Social Welfare in Combinatorial Markets

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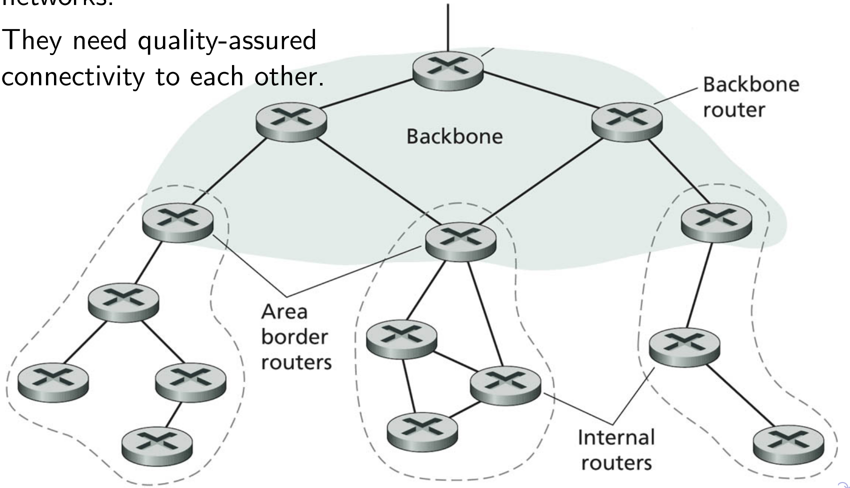
How to share a scarce good between people with competing interests?

How to share limited bandwidth between users on a network?

Engineering application in networking

- Internet is made up of smaller independent networks.
- They need quality-assured connectivity to each other.
- Network owners are willing to sell transit.

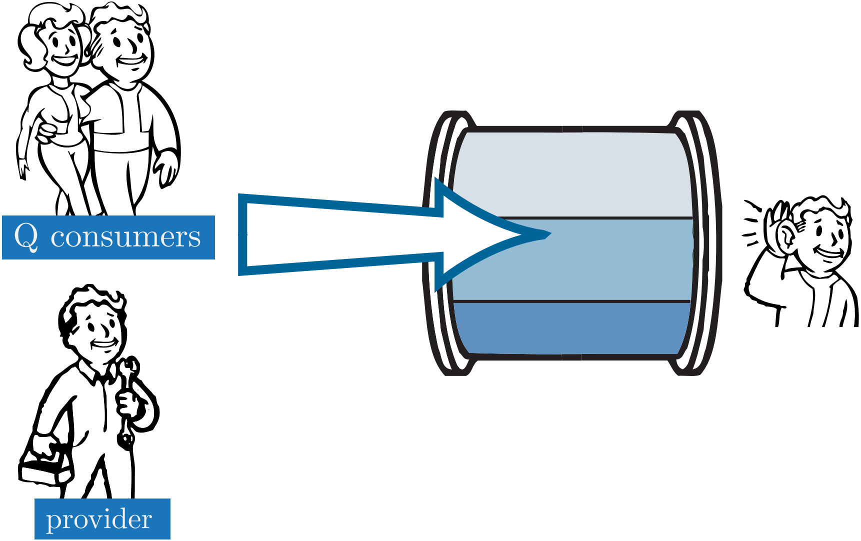


What is a good system for coordinating supply and demand in this market?

Mechanism requirements

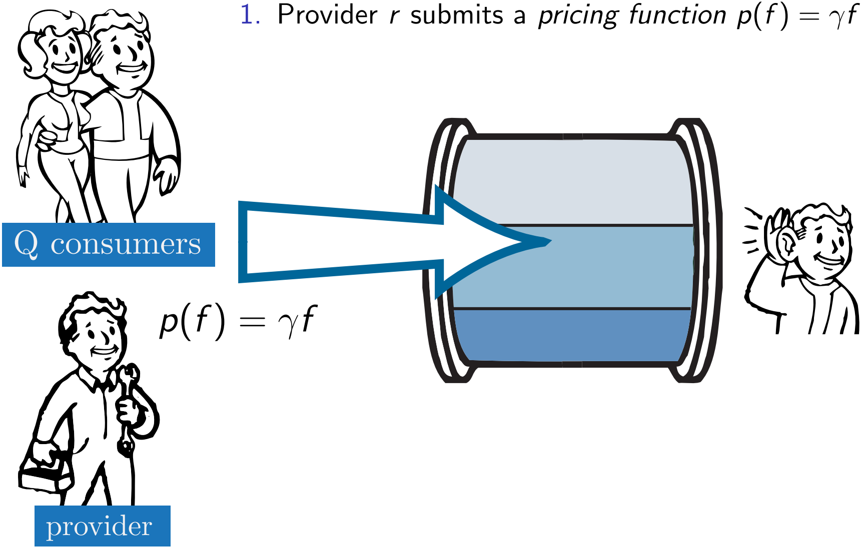
1. Easy enough to use by actual providers
2. Scalable to large networks
3. Resistant to selfish manipulation by users

