Excursions in Computing Science:
Week ii. Powers and Trees

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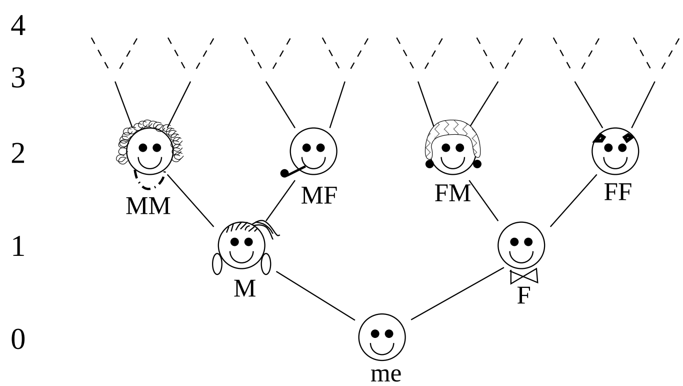
I. Prefatory Notes

1. Powers.

Teacher, invite your grade scholar to check and extend the following calculations. Start with $n = 1$ and work up. When that is secure, work down from $n = 1$. Point out that sums of powers of two are powers of two minus 1, and are called *Mersenne numbers*.

n	0	1	2	3	4	5	..
2^n	1	2	4	8	16	32	..
3^n	1	3	9	27	81	243	..
10^n	1	10	100	1000	10000	10000	..
$\Sigma 2^n$	1	3	7	15	31	63	..
$\Sigma 3^n$	1	4	13	40	121	364	..
$\Sigma 10^n$	1	11	111	1111	11111	..	
$2^{n+1} - 1$	1	3	7	15	31	63	..
$\frac{3^{n+1}-1}{2}$	1	4	13	40	121	364	..
$\frac{10^{n+1}-1}{9}$	1	11	111	1111	11111	..	

2. Trees

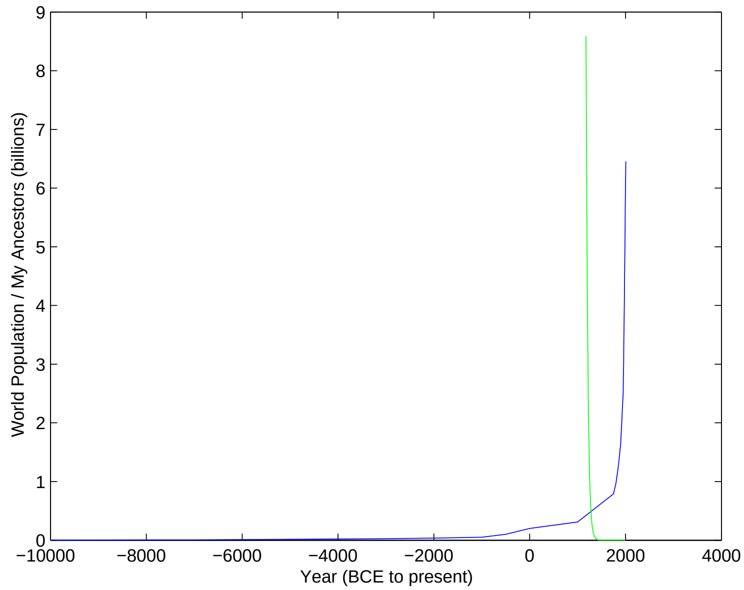


Family tree

 The author gratefully acknowledges support from the taxpayers of Québec and of Canada who have paid his salary and research grants while this work was developed at McGill University, and from his students and their funding agencies. The present Notes are based on the teaching and ideas of Eugene Lehman.

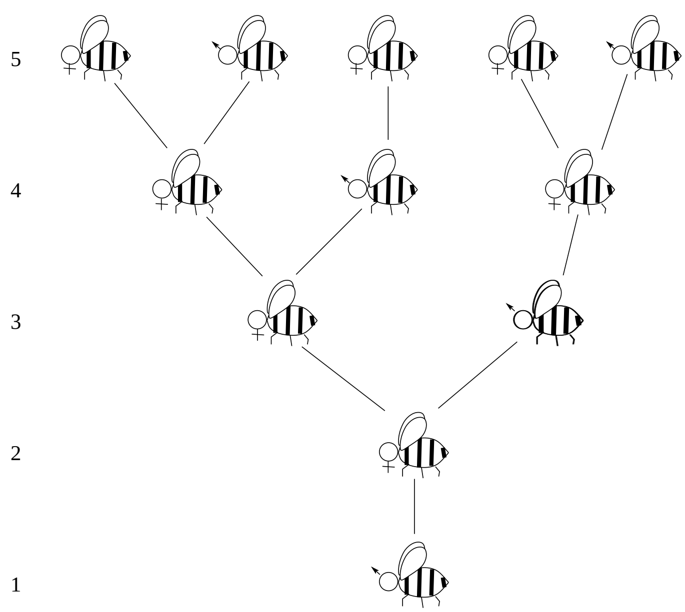
How many parents do you have? How many grandparents? How many great-grandparents? How many great-great-..-great-grandparents (10 times)?

These numbers grow very fast. Here they are in green, compared with the world population, shown in blue.



You can see that at some time during the Middle Ages your ancestors equalled the whole population of the world, and before that there were not enough people to provide you with all different ancestors.

Bees have fewer ancestors than we do, even if they are all different. (So do ants and spiders, among others.) A female bee is "diploid" like us and has a father as well as a mother. But a male bee is "haploid" and has only a mother. Here is a bee family tree.



Bee tree

One way to find out how many ancestors at any given level the male bee at the root of the tree has is to notice that the number of ancestors at any level is the sum of the numbers of ancestors at the previous two levels.

Such a sequence is called the "Fibonacci sequence" and its integers are "Fibonacci numbers".

n	0	1	2	3	4	5	6	..
f_n	0	1	1	2	3	5	8	..
$\phi_n = f_n/f_{n-1}$			1	2	1.5	1.666..	1.6	..

In the second row, "$\phi_n = f_n/f_{n-1}$", I have written what may be a way to find the next Fibonacci number, f_n, given only *one* previous Fibonacci number, f_{n-1}. The ratio, ϕ_n, of two successive Fibonacci numbers might just possibly settle down to a single number, which we can call ϕ, if we go far enough, i.e., make n big enough. Calculate some more f_n, say up to $n = 20$, and from them find the corresponding ϕ_n.

The Bee tree might be thought of as a tree like the Family tree, in which each individual has ϕ parents, so the Bee tree explores powers of ϕ in the same way that the family tree explores powers of 2. Powers of ϕ *approximate* the Bee tree, with the approximation becoming more and more accurate the higher up the tree we go.

3. Inverting trees.
In Week i we found inverse operators for addition and multiplication. What about powers?

Computer scientists take advantage of the extraordinarily rapid growth of powers of numbers, especially powers of 2, to turn the process upside down in order rapidly to find items in very large lists.

Suppose we have some way to represent all the stars in an enormous list, and we want to be able to find one particular star in this list. If we can find the middle star in the list, we could look there first. If it is the star we want, we're done. Otherwise we could check whether the star we want should come before the star we found, or after it, in the list (the list would have to be ordered on star position or star name or some other significant property of stars). If before, we find the middle star in the first half of the list and check that. If after, we find the middle star in the second half of the list.

We would repeat this process until we find our star, or until we find that it is not in the list. Let's work a simple example from which we can see that if the list has $2^n - 1$ stars in it, then we can find any star in at most n comparisons with the middle star in ever smaller sublists of stars. In this search we drop from about 2^n, the number of stars, to n, the number of search steps.

Suppose the list contains the following 7 ($= 2^3 - 1$) star names in alphabetical order.

`Altair, Arcturus, Castor, Procyon, Rigel Kentaurus, Sirius, Vega`

Remember that this list is supposedly stored as "bits" in a computer memory, not sitting out for us just to see the way I have written it above. Then, if we want to find `Castor` (one of the two "twins" in the constellation Gemini), we would:

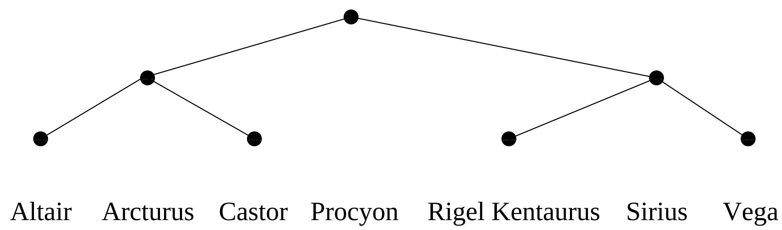
1. find `Procyon` in the middle of the whole list; decide that `Castor` comes *before* `Procyon` alphabetically, and so move to the sublist before `Procyon`;

2. find `Arcturus` in the middle of this sublist; decide that `Castor` comes *after* `Arcturus` alphabetically, and so move to the sublist after `Arcturus` and before `Procyon`;

3. find `Castor` in the middle of this sublist—indeed, `Castor` is the only name in the sublist, so we are done in any case; decide that `Castor` is indeed what we want, so finish successfully.

Three steps to find one star in a list of seven.

Here is the tree describing this process for all possible searches.



Note that computer scientists like to draw their trees upside down, maybe because they always use trees to go from large numbers to small numbers, not the other way around. The small number, 3 in this case, is the "height" of the tree, and is the worst-case cost of searching in the way described by using the tree.

We can see that to have a perfect "binary search tree" on a list of items, there must be a Mersenne number of items so that the list and each sublist has a middle item and so that the smallest sublists consist of only single items. Computer scientists can adapt the binary search strategy for lists of any size, but we won't follow them here.

The inverses of power operations are called "logarithms". They are written log() in general. Most calculators and computer programming languages use `log` to mean $\log_{10}()$, i.e. "log to the base 10," which is the inverse of powers of 10. The inverse of powers of 2, $\log_2()$, is often called `lg`. There is a third frequently-used logarithm, `ln`, which we'll come to in Note 5.

Here are the integer-valued logarithms to bases 2, 3 and 10.

n	1	2	3	4	8	9	10	16	27	32	64	81	100	..
$\lg(n)$	0	1		2	3			4		5	6			
$\log_3(n)$	0		1			2			3			4		
$\log(n)$	0						1						2	

Use a calculator or write a program to find the values I've left out. These will not be integers, but check that they increase gradually in size without ever decreasing.

4. Properties of powers.

- Product of powers is power of sum.

  For example, $2^2 \times 2^3 = 4 \times 8 = 32 = 2^5 = 2^{2+3}$.

  This was an extremely handy result in the days before computers, because it meant that one could use addition to multiply, via logarithms and powers. (The latter are called "antilogarithms" in tables of logarithms printed for this purpose.)

  For example,

$$
\begin{aligned}
&729 \times 27 \\
&\quad = 3^6 \times 3^3 && \text{look up the logarithms} \\
&\quad = 3^{6+3} && \text{product of powers is power of sum} \\
&\quad = 3^9 \\
&\quad = 19683 && \text{look up the antilogarithm}
\end{aligned}
$$

  Here I used integer powers of 3 to be precise, but one would normally not be able to use only integers, and normally logarithm tables are based on 10.

$$
\begin{aligned}
&729 \times 27 \\
&\quad = 10^{2.8627275} \times 10^{1.4313638} && \text{look up the logarithms} \\
&\quad = 10^{4.2940913} \\
&\quad = 19683 && \text{look up the antilogarithm}
\end{aligned}
$$

- Powers have fixed doublings.

  This is clearly true for powers of 2. Each step is a doubling: 1 then 2 then 4, and so on.

  It is also true for powers of ϕ. Each doubling happens after a little bit over one step.

  This is true for any sequence of powers. Sequences that grow in this way, with a fixed doubling step, are called "exponential".

5. Interest.
How can you turn ten dollars into a million? Here is a get-rich scheme which really works.

Invest it at eight percent compound interest for 150 years.

Compound interest is what you get when you invest some money, say 10$, and do not spend the interest you get at the end of the year, but add it to your investment.

Eight percent on 10$ would get you eighty cents, 0.80$, at the end of the year. That would hardly buy you a cheap hot chocolate, so put it back into the bank instead of spending it.

