{R-1}\right) & \geq \quad p \; if \; 0 \leq s_r < 1 \end{cases}$$

PROOF TECHNIQUE

2. Show that the worst case occurs with linear utilities and marginal costs.



PROOF TECHNIQUE

$$\text{minimize} \quad \frac{d_1^{NE} + \sum_{i=2}^{Q} \alpha_i d_i^{NE} - \frac{1}{2}\sum_{j=1}^{R} \beta_j (s_j^{NE})^2}{\sum_{j=1}^{R} s_j^{OPT} - \frac{1}{2}\sum_{j=1}^{R} \beta_j (s_j^{OPT})^2} \quad (1)$$

Price of anarchy

$$\text{such that} \quad \alpha_i \left(1 - \frac{d_i^{NE}}{R}\right) \geq \mu \ \forall i \text{ s.t. } d_i^{NE} > 0 \quad (2)$$

$$\alpha_i \left(1 - \frac{d_i^{NE}}{R}\right) \leq \mu \ \forall i \quad (3)$$

$$\beta_j s_j^{NE} \left(1 + \frac{s_j^{NE}}{R-1}\right) \leq \mu \ \forall j \text{ s.t. } 0 < s_j^{NE} \leq 1 \quad (4)$$

$$\beta_j s_j^{NE} \left(1 + \frac{s_j^{NE}}{R-1}\right) \geq \mu \ \forall j \text{ s.t. } 0 \leq s_j^{NE} < 1 \quad (5)$$

Nash eq. conditions

$$\sum_{i=1}^{Q} d_i^{NE} = \sum_{j=1}^{R} s_j^{NE} \quad (6)$$

Supply = demand

$$\beta_j s_j^{OPT} \leq 1 \ \forall j \text{ s.t. } 0 < s_j^{OPT} \leq 1 \quad (7)$$

$$\beta_j s_j^{OPT} \geq 1 \ \forall j \text{ s.t. } 0 \leq s_j^{OPT} < 1 \quad (8)$$

Optimality conditions

$$d_i^{NE} \geq 0 \ \forall i \quad (9)$$

$$0 \leq s_j^{NE}, s_j^{OPT}, \alpha_i \leq 1 \ \forall i, j \quad (10)$$

$$0 \leq \mu \quad (11)$$

Non-negativity

PROOF TECHNIQUE

$$\text{minimize} \quad \frac{(1-\mu)^2 R + \mu \sum_{j=1}^{R} s_j^{NE} - 1/2 \sum_{j=1}^{R} \beta_j (s_j^{NE})^2}{\sum_{j=1}^{R} s_j^{OPT} - 1/2 \sum_{j=1}^{R} \beta_j (s_j^{OPT})^2}$$

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} Nash eq. conditions

$$\beta_j s_j^{OPT} \leq 1 \; \forall j \text{ s.t. } 0 < s_j^{OPT} \leq 1$$

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$$0 \leq s_j^{NE}, s_j^{OPT} \leq 1 \; \forall j$$

$$0 < \beta_j \; \forall j$$

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PROOF TECHNIQUE

$$\text{minimize} \quad \frac{(1-\mu)^2 R + \mu \sum_{j=1}^{R} s_j^{NE} - 1/2 \sum_{j=1}^{R} \beta_j (s_j^{NE})^2}{\sum_{j=1}^{R} s_j^{OPT} - 1/2 \sum_{j=1}^{R} \beta_j (s_j^{OPT})^2}$$

} Price of anarchy

$$\text{such that} \quad \beta_j s_j^{NE} \left(1 + \frac{s_j^{NE}}{R-1}\right) = \mu \ \forall j$$

} Nash eq. conditions

$$s_j^{OPT} = \min(1/\beta_j, 1) \ \forall j$$

} Optimality conditions

$$0 < s_j^{NE}, s_j^{OPT} \leq 1 \ \forall j$$

$$0 < \beta_j \ \forall j$$

$$0 \leq \mu < 1$$

} Non-negativity

PROOF TECHNIQUE

$$\text{minimize} \quad \frac{\sum_{j=1}^{R}\left((1-\mu)^2 + \mu s_j^{NE} - \mu/2 \frac{s_j^{NE}}{1+s_j^{NE}/(R-1)}\right)}{\sum_{j=1}^{R}\left(\min(1/\beta_j, 1) - \mu/2 \frac{\min(1/\beta_j,1)^2}{s_j^{NE}(1+s_j^{NE}/(R-1))}\right)}$$

$$\text{such that} \quad 0 < s_j^{NE} \leq 1 \; \forall j$$

$$\beta_j = \frac{\mu}{s_j^{NE}\left(1 + s_j^{NE}/(R-1)\right)} \; \forall j$$

$$0 \leq \mu < 1$$

PROOF TECHNIQUE

$$\begin{array}{ll} \text{minimize} & \dfrac{(1-\mu)^2 + \mu s - {}^{\mu}\!/_{2} \frac{s}{1+s/(R-1)}}{\min(\frac{s(1+s/(R-1))}{\mu}, 1) - \frac{\mu}{2s(1+s/(R-1))} \min(\frac{s(1+s/(R-1))}{\mu}, 1)^2} \\ \text{such that} & 0 < s \leq 1 \\ & 0 \leq \mu < 1 \end{array}$$