Mechanism definition



Mechanism definition

1. Provider r submits a *pricing function* $p(f) = \gamma f$



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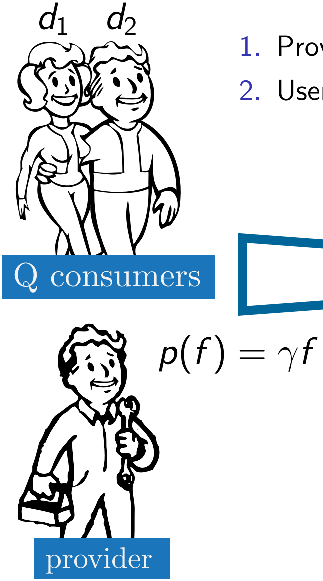
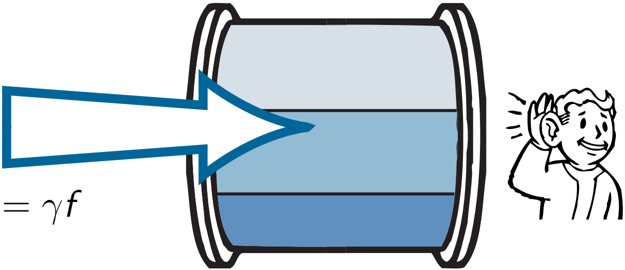
1. Provider r submits a *pricing function* $p(f) = \gamma f$
2. User q chooses a rate d_q to transmit





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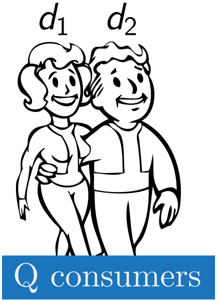

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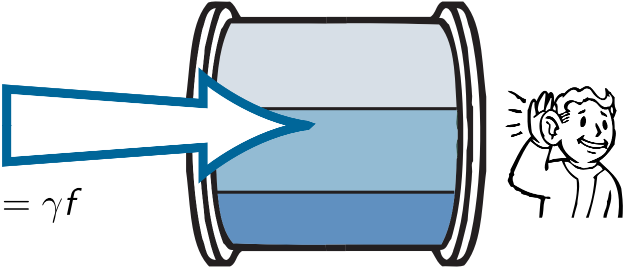
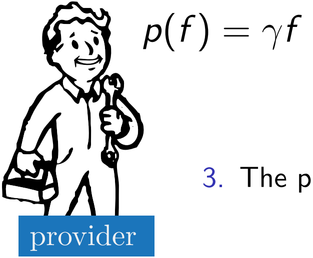
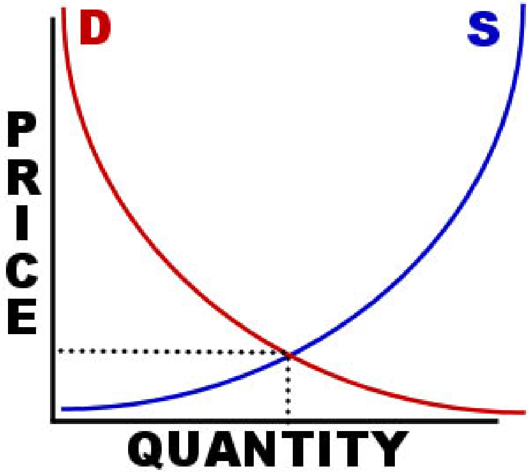




3. The provider receives $p(f)d_q$ from consumer q.

It has a simple interpretation



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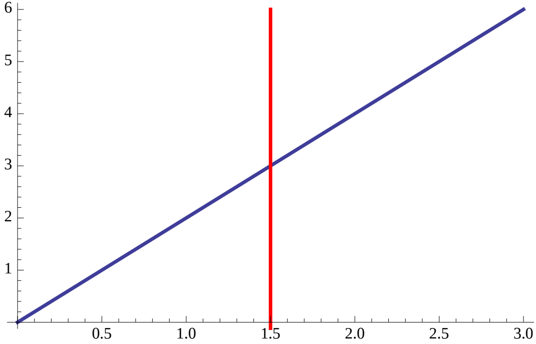

$$\text{At equilibrium, price} = p(\sum_q d_q)$$
$$= \gamma f$$

How do me measure social welfare?

- User q has a utility function of the form

$$U_q(d_q) = \underbrace{V_q(d_q)}_{\text{value}} - \underbrace{pd_q}_{\text{expenses}}$$

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- The *social welfare* is the sum of the utilities:

$$W(\mathbf{d}, \gamma) = \underbrace{\sum_{q \in Q} V_q(d_q)}_{\text{valuations}} - \underbrace{C(f)}_{\text{costs}}$$

How do we measure social welfare?