At the end of the second year, you would have 10.80$ plus eight percent interest on *that*, or 10.80×(1 + 0.08)$ = 11.66$. So now you have 1.66$ to spend instead of just 2×0.80$ = 1.60$.

You like that extra six cents, so you resist the temptation to buy two slightly less cheap hot chocolates or one really good one, and put the whole lot back in the bank. After three years you now have

$$10\$ \times (1 + 0.08)^3 = 12.60\$$$

After 150 years you will have your million, which, even then, should buy you quite a few very nice hot chocolates. Or maybe an economy spaceship, technology having also improved remarkably in that period of time.

Interest is often paid more than annually. For instance, eight percent annual interest compounded every six months gives you four percent twice a year.

$$(1 + 0.04)^2 = 1.0816$$

so this is actually a little bit better than eight percent per year. It does not seem much, but we are getting to see that even little bits make a big difference eventually.

Even so, it will take 147 years to make your million instead of 150. Let's try eight percent annually compounded every quarter.

$$(1 + 0.02)^4 = 1.0824$$

145 years.

Well, we are not making as much progress as we might have hoped. Let's compound *continuously*. We'll take $(1 + 0.08/n)^n$ as n gets arbitrarily big.

It would seem that this might become an infinitely large interest rate, but it doesn't. It actually becomes another exponential, $e^{0.08} = 1.0832871$, where $e = 2.7182818$, "Euler's number".

The inverse of e^x is another logarithm, the "natural" or "Napierian" logarithm. This is the one mentioned above (Note 3) and usually called `ln`.

$$\ln(e^x) = x = e^{\ln(x)} \text{ just as } \lg(2^x) = x = 2^{\lg(x)} \text{ and } \log(10^x) = x = 10^{\log(x)}.$$

6. Binomial coefficients.

Here is a relationship between powers and the triangular, tetrahedral and other simplex numbers of Week i.

What is the value of $(1+x)^n$ for any x and any n? We have already (Note 5) noticed that $(1+x)^n \approx 1+nx$ for small x.

We can start with any x and $n=2$.

$$(1+x)^2 = (1+x)(1+x) = 1+2x+x^2$$

Let's see how this was built.

$\times$	1	x
1	1	x
x	x	x^2

Now $n=3$

$\times$	1	$2x$	x^2
1	1	$2x$	x^2
x	x	$2x^2$	x^3

So $$(1+x)^3 = 1+3x+3x^2+x^3.$$

There's a pattern. We can see it even better if we go back to $n=1$ and even $n=0$.

$$(1+x)^1 = 1+x$$

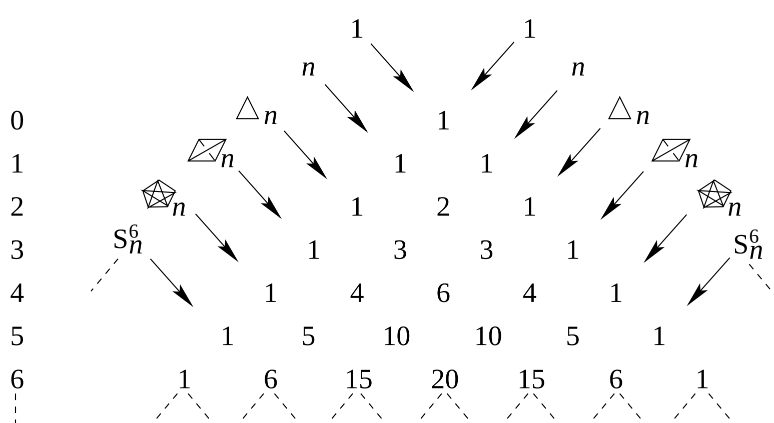
$$(1+x)^0 = 1$$

Let's write all the "coefficients" down without the x.

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \end{array}$$

Can you see why each coefficient must be the sum of the two coefficients immediately above it (up-left and up-right)? What will the next row be?

Here is "Pascal's Triangle", showing the relationship with the triangular, tetrahedral and other simplex numbers of Week i.



The numbers in the triangle are called *binomial coefficients*.

7. Factorials.

Some sequences grow even faster than exponential. Consider

n	0	1	2	3	4	5	6	..
$n!$	1	1	2	6	24	120	720	..

Note that there is one doubling between 2 and 6, two between 6 and 24, two between 24 and 120, three between 120 and 720, .. Small wonder that the exclamation point is used to specify this sequence. The numbers are called *factorials*.

What rule generates factorials? $n! = n \times (n-1)!$ It starts with $1! = 1$.

As before, we must work backwards to get 0!: $(n-1)! = n!/n$. We cannot go below zero.

Factorials are related to binomial coefficients. Define

$$n!m = \frac{n!}{(n-m)! \times m!}$$

and check that this gives the numbers in Pascal's triangle when n is the number of the row and m is the number of steps from the leftmost coefficient in row n (starting at $m = 0$). Check that $n!2$ are the triangular numbers, that $n!3$ are the tetrahedral numbers, and so on.

When n is an integer, $n!m$ is taken to be zero for $m < 0$ or $n < m$. The sequence is finite: just the binomial coefficients in the nth row of Pascal's triangle.

$$(1+x)^n = \sum_{m=0}^{n} n!mx^m = 1 + \frac{n}{1}x + \frac{n(n-1)}{2 \times 1}x^2 + \frac{n(n-1)(n-2)}{3 \times 2 \times 1}x^3 + ..$$

What if n is not an integer? Let's explore $n = 1/2$.

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3..$$

What does this mean? Let's see what happens when we multiply it by itself.

$\times$	1	$x/2$	$-x^2/8$	$x^3/16$	..
1	1	$x/2$	$-x^2/8$	$x^3/16$	..
$x/2$	$x/2$	$x^2/4$	$-x^3/16$	..	
$-x^2/8$	$-x^2/8$	$-x^3/16$	..		
$x^3/16$	$x^3/16$	..			
$\vdots$					

The terms in x^2 cancel. The terms in x^3 cancel. If we were to take the calculation further we would find that all terms in x^m cancel for $m > 1$.

This leaves $1 + x$:

$$((1+x)^{1/2})((1+x)^{1/2}) = ((1+x)^{1/2})^2 = 1 + x$$

So $(1+x)^{1/2}$ must be the same as $\sqrt{1+x}$.

We would also find that $(1+x)^{1/3}$ is $\sqrt[3]{1+x}$, that $(1+x)^{2/3}$ is $(\sqrt[3]{1+x})^2$, and so on.

This introduces us to *fractional powers*.

What about negative n? Let's try $n = -1$.

$$(1+x)^{-1} = 1 - x + x^2 - x^3..$$

What does this mean? Let's multiply it by $1 + x$:

$$\begin{array}{c|ccccc} \times & 1 & -x & x^2 & -x^3 & .. \\ \hline 1 & 1 & -x & x^2 & -x^3 & .. \\ x & x & -x^2 & x^3 & .. & \end{array}$$

This time everything cancels except 1: $(1+x)^{-1} \times (1+x) = 1$. So $(1+x)^{-1} = \frac{1}{1+x}$.

Clearly now $(1+x)^{-2} = \frac{1}{(1+x)^2}$ and so on.

This introduces us to *negative powers.*

When n is a non-negative integer (0, 1, 2, ..) the binomial coefficients are the terms of $(1+x)^n$ and form a finite sequence (or at least a sequence with a finite number of non-zero terms). When n is a fraction or negative, the sequence of coefficients in infinite.

The sum of set of terms, which may include a sequence of coefficients, is called a *series.* Infinite sequences of coefficients such as the above give rise to infinite series.

Infinite series raise questions about whether they *converge* or not. Will we actually get a finite number should we manage to sum the whole infinite series? All we will do for now is take note that infinite series can be tricky.

8. Really big numbers.

What power of 2 is also, almost, a power of 10?

$2^{10} = 1024 \approx 1000 = 10^3$

This is a useful question because the two number systems we are now most likely to encounter are based on 2 or 10. Computers use 2 because "on" and "off" are the two states that a computer circuit can be in (neglecting the sometimes important but usually transient state of switching from one to the other). Humans use 10 because we count on ten fingers.

This quantity of a thousand (more or less) is so important it has a name: "kilo" (for 1000) or "Kilo" (for 1024). It has an abbreviation: "K" (for both cases, so watch out!).

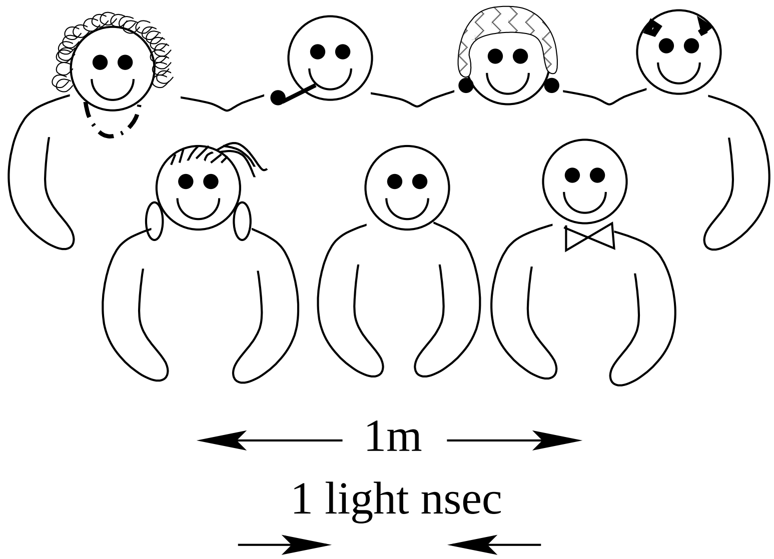
Powers of K are also important, and few enough of them are normally needed that each power can have its own name. Here they are.

kilo	1000	10^3	Kilo	1024	2^{10}
mega	1000K	10^6	Mega	1024K	2^{20}
giga	1000M	10^9	Giga	1024M	2^{30}
tera	1000G	10^{12}	Tera	1024G	2^{40}
peta	1000T	10^{15}	Peta	1024T	2^{50}
exa	1000P	10^{18}	Exa	1024P	2^{60}
zetta	1000X	10^{21}	Zetta	1024X	2^{70}
yotta	1000Z	10^{24}	Yotta	1024Z	2^{80}
nona	1000Y	10^{27}	Nona	1024Y	2^{90}
dica	1000N	10^{30}	Dica	1024N	2^{100}

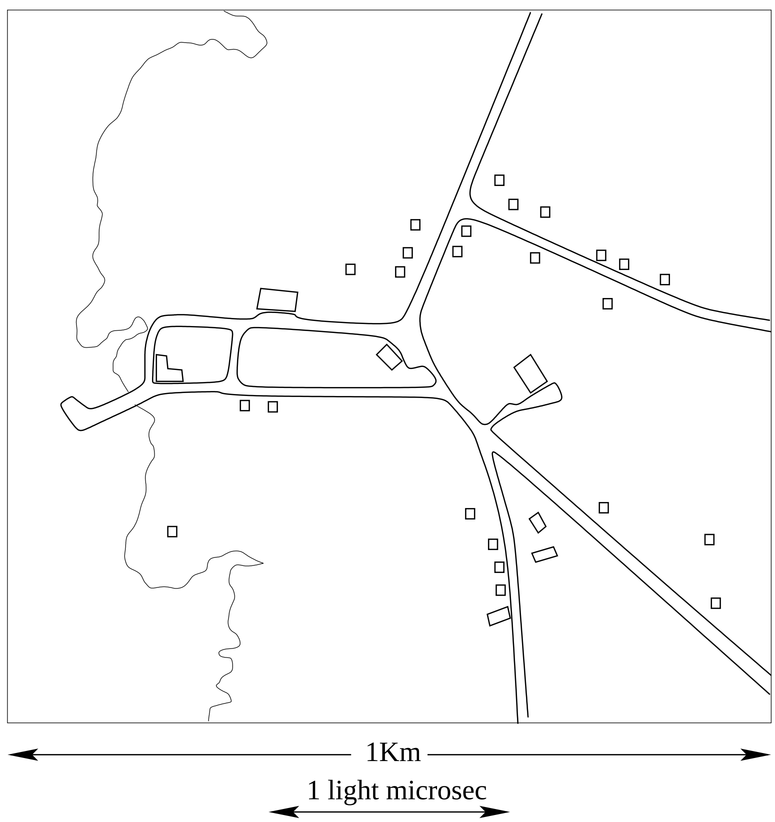
Note that the abbreviations, K, etc., mean different amounts depending on which column they are in.