PROOF TECHNIQUE

$$\begin{aligned}
&\text{minimize} && \frac{(1-\mu)^2 + \mu s - {}^{\mu}\!/_{2}\,\frac{s}{1+s/(R-1)}}{\frac{s(1+s/(R-1))}{2\mu}} \\
&\text{such that} && s(1+s/(R-1)) \leq \mu \\
& && 0 < s \leq 1 \\
& && 0 \leq \mu < 1
\end{aligned}$$

PROOF TECHNIQUE

$$
\begin{aligned}
\text{minimize} \quad & \frac{(1-\mu)^2 + \mu s - \frac{\mu s}{2(1+s/(R-1))}}{1 - \frac{\mu}{2s(1+s/(R-1))}} \\
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\end{aligned}
$$

PROOF TECHNIQUE

$$\begin{aligned}
\text{minimize} \quad & \frac{s^2((R-1)^2 + 4(R-1)s + 2s^2)}{(R-1)(R-1+2s)} \\
\text{such that} \quad & 0 < s \leq 1 \\
& 0 \leq \mu_{1,2} < 1 \\
& s(1 + s/R - 1)) \geq \mu_{1,2} \\
& \mu_{1,2} \in \mathbb{R}
\end{aligned}$$

PROOF TECHNIQUE

Theorem. *The price of anarchy of the two-sided market involving $R > 1$ suppliers equals*

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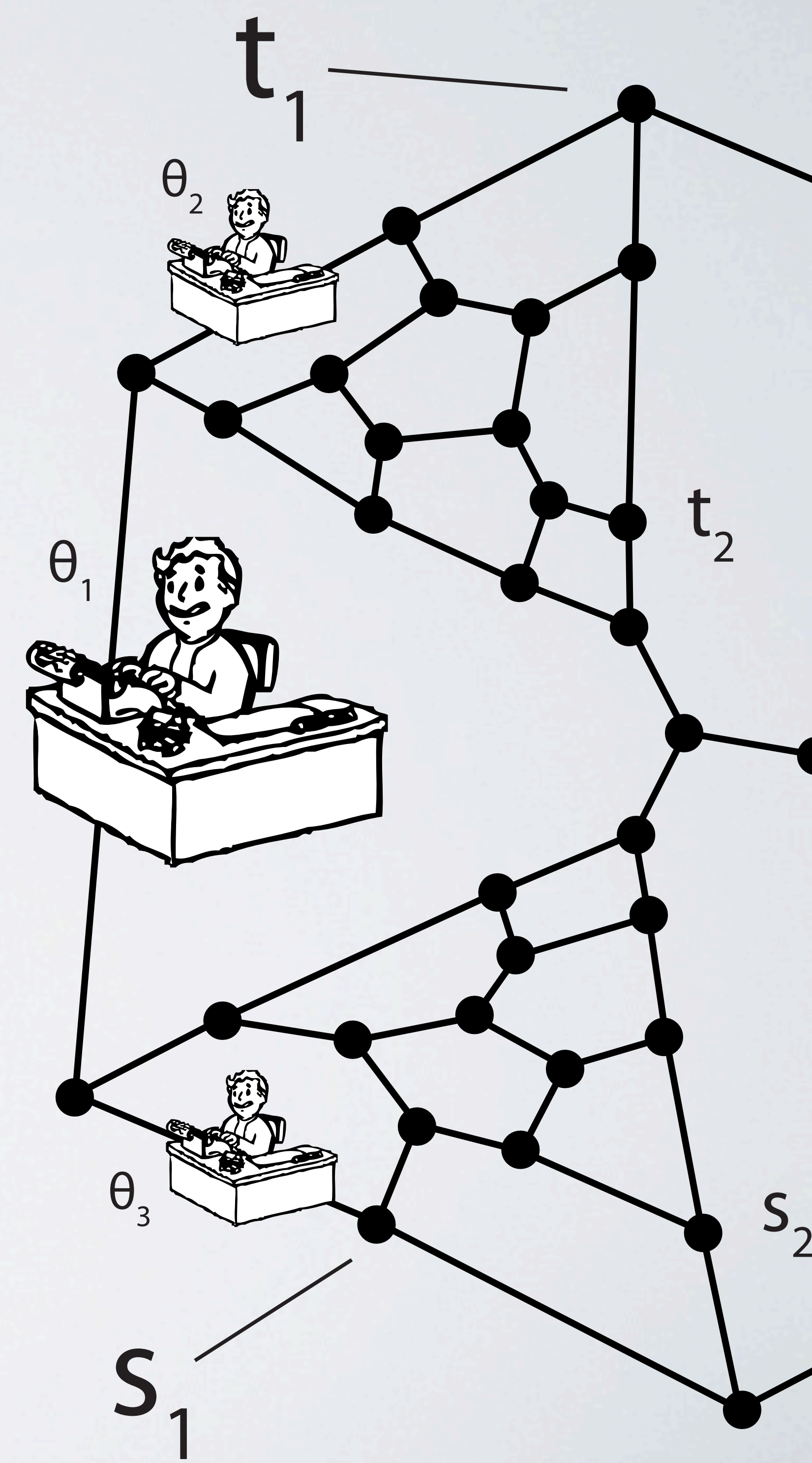
where $S = R - 1$, and s is the unique positive root of the polynomial