- Assume that any user's action is always the best he/she can do given what everyone else is doing:
- A *Nash equilibrium* is a combination of actions $(\mathbf{d}^{\mathsf{NE}}, \gamma^{\mathsf{NE}})$ that satisfies this.

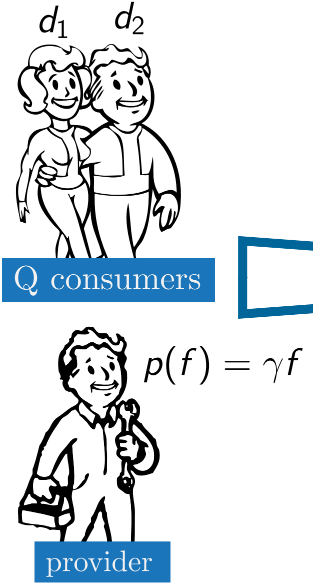
How do we measure social welfare?

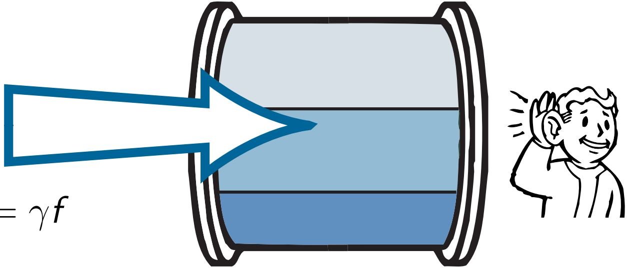
- Assume that any user's action is always the best he/she can do given what everyone else is doing:
- A *Nash equilibrium* is a combination of actions $(\mathbf{d}^{\mathsf{NE}}, \gamma^{\mathsf{NE}})$ that satisfies this.

- We will measure the loss of welfare due to the users' selfishness using the *price of anarchy*:

$$\frac{\text{welfare at worst Nash equilibrium}}{\text{best possible welfare}}$$

The mechanism, again





There are two sources of economic inefficiency.

Separating demand and supply side inefficiency

Theorem (Johari and Tsitsiklis, 2005)

The price of anarchy on the demand side of the market is $2/3$. □

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The price of anarchy on the demand side of the market is $2/3$*.* □

Lemma

The price of anarchy of the two-sided market equals

$$\frac{2\rho(\rho-2)}{\rho-4}$$

where $0 \leq \rho \leq 1$ *is an* overcharging parameter. □

Separating demand and supply side inefficiency

Lemma

The worst efficiency occurs with quadratic cost functions of the form

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Lemma

For these cost functions, the overcharging parameter equals

$$\rho = \frac{\beta}{\gamma}$$

□

Elasticity of demand

Definition

The elasticity of the flow f with respect to γ is defined to be

$$\epsilon = \frac{\%\Delta f}{\%\Delta\gamma} = \frac{\text{percentage change of f}}{\text{percentage change of } \gamma}$$

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Definition

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$$\epsilon = \frac{\%\Delta f}{\%\Delta \gamma} = \frac{\text{percentage change of f}}{\text{percentage change of } \gamma}$$

Lemma

At equilibrium, the overcharging parameter equals

$$\rho = 2 - \frac{1}{\epsilon}$$

□

Results for one link: user demand

Proposition

When consumer valuations are linear,

$$V_q(f) = a_q f$$

then $\epsilon = 1$ *and the price of anarchy is 2/3.* □

Proposition

When consumer valuations are monomials of degree d*:*

$$V_q(f) = a_q f^d$$

then $\epsilon = 1/(2-d)$ *and the price of anarchy is* $\Omega(1/d)$. □

Results for one link: supplier competition

Proposition

As the number of providers increases, $\rho \rightarrow 1$ *and the price of anarchy goes to one at a rate of* $\Omega(1 - 1/d)$*.* □

Results for one link: pricing function

Proposition

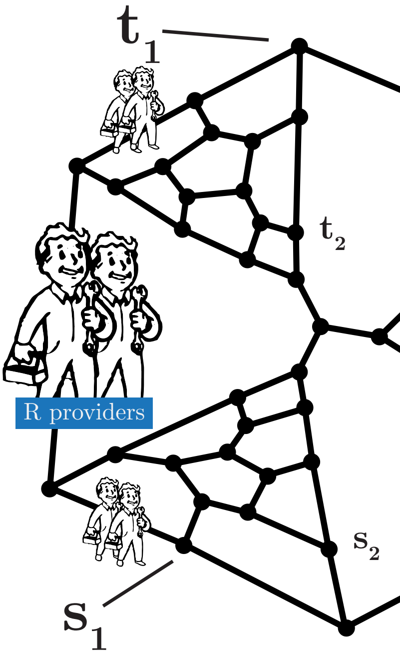
When the provider submits a monomial pricing function

$$p(f) = \gamma f^d$$

then $\epsilon = 1/d$ *and the price of anarchy is* $\Omega(1/d)$. □

Markets over general graphs

- Each user owns a pair of nodes (s_q, t_q).
- Users choose bandwidth over each path.
  They get utility from the total flow.
- There are at least two providers per link.
  They behave like in the single-link market