(Beyond this, should you ever want to go there, there would be dikilo, dimega, .., bidica, bidikilo, .. tridica, .. quardica, .. up to centa, and then up to kila, but that really seems excessive.) Not having found names beyond yotta, I have made some of these up.

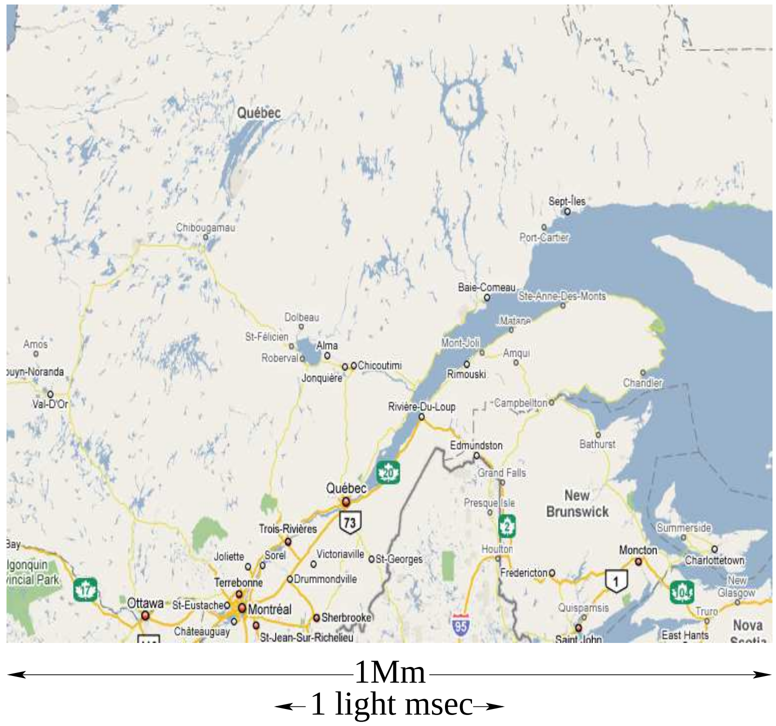
To illustrate the scales of these really enormous numbers, here is a set of images for distances at each scale. The unit is meters, so we talk of Km, Mm, Gm, etc. I have also listed times, which are the times light needs to travel comparable distances. These times involve some further abbreviations which we'll encounter in the next section, on small numbers. The basic unit is seconds.



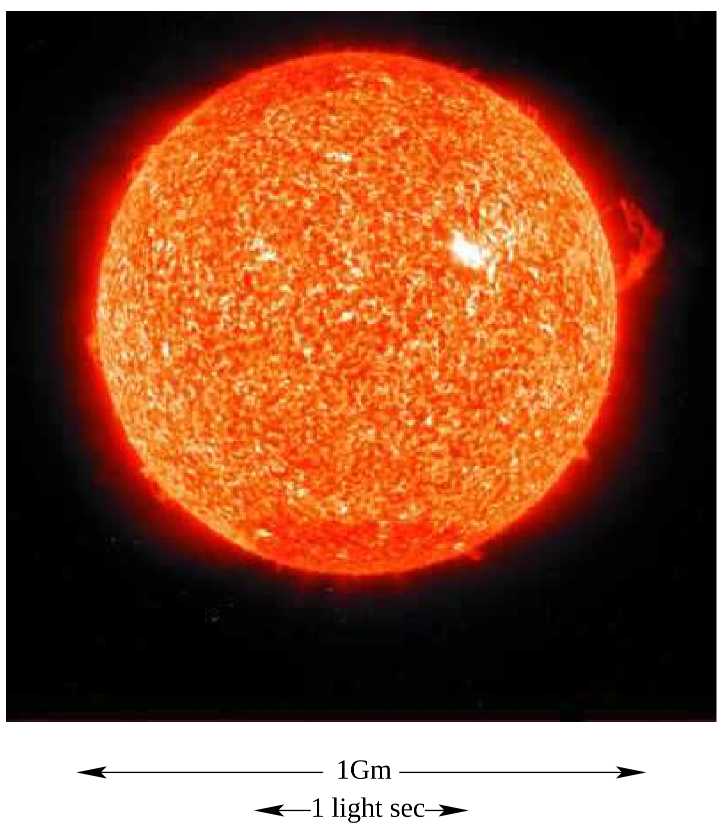
A family portrait.



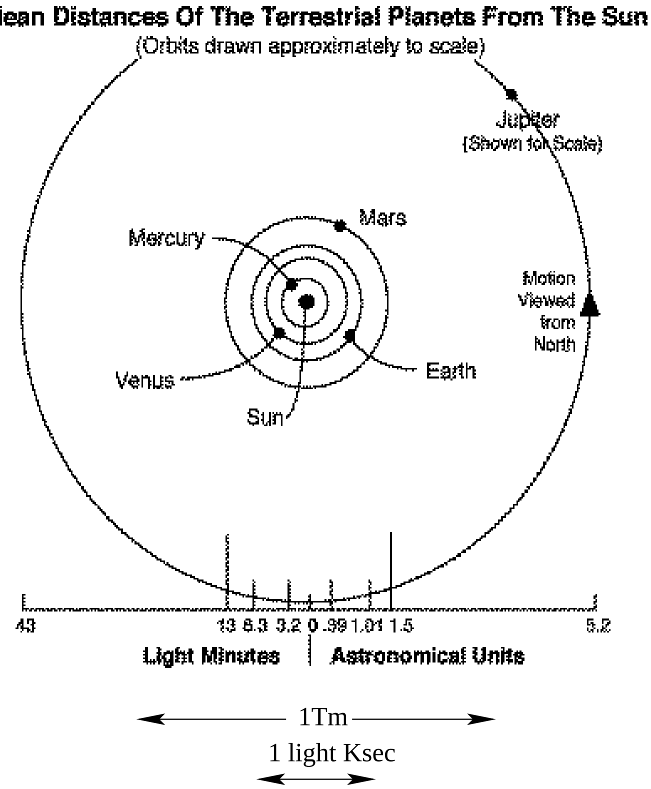
Hometown. (A tracing based on Google Earth. 08/7)



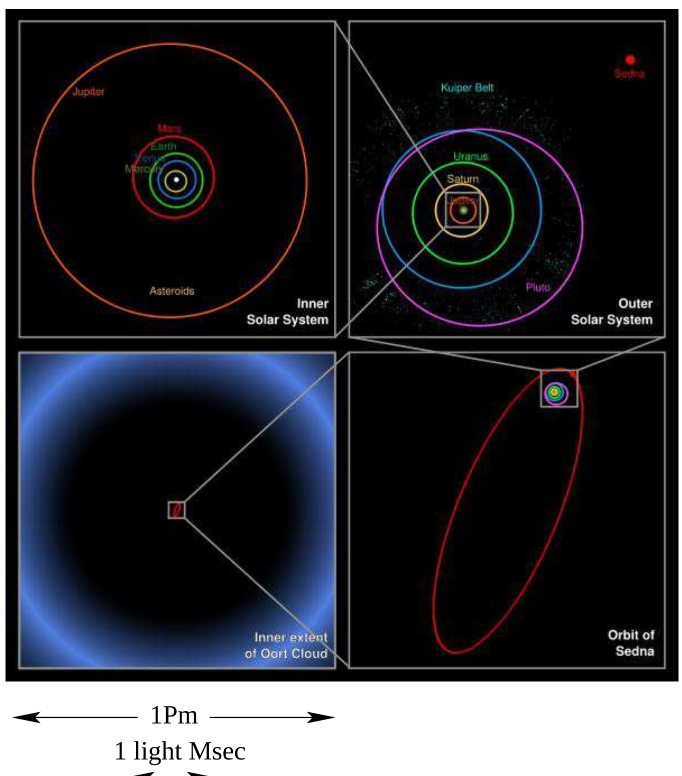
Map of Quebec and parts of neighbouring provinces. (From Google Maps. 08/7)



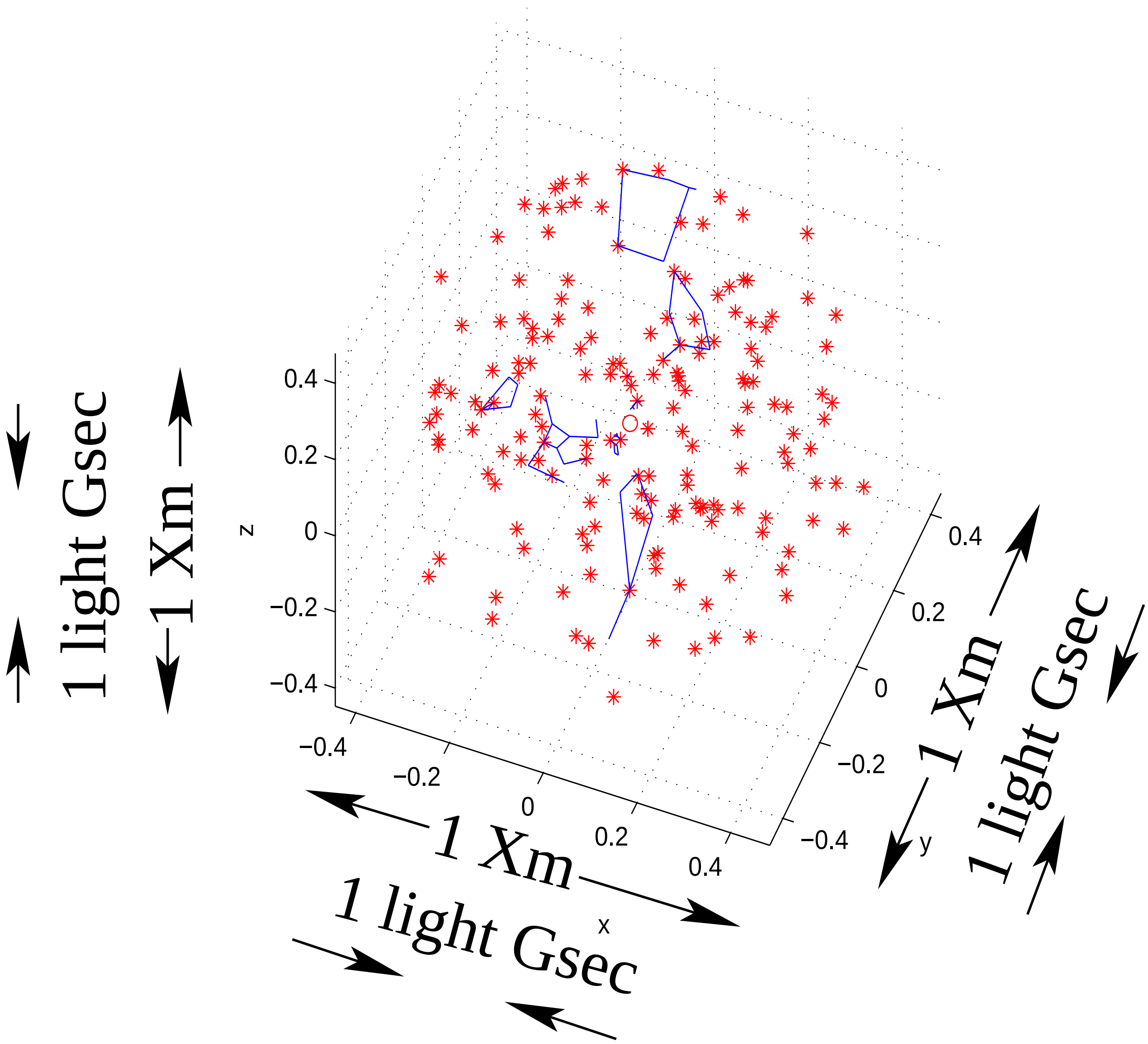
Sol, our Sun. (Source: stardate.org/images/gallery/sun5.jpg 08/7/31.)



Inner planets of the solar system. (Source: www.solarviews.com/eng/solarsys.htm 08/7/31. University of Phoenix.) The asteroids are not shown.



The Oort and Kuiper cometary clouds. (Source: en.wikipedia.org/wiki/Image:Oort_cloud_Sedna_orbit.svg 08/7/31.)

