Excursions in Computing Science: Book 11d. Forces and Invariants Part V. Functional Integrals

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I. Prefatory Notes

31. Path amplitudes. Quantum field theory in Part IV provides important insights not available before Dirac's integration of special relativity and quantum physics. But it did not make easy the calculations that challenged physics in the 1940s, such as the "Lamb shift", a tiny difference in the energy levels of the hydrogen atomic states $^2S_{1/2}$ and $^2P_{1/2}$ caused by polarization of the cloud of

virtual electrons and positrons that constitutes the "quantum vacuum"—the higher-order terms in the perturbation calculations of the U(1) theory, QED—and discovered by Lamb and Retherford in 1947.

It took a whole new approach to quantum physics to produce by the 1970s a quantum field theory which could do these calculations reliably. (Well, Schwinger in 1948 calculated the Lamb shift by a tour de force; and Dirac's 1966 "Lectures on Quantum Field Theory" does it in five lectures using the earlier techniques.)

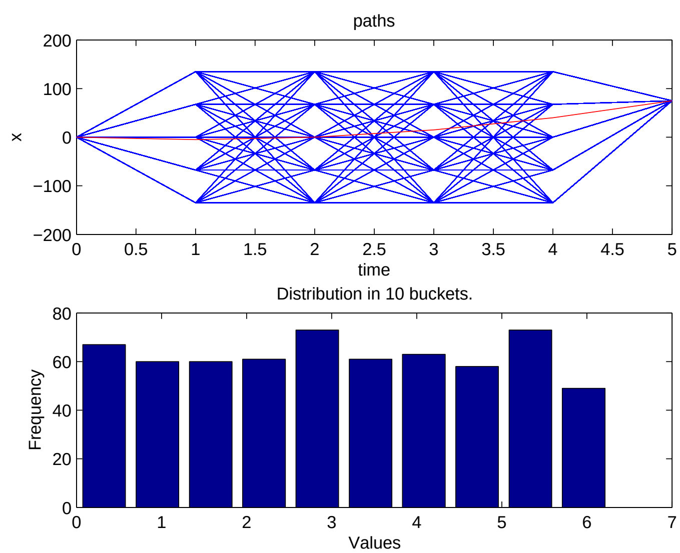
The third formulation of quantum physics (in addition to Heisenberg's matrix mechanics and Schrödinger's wave mechanics) is due to Richard Feynman, starting with his 1942 Ph.D. thesis.

In his own words, "A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of a particle at a particular time" [Fey48]. Feynman goes on (in sections 5 and 6 of that 1948 paper summarizing his thesis) to derive Schrödinger's equation from this new approach.

This is a significant departure from earlier views in which the Heisenberg uncertainty principle forbade a particle to have both a precise position and a precise velocity, i.e., a path. Instead a particle is now viewed as having a path, but *which* path is what is uncertain.

A computer program can illustrate what Feynman means, although the actual calculation must be done analytically. Here are $625 = 5^4$ paths through 1-dimensional space connecting point $x_a = 0$ at time 0 with point $x_b = 75$ at time 5. I've allowed the x-values to range from -160 to 160 in five discrete positions, and six timesteps (two at the endpoints) for the particle to travel from x_a to x_b.

An example of one path would have x-values `[`x_a`,xmin + xstep*[2,3,0,4],` x_b`]` with `xmin =` `−160` and `xstep = (160 − (−160))/4`



You can see that the particle is not forbidden any location in space (apart from the finiteness of the simulation). It can move from x_a towards x_b or away from it, and similarly for each step until the last, when it must arrive at x_b.

The red line shows the classical path: this is a particle with initial velocity -10 and uniform acceleration $+10$. Let's look at the Lagrangian—we'll work relative to the mass m (or with $m = 1$ if you like).

$$\frac{L}{m} = \frac{1}{2}(\dot{x})^2 + ax$$

Euler-Lagrange gives the constant acceleration

$$\ddot{x} = a$$

so then, taking the antislope, with initial velocity v_0,

$$\dot{x} = at + v_0$$

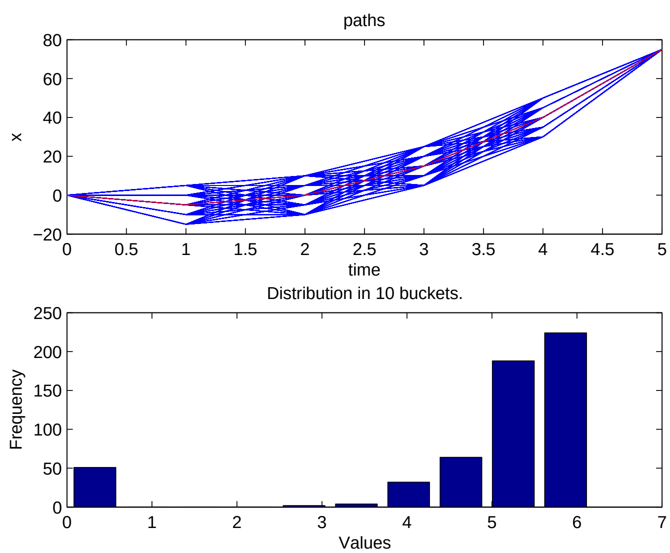
and

$$x = \frac{1}{2}at^2 + v_0 t + x_a$$

This last gives the path shown in red.