$$\gamma(s) = 16s^4 + 10S^2 s(3s - 2) + S^3(5s - 4) + Ss^2(49s - 24)$$

Furthermore, this bound is tight.

EXTENSION TO NETWORKS

- At each link, there is an independent instance of the single-link market.
- Consumers buy capacity in order to transmit flow from **s** to **t**.



EXTENSION TO NETWORKS

Theorem. *When extended to networks, the mechanism has the same price of anarchy as in the single link case – approximately* **0.588727**.

EXTENSION TO NETWORKS

Corollary. *When extended to a general economy of N goods, the mechanism has the same price of anarchy of* **0.588727**, *under some mild assumptions on costs and utilities.*

Theorem. *When extended to networks, the mechanism has the same price of anarchy as in the single link case – approximately* **0.588727**.

TWO-SIDED MARKET-CLEARING MECHANISMS

- A two-sided market-clearing mechanism is a pair of sets of functions: $\mathcal{D} = \{D(b, p) \mid b > 0\}$ and $\mathcal{S} = \{S(b, p) \mid b > 0\}$

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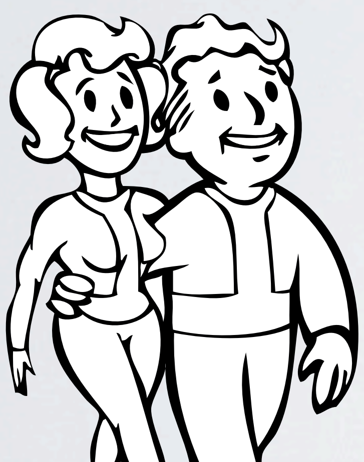
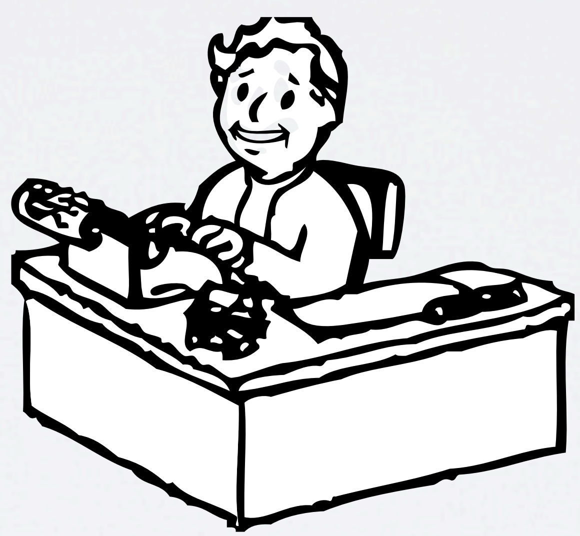


Q users



R providers

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TWO-SIDED MARKET-CLEARING MECHANISMS

- Consider the market-clearing mechanisms for which
  - The utility to each user in concave is his bid:

$$U_q(d_q) = V_q(D(p(\vec{b}), b_q) - p(\vec{b})D(p(\vec{b}), b_q)$$

  - D is bounded from below and S is bounded from above.
  - When users have no market power, the mechanism achieves an optimal allocation.

OPTIMALITY

Lemma. *Under some mild assumptions, every mechanism that accepts a scalar message* θ *from each user must allocate demand and supply according to:*

$$\begin{aligned} D(\theta, p) &= a(p)\theta \\ S(\theta, p) &= 1 - b(p)\theta \end{aligned}$$

where $a(p), b(p) \geq 0$ *are some functions of the price* $p > 0$.

OPTIMALITY

Theorem. *Among the mechanisms that have $a(p) = b(p)$ for all $p > 0$, the mechanism presented here is the only one that achieves the best possible price of anarchy of 0.588727.*

Lemma. *Under some mild assumptions, every mechanism that accepts a scalar message θ from each user must allocate demand and supply according to:*

$$\begin{aligned} D(\theta, p) &= a(p)\theta \\ S(\theta, p) &= 1 - b(p)\theta \end{aligned}$$

where $a(p), b(p) \geq 0$ are some functions of the price $p > 0$.

IN CONCLUSION

Our results were to:

- Extend the proportional allocation mechanism to two-sided markets.
- Establish a tight bound on the price of anarchy in both the single and multi-resource settings.
- Establish the optimality of the mechanism within a large class.