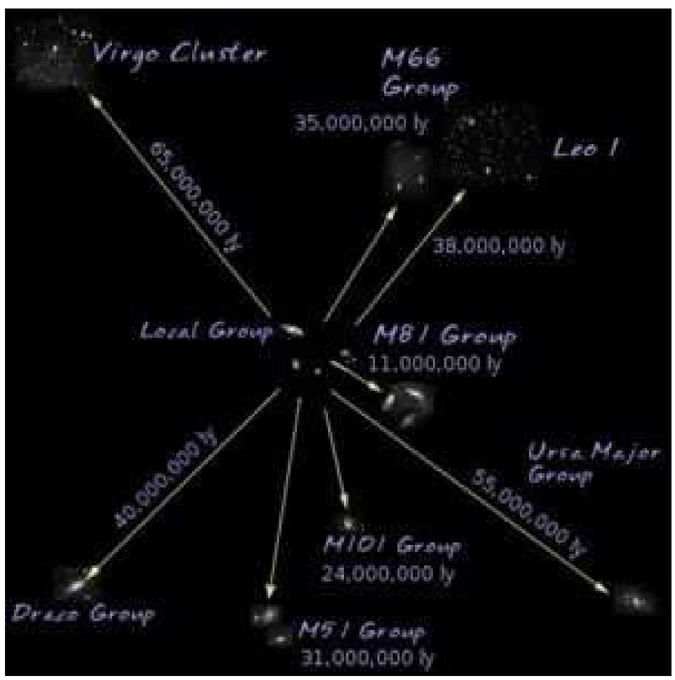


Stars within 0.5 exameters (52 light years) of Sol. (MATLAB program. Star data from Yale Bright Star Catalog (as of 08/8 at tdc-www.harvard.edu/software/catalogs/bsc5.html) and constellation data from [Pas00].)
Seen from outside. Sol is the red circle at the centre. All 178 stars are shown as asterisks, without distinction as to brightness, either intrinsic or as apparent from Earth.
The nine constellations shown have as a major star at least one of the 178 nearby stars. The other vertices are stars that are further away and so not shown.

1Zm

1 light Tsec

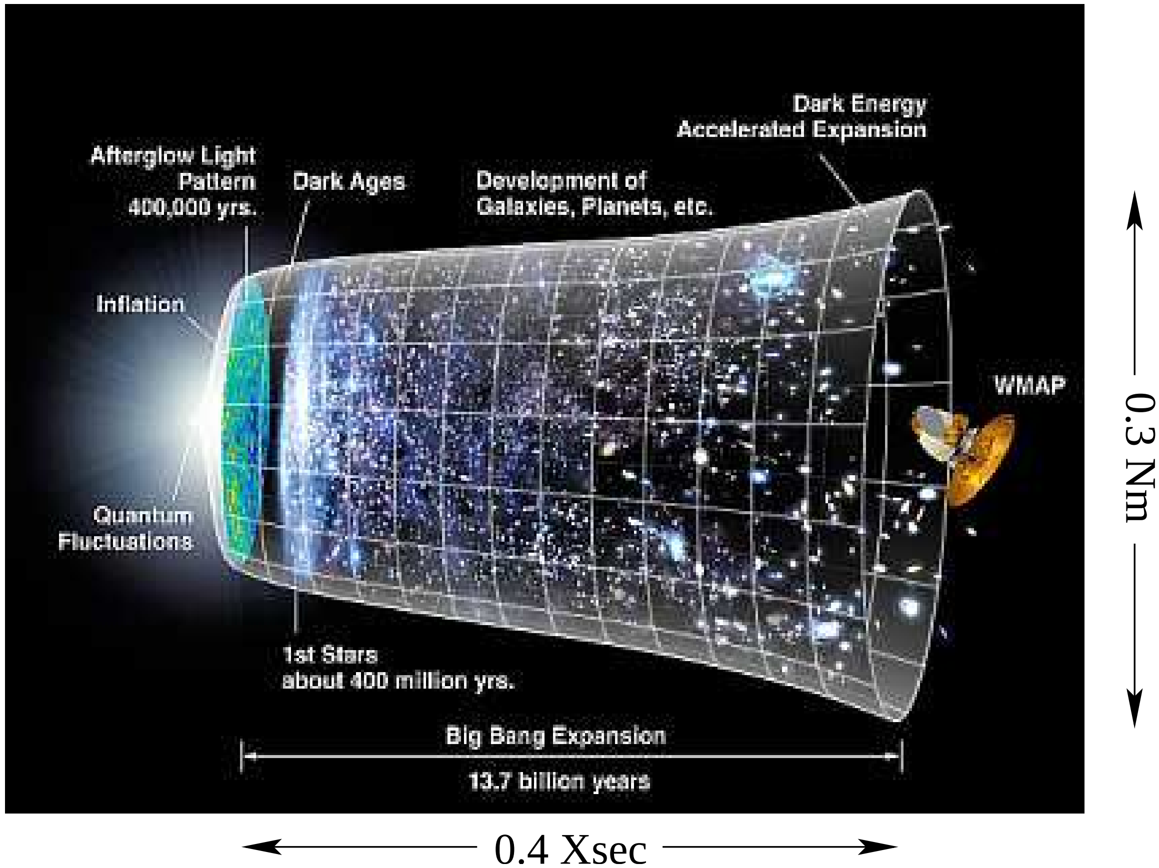
Artist's impression of the Milky Way, our galaxy, as a barred galaxy. (Source: www.solarviews.com/eng/solarsys.htm 08/7/31. University of Phoenix.)



1 Ym

1 light Psec

The Virgo Supercluster of galaxies, our local supercluster. (Source: www.answers.com/topic/virgo-supercluster 08/8/19.)



The history of the visible Universe. (Source: www.cobankopegi.com/b/wmap.jpg 08/8/21.) A synopsis of data gathered by NASA's Wilkinson Microwave Anisotropy Probe, launched 2001. "From a stable orbit 1 million miles [1.6Gm] from the Earth, WMAP can measure fluctuations that amount to less than a millionth of a degree Fahrenheit."

9. Really small numbers.
Since we now know about negative powers, we can also move "downwards" from human scale.

Again, we use thousands as the basic step, this time inverted. The abbreviation for "milli" or "Milli" is "m" (lower case) for each, with context again used to discriminate.

milli	1/1000	10^{-3}	Milli	1/1024	2^{-10}
micro	m/1000	10^{-6}	Micro	m/1024	2^{-20}
nano	μ/1000	10^{-9}	Nano	μ/1024	2^{-30}
pico	n/1000	10^{-12}	Pico	n/1024	2^{-40}
femto	p/1000	10^{-15}	Femto	p/1024	2^{-50}
atto	f/1000	10^{-18}	Atto	f/1024	2^{-60}
zepto	a/1000	10^{-21}	Zepto	a/1024	2^{-70}
yocto	z/1000	10^{-24}	Yocto	z/1024	2^{-80}
ninto	y/1000	10^{-27}	Ninto	y/1024	2^{-90}
tinto	ν/1000	10^{-30}	Tinto	ν/1024	2^{-100}

Note that there are some duplicate first letters, so Greek letters are used, μ ("mu") for micro and ν ("nu") for ninto.

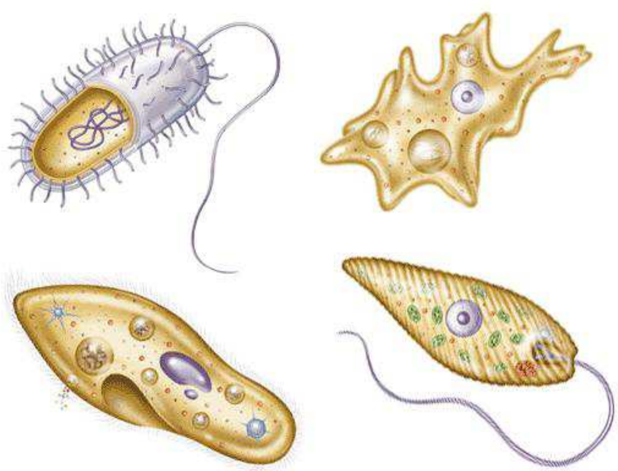
(To go beyond tinto, how about hatinto, thirtinto, fourtinto, fiftinto and so on to kento and milo?) I made up some of these, too, beyond atto.

Here is another sequence of images of distances, this time getting smaller and smaller.

1 mm

1 light psec

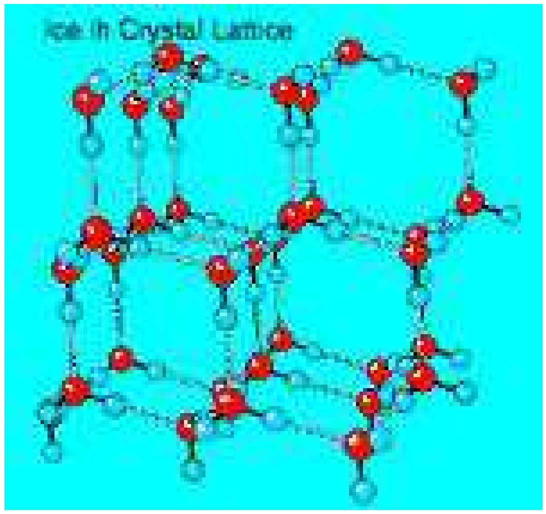
Eastern Canada's worst predator, the black fly. (Source: commons.wikimedia.org/wiki/Image:Black_Fly.gif 08/8/21.)



1 μm

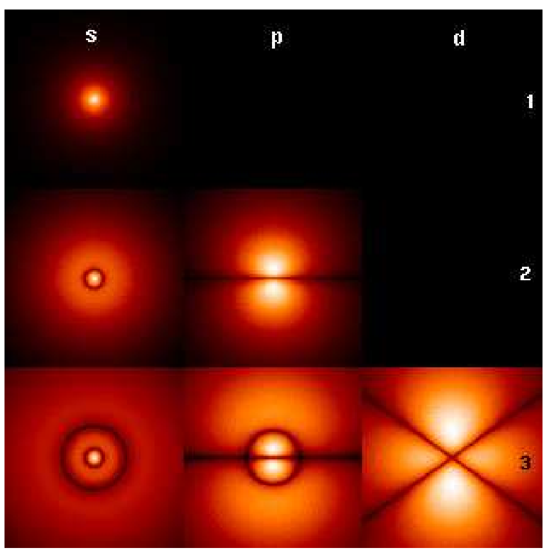
1 light fsec

Unidentified microbes. (Source: www.scientific-art.com/GIF%20files/Zoological/microbea.gif 08/8/21.) The top left drawing could be Escherichia coli, the bacterium which, living in our intestines, helps us digest, but which, in our drinking water, may be fatal.



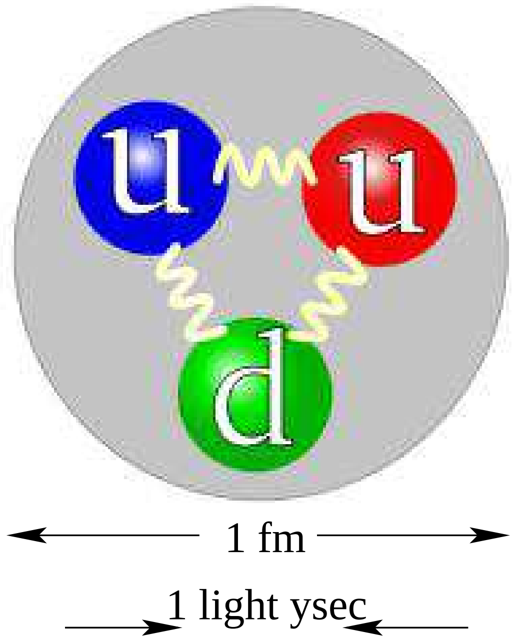
1 nm
1 light asec

The molecular structure of a hexagonal ice I crystal. (Source: www.its.caltech.edu/~atomic/snowcrystals/ice/ice.htm 08/8/21.) “Ice can assume a large number of different crystalline structures, more than any other known material. At ordinary pressures the stable phase of ice is called ice I, and the various high-pressure phases of ice number up to ice XIV so far. (Ice IX received some degree of notoriety from Kurt Vonnegut’s novel Cat’s Cradle.)”