Results

Theorem

Let G be a path. Suppose that

1. *There are at least three providers per link.*
2. *Consumers have linear valuation functions.*
3. *Providers' cost at a given link are within a factor of two.*

The the price of anarchy is about 0.39. □

Proof idea

Theorem (Johari and Tsitsiklis, 2005)

The price of anarchy on the demand side in an arbitrary graph market is $2/3$.

Lemma

The price of anarchy in an arbitrary two-sided graph market equals

$$\frac{2\rho(\rho-2)}{\rho-4}$$

where $0 \leq \rho \leq 1$ *is an* overcharging parameter.

Proof idea

1. Derive expression for overcharging coefficient ρ as a function of the topology of the graph, and users' demands
2. Derive constraints under which prices γ are at equilibrium.
3. Minimize ρ over all possible costs and prices, under above constraints.

Proof idea

Minimize overcharging coefficient ρ.

$$\min \frac{\sum_{i=1}^{n} \dfrac{1}{\dfrac{1}{\dfrac{1}{g_{i1}} - \dfrac{1}{g_{i2}+g_{i3}+1/\sum_{k\neq i}\Gamma_k}} + \dfrac{1}{\dfrac{1}{g_{i2}} - \dfrac{1}{g_{i1}+g_{i3}+1/\sum_{k\neq i}\Gamma_k}} + \dfrac{1}{\dfrac{1}{g_{i3}} - \dfrac{1}{g_{i1}+g_{i3}+1/\sum_{k\neq i}\Gamma_k}}}}{\sum_{i=1}^{n} \dfrac{1}{g_{i1}+g_{i2}+g_{i3}}}$$

$$\text{s.t.} \quad \frac{\dfrac{1}{g_{i2}} - \dfrac{1}{g_{i1}+g_{i3}+1/\sum_{k\neq i}\Gamma_k}}{\dfrac{1}{g_{j2}} - \dfrac{1}{g_{j1}+g_{j3}+1/\sum_{k\neq j}\Gamma_k}} \leq \Delta_b \text{ for all } i,j$$

$$\frac{1}{g_{i2}} - \frac{1}{g_{i1}+g_{i3}+1/\sum_{k\neq j}\Gamma_k} \geq 0 \text{ for all } i$$

$$g_{ij} \geq \text{ for all } i,j$$

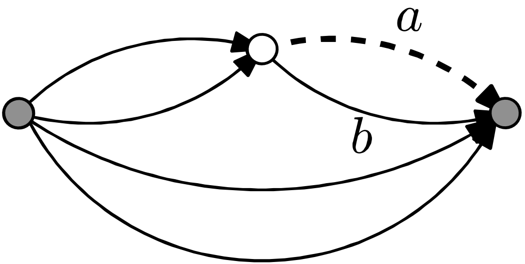
Proposition

Let G be a path. The price of anarchy decreases to zero as

- *The length of the path increases, when there are two providers per link.*
- *The competitor's effectiveness decreases.*
- *The curvature of consumer's valuation functions increases.*

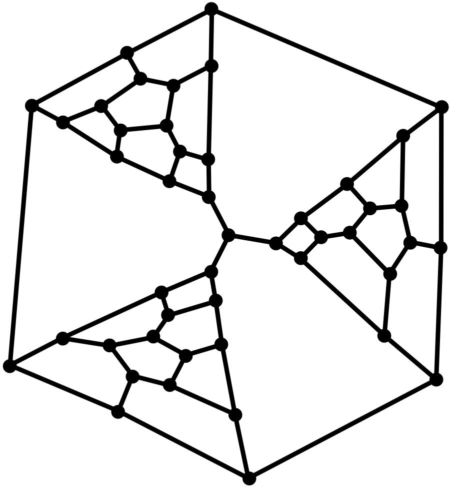
Rates of decrease are available for each of these cases.

Parallel-serial graphs



- The price of anarchy in a parallel-serial graphs is bounded by that of a path.
- As the number of paths increases, the price of anarchy goes to one, as long as competitors are competitive.

Arbitrary graphs



Conclusion

- Social properties of the mechanism vary a lot.
- We can now better decide if we want to use it in practice.