100 pm
333 light zsec

Probability densities for the three lowest-energy configurations of the electron in the hydrogen atom. (Source: en.wikipedia.org/wiki/Orbital_angular_momentum 08/8/21.) I have not thought of an example at 1 picometer scale, so 100 pm must do.



The quark structure of the proton. (Source: en.wikipedia.org/wiki/Proton.html 08/8/21.)

10. Summary

(These notes show the trees. Try to see the forest!)

- Powers of 2, 3, 10. As “trees”.
- “Bee tree”: powers of ϕ and Fibonacci numbers.
- Search tree and inverse powers: logarithms.
- Interest and Euler’s number.
- Powers of $1 + x$ and Pascal’s triangle.
- Fractional and negative powers.
- Really big and really small numbers in the universe.

II. The Excursions

You’ve seen lots of ideas. Now *do* something with them!

1. Fold a piece of paper in half eight times (that is, to half-size, to quarter-size, and so on), then tear it down the middle.

2. (Eugene Lehman.)
   a) Look up Père Marin Mersenne (1588–1648). What happened in history the year he was born? What happened throughout the whole second half of his life?
   Find which of the first 17 Mersenne numbers are prime. (Helpful hints for composite-checking: an integer whose digits sum to a multiple of 3 is itself a multiple of 3 (why?); same with 9 (why?); $2^p - 1$ is always composite if p is composite (why?).)
   b) *Perfect numbers* are those whose divisors, apart from themselves (but including 1), sum to themselves. For example $6 = 1 + 2 + 3$; $28 = 1 + 2 + 4 + 7 + 14$. Can you see a pattern?

A Mersenne prime times the next smaller power of 2 is always perfect. Why? What are the next two perfect numbers after 6 and 28? Is there a perfect number of five digits? If not, how many digits has the next perfect number? Are there any perfect numbers that are not products of a Mersenne prime with the next smaller power of 2? Specifically, are there any odd perfect numbers? Why do Mersenne perfect numbers always end in 6 or 8?

3. a) According to the family tree in Note 2, how many parents do you have? How many grandparents? Great-grandparents? Great-great-grandparents? And so on.
   b) Let's warp the usual meaning of "ancestor" to include yourself (as your "0th ancestor"), your parents (as your "1st ancestors") and so on. With this definition and according to the family tree in Note 2, how many ancestors do you have back as far as your parents? As far as your grandparents? Great-grandparents? Great-great-grandparents? And so on.

4. I made some assumptions in calculating the number of your ancestors. First I assumed that a generation is 25 years: that is, typically, each child is born when heir parents are 25. Second I assumed that every one of your ancestors is so in only one way. That is, your parents were not siblings, which would give you only two grandparents, or they were not cousins, which would give you fewer than eight great-grandparents, and so on. Back far enough in time, every ancestor cannot be distinct from every other.
   Imagine and describe some situations in which you have fewer than 2^n n^{th} ancestors. Work out how many ancestors you have in each case.

5. Here is the MATLAB program I used to draw the ancestor and world population graphs.

```
  % worldPopMyAnc.m                THM             080729
  % Plot of world population and ancestor count.
  % Data from en.wikipedia.org/wiki/World_population
  X=[-10000,-9000,-8000,-7000,-6000,-5000,-4000,-3000,-2000,-1000,-500,1,1000,
1750,1800,1850,1900,1950,1955,1960,1965,1970,1975,1980,1985,1990,1995,2000,2005];
  Y=[1000,3000,5000,7000,10000,15000,20000,25000,35000,50000,100000,200000,310000,
791000,978000,1262000,1650000,2518629,2755823,2981659,3334874,3692492,4068109,
4434682,4830979,5263593,5674380,6070581,6453628];
  hold off
  plot(X,Y/1000000)
  hold on
  Xanc=[2000:-25:1175];
  Yanc=[2.^((2000.-Xanc)./25)];
  plot(Xanc,Yanc/1e9,'g')
  xlabel('Year (BCE to present)')
  ylabel('World Population / My Ancestors (billions)')
  hold off
```

   The numbers in this program mean that the population was `Y` thousand in year `X`: 1000 thousand (1 million) in year 10 000 BCE, .. 310 000 thousand (310 million) in year 1 CE, .. 6 453 628 thousand (6.45.. billion) in 2005. Play with the program until you can find out to the nearest decade when it claims that your ancestors outnumber the population of the world.
   (One thing to try is to use MATLAB's `semilogy()` plotting function instead of `plot()`.)

6. Looking at the numbers in the above program, and particularly at the lengths of time recently needed to double the world's population, show that population growth has lately been faster than exponential. (See Note 4.)

7. Rabbit hunters (Eugene Lehman.) Mrs C went home with 1/2 rabbit more than 50% of the total number of rabbits caught. After Mrs C left, Ms B went home with 1/2 rabbit more than

50% of the remaining uncaught rabbits After Ms B went home, Dr A went home with 1/2 rabbit more than 50% of the remaining uncaught rabbits. But there were no more uncaught rabbits! How many did each catch? How many rabbits would Mr D catch?

8. Richard Muller [Mul08, p.126] says that 81 generations of uranium 235 nuclear fissions occurred in the Hiroshima bomb (1945/8/6). Fission is caused by a neutron penetrating the nucleus, and each U-235 fission in turn produces two neutrons. How many nuclei split in that explosion? The "yield" was equivalent to 13 kilotons of TNT: if the uranium had held together for 82 instead of 81 doublings, what would the yield have been?
   Plutonium fission releases three neutrons. How many generations of plutonium fission, as in the Nagasaki bomb (1945/8/9), would have been needed to split at least the same number of nuclei as the U-235 nuclei split above?

9. Every power of three is one plus twice the sum of all previous powers of three (including $3^0 = 1$).
   a) Check this out for some low powers of 3.
   b) Figure out why it is always true.
   c) How does this rule adapt to powers of integers other than three?

10. Show that powers of 11, up to a certain point, are palindromes, i.e., they are the same read from right to left as from left to right. At what point does this stop? How do these palindromes relate to Pascal's triangle and why?

11. Every positive number when multiplied repeatedly by itself becomes either arbitrarily large or arbitrarily small, excepting one. What is that one? What about non-positive numbers?

12. Show that $(1 + a^2 + a^3 + .. + a^n)(1 - a) = 1 - a^{n+1}$. What does this tell us about the sum of the 0th to the nth powers of a number? What is that sum if $a < 1$ and n becomes arbitrarily large ("goes to infinity")?

13.
```
As I was going to St Ives
I met a man with seven wives
Every wife had seven sacks
Every sack had seven cats
Every cat had seven kits
Kits, cats, sacks, wives
How many were going to St Ives?
```

14. Find an argument that the number of bees at *any* level in the Bee tree is the sum of the numbers of bees at the two immediately lower levels. Hint: how many males and how many female bees are there at each level?

15. a) Use a calculator, or write a program, to find ϕ to four decimal places. What is meant by "convergence"?
    b) Multiply this number by each of the Fibonacci numbers, f_n, and round to the nearest integer. When does this result not equal the next Fibonacci number, f_{n+1}?

16. **Golden ratio.** The number ϕ is called the "golden ratio" and appears in many interesting places.
    a) Show that $1 + \phi = \phi^2$ by applying $f_{n-1} + f_n = f_{n+1}$ to show $1 + \frac{f_n}{f_{n-1}} = \frac{f_{n+1}}{f_n}\frac{f_n}{f_{n-1}}$.
    b) Show that $\phi - 1 = 1/\phi$ by similar reasoning or by working directly from the result in (a).
    c) Show that $\phi = (1 + \sqrt{5})/2$ satisfies both of the relationships in (a) and (b). So does $\phi = (1 - \sqrt{5})/2$. Because ϕ can be written this way, in terms of the algebraic operations $(+, \times, \sqrt{})$ on regular numbers, it is called an *algebraic* number. ϕ is neither an integer nor

a "rational" number, i.e., a number which is a *ratio* of two integers. Integers and rationals have either a finite number of decimal places before becoming zeros (1, 1/2=0.5, 1/4=0.25, 1/5=0.2, ..) or an infinite repetition of a finite number of decimal places (1/3=0.333..., 1/27=0.037037037..., ..). These are what I just now called "regular numbers". The decimal places in ϕ never stop and do not form a repeating pattern.
d) Here is a sunflower head, in which Wikipedia says (en.wikipedia.org/wiki/Sunflower) we can see "florets in spirals of 34 and 55". Can you find other successive Fibonacci numbers in this picture? Find a nice, complete sunflower head and look for Fibonacci numbers. Or try a daisy or a pineapple.

e) Look through [Liv02] for many other ideas.

17. a) Compare the Fibonacci sequence with the sequence of powers of 2 up to 2^8 and see the regularity of the doublings.
b) Suppose the size of the doubling step for the Fibonacci sequence is d. Then $2 = \phi^d$ and you can use logarithms to find d:

$$\log(2) = \log(\phi^d) = d\log(\phi).$$

(Why does this last step follow from what you have just learned about powers and logs?) So

$$d = \log(2)/\log(\phi) = 0.301/0.209 = 1.44$$

and powers of ϕ take 1.44 steps to double.
c) What is the size of the doubling step for powers of 10?

18. Look up Fibonacci (Leonardo of Pisa, 1170?–1250?). What important landmark was constructed during his lifetime? Where did his father, Gugliermo Bonaccio, take the family on a protracted business trip? What small animals' generations did Fibonacci abstract his sequence from?

19. a) Why is the following always true of a power sequence?
$$p_n^2 = p_{n-1}p_{n+1}$$
(And also $p_n^2 = p_{n-2}p_{n+2}$, $p_n^2 = p_{n-3}p_{n+3}$ and so on.)
In 1, 2, 4, 8, 16, .., try $4^2 = 2 \times 8$ and $4^2 = 1 \times 16$.
b) Is this approximately true of the Fibonacci sequence? Compare f_n^2 with $f_{n-1}f_{n+1}$ in .., 1, 2, 3, 5, 8, 13, .., say 5^2 vs 3×8 or 8^2 vs 5×13. What is the pattern?
c) Look up [Gar82, pp.64–6] for a paradox based on this approximate relationship in the Fibonacci sequence.

20. The Australian tongue orchid takes remarkable advantage of the haplodiploidy of wasps. Orchids often mimic female insects of particular species in order to attract the male into

attempting to mate with the orchid. In doing so the frustrated male rubs against the orchid's pollen sacs (or occasionally has them stuck onto him by the orchid) and then carries the pollen to another orchid, far enough away that he has forgotten his frustration. Thus the orchid reproduces instead of the insect. (Orchids that propagate in this way must be rare or else the pollinators stop being fooled.)

The Australian tongue orchid actually gets the male wasp of its pollinating species to ejaculate. This wastes sperm that could otherwise fertilize a female which would then produce an equal number of males or females. The unfertilized female wasp produces only males, an advantage for the orchid.

Some orchids, like most other flowers, put out enough energy to create *nectar* to attract pollinators. Another possible disadvantage of this method is that it can attract riff-raff species of insect who acquire pollen but do not propagate it effectively. It could be advantageous to limit pollinators to a specialist species.

So when Darwin learned that the star orchid of Madagascar secrets its nectar in a spot a foot deep into a narrow spur of the flower, he made a prediction which was vindicated only a couple of decades after his death. What was that prediction?

My source for this excursion is [PM09]. Learn enough evolutionary biology to join in the discussion of the interpretations I have taken from this article.

21. In the search tree in Note 3, how many steps are needed to search for `Rigel Kentaurus` (the closest star to us apart from `Sol`, our own sun)? To search for `Sirius` (the "dog star", visible, in the northern hemisphere, in winter and in the "dog days" of summer)? To search for `Procyon`? What is the maximum number of steps? What is the average number of steps?

22. The game of 20 questions allows "yes" or "no" answers to questions invented by the person who is trying to guess what object another person is thinking of. Explain why 20 questions are enough, if effectively used, to distinguish among a million different objects.

23. What is the reference in the following story? When Noah, who had saved all the creatures of the Earth from the great flood by securing male and female of each species in his boat, released them all after the waters had receded and the rainbow come and gone, he told them to go forth and multiply. A few weeks later he came across the two adders with no signs of progeny. "Why have you not multiplied?" he asked them. "We can't multiply: we're only adders." However, several weeks after that, Noah found the adders again, this time near a stack of firewood, with many baby adders swarming about them. "Aha" said Noah "I see that adders *can* multiply by logs".

24. Use a calculator or write a program to persuade yourself that $\log_3(n)$ for any number n (not just integer) is equal to $\log(n)/\log(3)$.

25. Ask an older engineer or scientist if hey has a *slide rule* hey can show you. Explore how the slide rule adds two numbers by mapping each into a distance along a straight edge, one number on the fixed part of the slide rule and one on the sliding part. Notice that the numbers it adds are not the numbers written on the two parts of the rule but the logarithms of those numbers. Make your own slide rule by calculating the logarithms (base 10) of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 and measuring them onto two edges of paper. Then you can multiply, say, 2×50 by placing the 1 of one edge at the 50 of the other and reading the number on the second that is opposite the 2 of the first.
    This works equally well for 2000×50 or 20×500: you just must keep track of the number of zeros.

26. Show that $(1 + 0.08)^{150}$ is over 100 000, so that this times ten \$ is over one million \$. Hence you have turned ten into a million dollars.
    Compound interest is also an exponential process. What is the doubling time for interest compounded at eight percent?

27. Use a calculator or write a program to calculate $(1 + 0.08/n)^n$ for a sequence of ever-larger n, say 1, 2, 4, 8, 16, .., and compare each answer with $2.7182818^{0.08}$.

28. a) What is the doubling step for powers of $e = 2.7182818$? Show that this corresponds to a continuously compounded interest rate of 69%.
b) A handy rule of thumb comes out of this for doing quick calculations on small interest rates. The doubling time for interest rate i% is $69/i$ years, since i times this many years gives the 69%. Now 69 is a difficult number to divide in your head by anything except 3 or 23, so let's go to a slightly higher percentage—which we should do anyway since interest is never compounded continuously in practice—say 72%.
Now you can astonish your friends and teachers by being able to calculate investments in your head. For instance, how long would it take to double money invested at 8%? $72/8 = 9$ years. How many such doublings are needed to multiply it by 100 000? Well, a million is 20 doublings and ten times less is close enough to $8 = 2^3$ times less, so $20 - 3 = 17$ doublings in a factor of 100 000, and $17\times 9 = 153$ years, not too far off the 150 years we originally found.
c) Eight percent is getting a little high for good accuracy with these mental interest calculations, Do some comparisons between the trick of (b) with the true interest calculations for interest rates of 1, 2, 3, 4, 6, 8 and 12 percent.

29. Since $(1 + i/n)^n \approx 1 + i$ for small i (as we have seen; and the approximation gets better and better as n gets larger and larger—just as ϕ is a better and better approximation to ϕ_n as n gets larger and larger) so the product of two interest-rate expressions

$$(1 + i/n)^n \times (1 + j/n)^n \approx 1 + i + j \approx (1 + (i + j)/n)^n$$

What kind of function of i and j gives a product which is the same function of $i + j$? What is the special significance of $(1 + 1/n)^n$ as n gets arbitrarily large? Calculate this sequence and see if it converges to anything you have seen in the Notes.
Euler's number, e, is neither integer, rational nor algebraic. Like the algebraic numbers ($\sqrt{2}$, $\sqrt[3]{5}$, ϕ, ..), e has an infinite number of digits after the decimal place which do not become any repeating pattern. Unlike the algebraics, it cannot be expressed using algebraic operations on integers or rationals, but only as a *limit*—the convergence as n becomes arbitrarily large. Euler's number is one of a fourth category, the *transcendental* numbers. (Fascinatingly, e does have an algebraic relationship with another transcendental number, π,

$$e^{i\pi} + 1 = 0$$

where i in this equation is no longer the interest rate but a new special number, but we'll have to wait for trigonometry and for "two-dimensional" numbers to pursue that: Week 4.)

30. Look up Leonhard Euler (1707–1783) and Euler's number. (Pronounced "Oiler".)

31. Many natural laws are exponential. Boltzmann's law, as we will see in Book 9c, says that the number of air molecules per unit volume decreases exponentially with height h: proportional to $e^{-mgh/kT}$ where mgh is the potential energy of the molecules (of mass m at height h in the Earth's gravitational field of strength g near the Earth's surface) and kT is the average thermal energy of each molecule (temperature T is a measure of average thermal energy but it must be converted from degrees Absolute to energy units by Boltzmann's constant k). Radioactive elements lose half their nuclei in a fixed length of time called the *half-life*.
A remarkable exponential law is Gordon *Moore's law* (1965, 1975) which says that the "complexity for minimum component costs" doubles every two years. The usual interpretation is that the number of transistors on integrated circuits doubles every two years. This rate has been maintained for over forty years of computer and digital component manufacture. This is remarkable because it does not describe a law of nature but rather prescribes technological/commercial human behaviour. It says that digital innovation and marketing will continue

to support this doubling in performance.
a) How many years from now will a computer that costs as much as your present computer will have 1000 times the performance?
b) Of course, all exponential increases must come to an end and so will Moore's law. Make a study of the present-day sizes of integrated circuit components and try to forecast when the limitations of light speed and quantum mechanics will wreck Moore's law for conventional electronics.

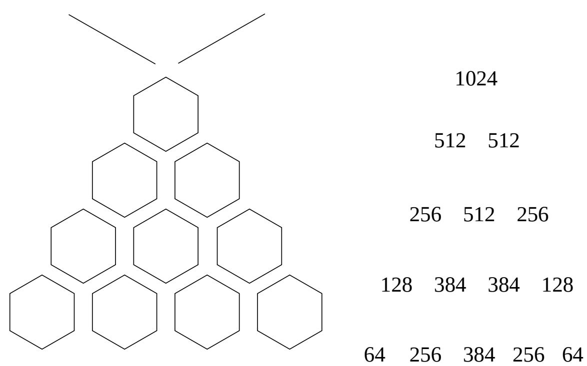
32. Look up John Napier of Merchistoun (1550–1617) and Napierian logarithms. (Rhymes with "rapier".)

33. Why do relationships such as the following hold for every row of Pascal's triangle?

$$\begin{array}{rcl} 1 & = & 2^0 \\ 1+1 & = & 2^1 \\ 1+2+1 & = & 2^2 \\ 1+3+3+1 & = & 2^3 \\ & \vdots & \end{array}$$

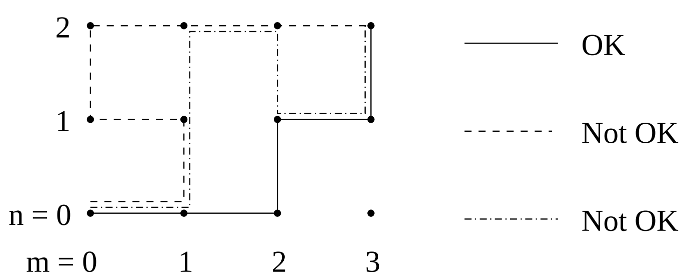
a) If you drop 1024 pachinko balls (the small, shiny ball-bearings used in the Japanese arcade game) through a *Galton board* and exactly half go each way at each hexagonal top apex, you will get the distributions shown at each layer. Relate these to Pascal's triangle.

34. b) Write a MATLAB program using `rand(1) > 0.5` (`rand(1) < 0.5` if you write right to left) to make a coin-flip choice about whether each ball goes left or right at each hexagon apex and find the distribution of balls after 1024 of them have dropped through a 4-level Galton board.



c) Try the above for a tetrahedral champagne-glass fountain.
d) Look up Francis Galton (1822–1911). Who was his famous cousin whose views he challenged effectively more than once?

35. **Mhonotonic paths.** Show that $(m+n)!m$ paths can be formed in an m by n rectangle which never back up horizontally or vertically.
For instance, in this $m \times n = 3 \times 2$ example, the solid path is OK because it never decreases in either m or n, but the dashed path backs up from $m = 1$ to $m = 0$ (horizontal) and the dash-dot path backs up from $n = 2$ to $n = 1$ (vertical).



36. Use the binomial theorem to find the slopes we were exploring in Note 12 of Week i. The slope at value x_0 of some expression $f(x)$ is approximately

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

(why?) where Δx is a small increase in x beyond x_0, and where the smaller Δx is, the better the approximation.
So for $f(x) = x^2$, $(x + \Delta x)^2 - x^2 = (x^2 + 2!1x\Delta x + ..) - x^2 \approx 2x\Delta x$ and so the slope is $2x$ (and in this case, for technical reasons we do not get into until Week 12, the approximation is exact).
Notice how this slope depends on x. Check the positions on x^2 in Week i Note 12 where the slopes are those I gave there. Find the slope at $x = 1$. Calculate the constant, c, in $y = sx + c$ for the straight line of that slope, s, and touching x^2 at that point.
Find slopes of the other nonlinear curves in Note 12 of Week i using arguments similar to that above.

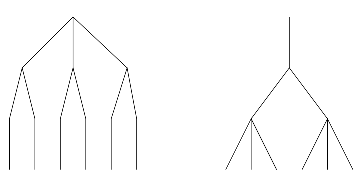
37. The way I wrote ! in Note 6 as both a unary (one operand) and a binary (two operands) operator is potentially ambiguous. What is the ambiguity and how can it be fixed so that ! can be used safely in these two ways for a computer programming language?

38. a) As n gets very large, what happens to $n!/n^n$? Try calculating the first few by hand, then, if you like, by calculator. Can you make a general argument?
b) If we define $(n)_k \stackrel{\text{def}}{=} n(n-1)..(n-k+1)$ i.e., the first k factors of $n!$, what happens to $(n)_k/n^k$ as n gets very large while k stays fixed? Again, try calculating the first few by hand, then, if you like, by calculator. By showing that $(n)_k/n^k = (n-1)_{k-1}/n^{k-1}$ make a general argument.

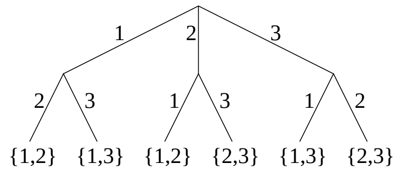
39. (Zeno's dichotomy paradox.) "That which is in locomotion must arrive at the half-way stage before it arrives at the goal." (Aristotle, Physics VI:9, 239b10, cited from Wikipedia 2008/10/15.)
A block ahead of you the schoolbus has just stopped, so you start to run. Unfortunately, before you get to the end of the block, you must get half-way. Then, before you get to the end of the block, you must get three-quarters-way (half of the remaining half-way). Then seven-eights (half of the remaining half-half-way) and so on. So you will never catch the bus.
a) What is wrong with this argument? Explore the series
$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ..$$
b) You are in a room with something you really want, but you have been told that the physics of the room are such that you can only get halfway to your goal with each step. Zeno would give up. What would you do?

40. In a family of four children, is it more likely that there are two of each sex or that there are three of one sex and one of the other? Try writing down all the possibilities explictly, or try writing them all down as a tree.

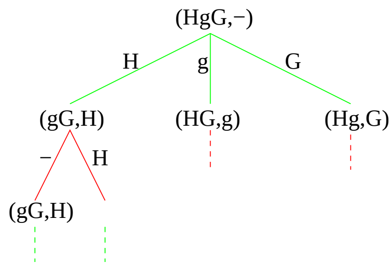
41. **Factorial trees.** Here are two trees related to 3!. How many leaves do they have? How many other trees of height 3 can be constructed with thus number of leaves?



42. **Choose trees.** Here is a tree showing how to choose two things out of three. Note that each pair of things is chosen twice using this tree, so the number of different ways of choosing is $3 \times 2/2$, half the number of leaves in the tree. How many ways can three things be chosen from four? How do these numbers relate to Pascal's triangle?



43. **Ways trees.** Two grumpy people, g and G, and one happy person, H, are on one side of a river and want to cross. The boatman has room for only one person in the boat. If H is ever left alone with either g or G, H will become unhappy because the grump does not have the other grump to grump at. How can they all cross the river without making H unhappy?
Here is the beginning of a "ways tree" for this problem. (Factorial trees and choose trees are also ways trees, but special cases.) Green lines show the outgoing boat trips and red show the return trips. The configurations show who is on which side: e.g., (gG,H) means g and G are on the original side and H is on the other side. Complete it to solve the puzzle.



What is the ways tree for a problem with a 2-place boat, no boatman, an equal number of nice and nasty people and the nasty must never outnumber the nice? Couples, and no woman may be with any man in the absence of her husband?

44. (e trees, π trees.) A tree whose fanout converges on Euler's number e the way the Bee-tree fanout converges on ϕ would have in each level respectively 1, 3, 8, 22, 60, 163, 443, .. nodes. Each of these is the nearest integer to $e\times$ the previous integer.
Here are "rewrite rules" that generate the top of such a tree, where the root node has fanout 3.

3 => 8 2 => 6

Where do these rules break down?
Can you begin a π-tree, with 1, 3, 9, 28, 88, 276, 867, .. nodes on each respective level?
What is the difference between ϕ on one hand and e and π on the other that affects this?

45. What are the first few terms of the series for $(1+x)^{-1/2}$? What does it mean?

46. A Kilo is 1024 = 1000 (1 + 0.024) = (1 + 0.024)$\times$ a kilo: that is, a Kilo is 2.4% more than a kilo.
We can use binomial coefficients to show that a Mega is about 5% more than a mega: $1024^2 = 1000(1+0.024)^2 = 1000(1+2\times 0.024+0.024^2)$. But 0.024^2 is really tiny (how much is it?) and so we can forget about it.

$$1024^2 \approx 1000(1+2\times 0.024) = 1000(1+0.048)$$

and this is almost 5% more.
Do the same for Giga and giga, ignoring all powers of 0.024 higher than the first: you should

get about 7%.
Work out the percentages for the remaining large numbers. When do you think the difference becomes so much that it is not practical to think of the upper-case and lower-case versions as the same thing?

47. Look up Blaise Pascal (1623–62). What was his probabilistic argument for the existence of G-d?

48. **Multinomial coefficients.** a) Rework the beginning of Note 6 for $(w+x)^2, (w+x)^3$ and $(w+x)^4$.
b) Let's try $(w+x+y)^2$:

$$\begin{array}{c|ccc} & w & x & y \\ \hline w & w^2 & wx & wy \\ x & wx & x^2 & xy \\ y & wy & xy & y^2 \end{array}$$

gives $(w+x+y)^2 = w^2+x^2+y^2+2wx+2xy+2yw$
c) Now show $(w+x+y)^3 = w^3+x^3+y^3+3w^2x+3x^2y+3y^2w+3wx^2+3xy^2+3yw^2+6wxy$.
What about $(w+x+y)^4$?
d) Check that

$$\frac{3!}{\ell!m!n!}$$

is the number in front of the term

$$w^\ell x^m y^n$$

in the cube expansion in (c). Note that the powers sum to $\ell+m+n=3$ and that some of them can equal zero. Write down all the possible ℓ, m and n for this cube expansion.
e) If we move to four variables, $(w+x+y+z)^2$ and $(w+x+y+z)^3$ and so on are going to become complicated. So we should start looking for patterns.
One obvious one is that w^2, x^2, y^2 and z^2 will each appear only once in the square, w^3, x^3, y^3 and z^3 only once in the cube and so on.
Another obvious one is that the $wxyz$ term (which appears only when we get to the 4th power) will have $4! = 24$ in front of it. This is because there are that many permutations of these four letters: there are 4 ways of choosing the first letter (w, x, y or z); for each of these there are 3 ways of choosing the second letter (x, y or z if the first was w, and so on); for each of *these* 2 ways of choosing the third letter; and finally only one way of choosing the fourth and last; so $4\times3\times2\times1 = 4!$ ways in all, and that will be the number of $wxyz$ terms that must appear, in all these different orders of course, when we multiply $(w+x+y+z)$ by itself 4 times.
How do these special cases fit into and extend the pattern suggested in (d)?
f) Let's think of the 4th-power term wx^2y: it must come from three 3rd-power terms x^2y, wxy and wx^2. Your calculations in (e) should have shown that the multipliers of these terms are 3, 6 and 3, respectively, consistent with the pattern of (d). Is their sum, 12, also consistent with the pattern?
g) Write down all possible forms of 4th-power terms (this does not necessarily mean all 4th-power terms) and show a similar extraction of each from combinations of 3rd-power terms.
h) Show that

$$\frac{p!}{m_1!m_2!..m_\ell!} = \frac{(p-1)!}{(m_1-1)!m_2!..m_\ell!} + \frac{(p-1)!}{m_1!(m_2-1)!..m_\ell!} + ... + \frac{(p-1)!}{m_1!m_2!..(m_\ell-1)!}$$

where p is the power, ℓ is the number of letters, and the ℓ integers sum $m_1+m_2+..+m_\ell = p$, with the smallest of them being 1. (What happens to the expansion when, say, $m_2 = 1$?).

These fractions are called *multinomial coefficients*. What must ℓ be for the multinomial coefficients to reduce to the binomial coefficients of Note 6?

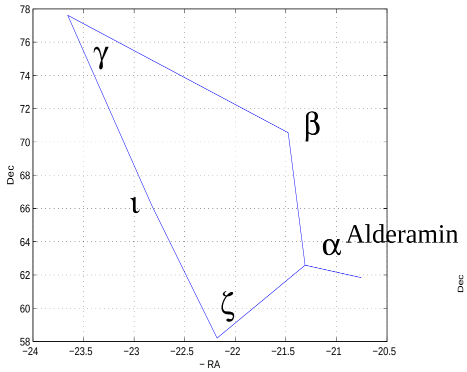
49. **Fermat's little theorem again.** Using the argument of Excursion *Fermat's little theorem again* back in Week i, show that the binomial coefficients $(m+n)!/(m!n!)$ (Note 6) and indeed the multinomial coefficients $(\ell + m + n)!/(\ell!m!n!)$ (Excursion *Multinomial coefficients*) are multiples of the numerator when this sum is prime, except when one of the factorials in the denominator equals the numerator. Or reverse the argument to get another proof of the theorem.

50. Here are the nine constellations shown in the exameter-scale image (next two pages).
 a) Find them in the image in Note 8. (Note that the closest constellations, Centaurus, Canis Major and Canis Minor, are very small in that image, because they are very close to Sol. Can you see them?)
 b) Find them in the night sky. (Note that you will be able to see some in summer and some in winter, and that you will also have to travel to the other hemisphere of the Earth to see them all.)
 Each constellation is shown with a grid of two coordinates, **RA** (right ascension, the angle in hours, minutes and seconds measured counterclockwise around the Earth's equator from one of the two points where that equator intersects with the "ecliptic" which is the plane in which the Earth and most other planets orbit Sol) and **Dec** (declination, the angle in degrees and minutes, measured northwards from the equator to $+90°$ and southwards to $-90°$). (For example, the black hole at the centre of our Milky Way Galaxy is in the constellation Sagittarius at 17h $45'$ right ascension and $-29°$ declination. Sagittarius can be seen in the spring in the southern hemisphere and in the autumn if you live not too far north in the northern hemisphere.)
 (In the sky, from north to south, the zero of RA separates the constellations of Cassiopeia and Cepheus, drops through the rear of Pegasus, separates Cetus from Aquarius, and then Phoenix from Grus.)
 (The constellations of the "zodiac" lie on the ecliptic, which is also the apparent path of the sun as the Earth orbits around it, so that astrologers can attribute importance to which constellation the sun was "in" the moment you were born.)
 The constellations from Note 8 are placed in the two figures here more or less in the positions given by their right ascensions and declinations.
 Note that important constellations are missing because all their major stars are too far away: Ursa Major (the Big Dipper)'s brightest stars are 7 to 13 exameters; Orion's bright stars are from 70 to 150 exameters; α Cassiopeia is 22 exameters and β Cassiopeia is just out of the 0.5 exameter range; Cygnus' brightest are 17–147 exameters; and Crux, the Southern Cross, is 9–32.
 c) Look up the constellations just mentioned in a star book or on a star map and try to see them in the night sky.

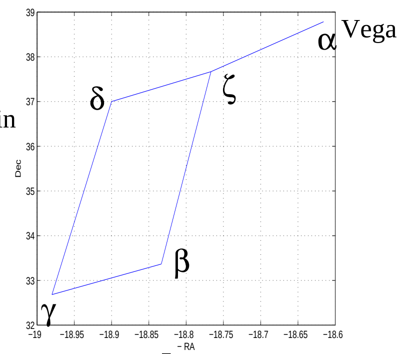
51. Use the programs drawStars.m, selectStarsUpto52ly and drawConst3D.m in MATLABpakfor this week to draw the 178 nearest stars as in Note 8 and the nine constellations as in Note 8 and in the previous excursion. Explore Larry Niven's (the science fiction author) "known space" within half an exameter from us using MATLAB's ability to rotate 3D images.
 (Note on drawing constellations: the distances in `parx` are true only for the stars within 0.5 exameters; I have interpolated `parx` for the other stars so that the 3D picture resembles the constellations as we see them from Earth. This obliges you also to do some manual work with the data.)

52. Find an image of the universe out to 5Ym (500 million light years) from us, ten times the size of our local supercluster of galaxies, and observe that the galaxies form a foam. The origin of this structure appears to be quantum foam at the time of the "big bang".

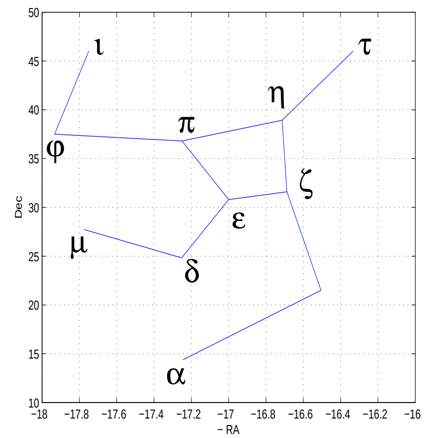
Northern Hemisphere, mainly Summer



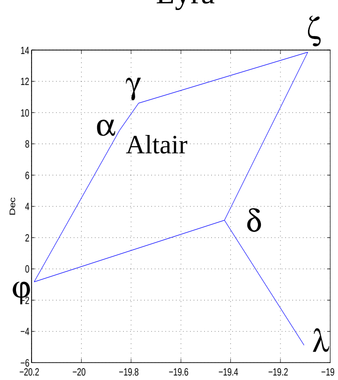
Cepheus



Lyra



Hercules

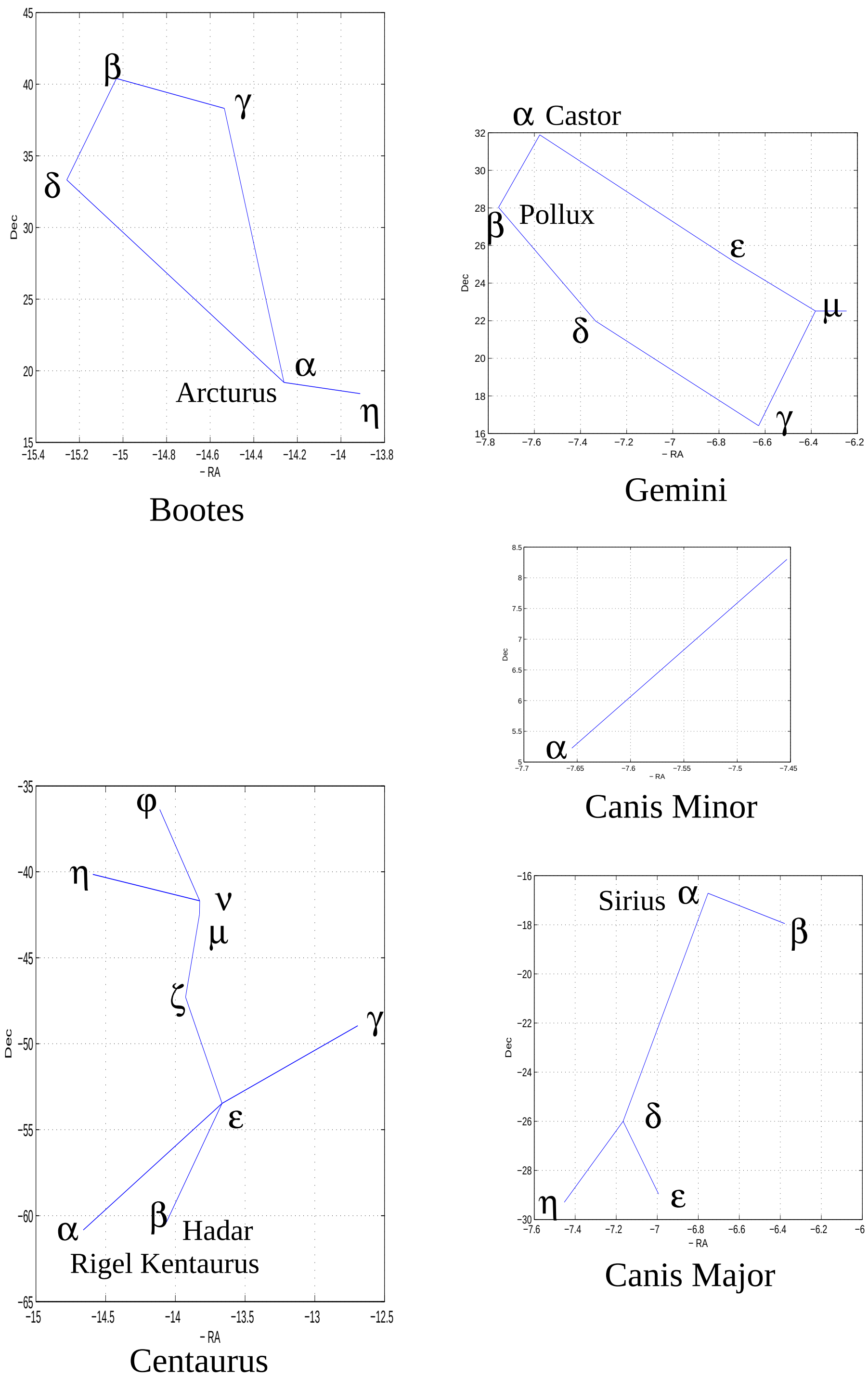


Aquila

Southern Hemisphere, mainly Winter

Figure 1:

Northern Hemisphere, mainly Winter



Southern Hemisphere, mainly Summer

Figure 2:

The "bubbles" in the foam are voids containing few galaxies and are typically 1Ym in diameter. However, there also appears to be one exceptionally large, 10Ym void, which cannot be accounted for by the cosmology of 2008. This and the nature of the "dark energy" causing the renewed expansion of the universe, may be the outstanding challenges to physical science in the early 21st century.

53. a) The number of particles in the universe is said to be between 10^{72} and 10^{87}. Express these two numbers using the vocabulary of Note 8.
b) Look up the word "googol". This is a name for 10^{100}. Express this using the vocabulary of Note 8.
c) How many protons can fit into the visible universe if protons could be stacked touching each other? (This could be a difficult question, so start with a simple approximation.) Express this number using the vocabulary of Note 8.

54. How fast are we moving in space? Try the following calculations in your head to a single significant figure.
a) The meter was defined by Napoleon's scientists to be 1/10 000 000 of the distance from pole to equator, so the circumference of the Earth is 40 000 000 meters (actually a little more than that since the meter was redefined as the distance light travels in 1/299 792 458 of a second). Since Earth turns a full revolution in one day, what is the speed (Km/sec) someone "standing still" is moving at relative to the centre of the Earth?
b) The Earth moves in an essentially circular orbit around the sun about 8 light minutes away from the sun. What is the speed (Km/sec) of the centre of the Earth relative to the sun?
c) At what speed (Km/sec) is someone "standing still" on the Earth's surface moving relative to the sun? Is the speed faster at noon or at midnight?
d) The sun moves at 220Km/sec in its orbit around the centre of the Milky Way galaxy. If the galaxy were a rigid disk (it is not) and if the sun were at the outer edge of the galaxy (it is close enough for one-figure accuracy), how long would it take for a complete revolution?
e) At what possible speeds (Km/sec) is someone "standing still" on the Earth's surface moving relative to the galactic centre?

55. Look up the stories associated with the constellations? For instance, how did Orion, the hunter, get into the sky?

56. Look up and learn Wade Hemsworth's "The Blackfly Song". (http://www.nfb.ca/film/blackfly Christopher Hinton 1991. Accessed 13/11/10.)

57. Read Kurt Vonnegut's "Cat's Cradle".

58. In familiar terms, how long is a kilosecond? A megasecond? A gigasecond?
In familiar terms, how long is a millicentury? A microcentury? A nanocentury?

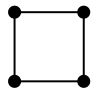
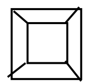
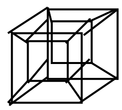
59. One litre is (almost) a milli meter-cubed (m m^3). How does this differ from a millimeter-cubed (mm^3)? That is, what is $milli^3$? What is the cube root of a nona? of a giga? What is the square root of a micro?

60. William Thomson (1824-1907, Lord Kelvin from 1892) said that molecules are so small that if you could mark a cupful of water molecules (say by getting a cupful of "heavy water", D_2O, where each hydrogen atom has been replaced by its 2-nucleon isotope deuterium, or by converting each oxygen molecule to its radioactive isotope, ^{15}O), dumped them into the ocean, waited for them to diffuse evenly among all the "seven seas", then drew out a cup of the mixture from anywhere in the oceans, you would find 100 of your marked molecules. Let's take the ^{15}O route, suppose the stirring takes place in much less than its 2 1/4-minute half-life, and work with a litre, which is about two cups (closer to US cups than Imperial, i.e., anachronistically, non-US) and, of course, 200 atoms in the resulting sample.

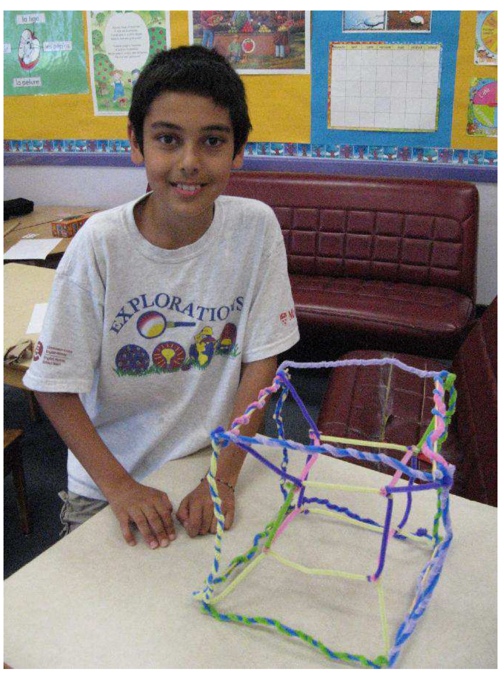
Using the fact that a litre of water masses 1 kg and that 17 kg of your ^{15}O water contains 0.6 nona molecules, calculate a) the linear dimensions of a water molecule (compare this to the ice-crystal spacing in Note 9) and b) how many litres Thomson reckoned are in the world's oceans.

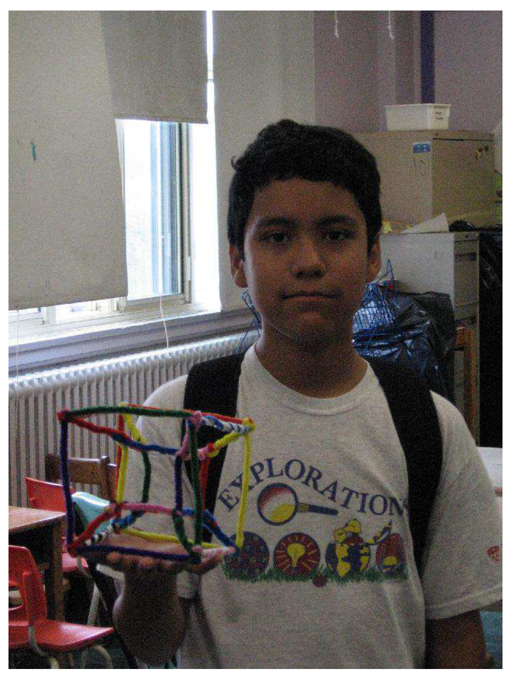
61. The number, 0.6 nona, of molecules that mass the same in kilograms as the atomic weight (number of nucleons) of each, is close to the largest of the very large numbers we considered in Note 8. The ranges of numbers in Notes 8 and 9 suffice to describe most of the *lengths* that we normally measure in the universe. But they do not convey even the corresponding range of *volumes*, which are the cubes of lengths. Since *masses* in our three-dimensional universe are proportional to volumes, the ranges of numbers in Notes 8 and 9 are not sufficient to describe the ranges of masses in the universe from the masses of protons to the masses of galaxies.
While we could possibly get away with volumes such as a few dm^3 (d = dica) or a few $\tau\mathrm{m}^3$ (τ = tinto), for masses we really need either to extend our ranges of numbers or to introduce new units.
I'd like to introduce new units. I have already used kilograms (kg) for human-scale masses, even though this could be confusing when we want to talk about kilo-kg (a Kkg is a megagram) or milli-kg (a mkg is a gram). At the atomic scale we can talk about daltons, d (named after John Dalton, 1766–1844) or "unified atomic mass units". One dalton is the mass of a proton. To be precise, it is defined as 1/12 the mass of an isolated carbon-12 atom, but we need only be precise "enough".
How many daltons is 1 kg?
An alternative atomic-scale mass is the electron-Volt, eV. This is actually a unit of energy, but Einstein showed (see Week 7a) that mass and energy are equivalent. Protons and neutrons each mass about 1GeV and electrons mass about 1/2 MeV.
How many eV in a dalton? In a kg?

62. **Hypercubes.** Powers of 2 can be illustrated graphically as a sequence of *hypercubes*. Hypercubes generalize squares and cubes to any number of dimensions. A d-dimensional hypercube has 2^d vertices. Thus a square is a 2-dimensional hypercube with $2^2 = 4$ vertices (points). A cube is a 3-dimensional hypercube with $2^3 = 8$ vertices. Going backwards, a straight line is a 1-dimensional hypercube with $2^1 = 2$ vertices, and a single point is a 0-dimensional hypercube with $2^1 = 1$ vertex.
A *tesseract* is a 4-dimensional hypercube. It can be visualized in 3-D in the same way that a cube can be visualized in 2-D. One way of drawing a cube on a flat sheet of paper is to draw a square with another square inside it and connect four pairs of corners. This sees the cube from end on, and in perspective, with the further face apparently smaller than the closer face. Similarly we will visualize a tesseract as a cube inside a cube, with eight pairs of vertices connected.
As with simplexes (Week i, Excursion "Simplexes") it is interesting to count edges, faces, etc. as well. Here is a start.

Dimensions	Hypercubes	Vertices	Edges	Faces
0	•	1		
1	•—•	2	1	
2		4	4	1
3		8	12	6
4		16	32	24
⋮		⋮	⋮	⋮
d		?	?	?

a) Check the numbers in each column.
b) If we define the *measure* of a geometric entity to be its length if it is an edge, its area if it is a face, its volume if it is a solid, etc., what are the measures of all hypercubes with sides of length 1?
c) Use pipe cleaners to build a 3-D model of a tesseract.





63. Add to the inventory of "personalities" of the integers (excursion in Week i) the categories covered in these Notes, and see if you can now distinguish any of the personalities you could not before.

	...	power of 2	power of 3	Mersenne number	Fibonacci number	lg	log	$n!m$	..
1		$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	..
:		:	:	:	;	;	:	:	

64. Consider a culture in which married women take their husbands' names and a population in which everyone is married and nobody has the family name Fibonacci. In what ways can you be descended from Fibonacci? Use the Bee tree of Note 2 to illustrate: if you are a woman, your father is your most recent progenitor who could have been named Fibonacci (Fibonacci was your maiden name but you are now married with another name)—your mother might on the other hand have been Fibonacci at birth; if you are a man, only your mother is your most recent progenitor who could have been Fibonacci at birth. If your parents were not Fibonaccis themselves, what about their progenitors?
This exercise goes on forever because Fibonacci never was a family name: the mathematician is more accurately known as Leonardo of Pisa; his father was Bonaccio, and Leonardo became known as the son of (figlio) Bonaccio.

65. Any part of the Preliminary Notes that needs working through.

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