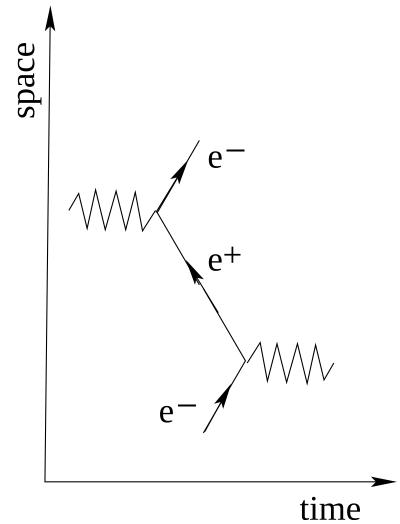
For the other paths, those in the simulation, we must sum up $v^2/2 + ax$ for each segment of the path. We can assume constant $v = (x - x_{\text{old}})/\texttt{tstep}$ where x_{old} is the previous value of x and `tstep` is the length of each time interval.

The bar chart, which is the lower plot in the figure, shows the distributions of total phases over all 625 paths, in the range from 0 to 2π. It seems to be effectively uniform: the phase contributions from each path will cancel each other out. Only the phase changes from the classical path and the paths near it will have a net effect, which is what we saw, even more crudely than here, in Note 7 of Week 5. If we tinker with the x-ranges to keep all 625 paths very close to the classical path, we see the very nonuniform distribution of phases.



This program might be useful if there were a way to extrapolate from a few sample paths to the continuum of all possible paths, but even so, the cost of refinement is exponential in the number of x-steps—there is one dimension for each—and the problem gets worse if we want y- and z-steps as well in three spatial dimensions.

The above illustrations do not show that the "particle" can also move backwards in *time*. Feynman also said [Fey49] "The various creation and annihilation operators in the conventional electron field view are required because the number of particles is not conserved, i.e., pairs may be created or destroyed. On the other hand charge is conserved which suggests that if we follow the charge, not the particle, the results can be simplified." He goes on in that paper to view the creation and subsequent annihilation of a positron as an electron zig-zagging in time, changing its sign (charge) as well as its direction in time for the positron "zag". This trajectory represents the creation of an electron-positron pair (upper part of the figure below), the positron travelling forwards in time, and then annihilating with (another) electron (lower part). Since the photon paths (the wiggly lines) are invisible in the detector observing these events, this same diagram can also be thought of as a single particle following the arrows on the lines but changing sign when it is moving backwards in time. Feynman gives the unforgettable image of a bombardier flying over a zigzag road: at one point in time there is one road, then there are three roads for a while, then one again.



32. Functionals. The mathematics that might help us deal with an infinite number of paths in Note 31 is the calculus of functionals. Whereas a *function* maps numbers to numbers, e.g., $x^2 : 1 \to 1, 2 \to 4, \cdots$, a *functional* maps functions to numbers, e.g., $F[f] = \int_0^1 dx f^2(x)$, some of whose mappings are given by the table

f	1	x	x^2	x^n	$\cos x$	e^x	$\cdots$
F	1	$1/3$	$1/5$	$1/(2n+1)$	$(1/4)\sin 1/2 + 1/8$	$(e^2-1)/2$	$\cdots$

The relationship between a path and the amplitude for the path (or the phase for the path) is functional. Path integrals become sums over all functionals and so might be approached through an extension of the idea of an antislope.

We start with slopes. We would be interested in another functional, say $f + \delta f$, differing only slightly from the given function f.

If f were a path between two points, and the functional $F[f]$ were given by a definite integral, say

$$F[f] = \int_0^1 dx(3f^2(x) - xf(x))$$

then δf must be zero at $x = 0$ and at $x = 1$.
Then the extension of the idea of slope of a functional $F[f]$ would be

$$\frac{F[f + \delta f] - F[f]}{\delta f}$$

Let's explore with the example

$$F[f] = \int_0^1 dx(3f^2(x) - xf(x))$$

then

$$F[f+\delta f] = \int_0^1 dx(3f^2(x) + 6f(x)\delta f(x) - x(f(x) + \delta f(x)))$$

if we consider $(\delta f)^2$ to be negligible. So

$$F[f+\delta f] - F[f] = \int_0^1 dx(6f(x)\delta f(x) - x\delta f(x))$$

Unfortunately, while we can move the difference between $F[f+\delta f]$ and $F[f]$ into the integral, we cannot move a division by $\delta f(x)$ into the integral. So we cannot find a "slope" in general.

This is like partial slopes versus slopes of 2-numbers. The latter are defined to be independent of direction (see Note 28) while partial slopes require a direction (see Note 1 of Book 11c).
We can extend the notion of "direction" from partial slopes by choosing a particular function for $\delta f(x)$

$$\delta f(x) = \epsilon\eta(x)$$

with ϵ the small multiplier whose limit we take to zero. The "slope" then is, for example.

$$\frac{F[f+\epsilon\eta] - F[f]}{\epsilon} = \int_0^1 dx(6f(x) - x)\eta(x)$$

and we can invert this to say that the *antislope* in the "direction" $\eta(x)$ of

$$\int_0^1 dx(6f(x) - x)\eta(x) \quad \text{is} \quad \int_0^1 dx(3f^2(x) - xf(x))$$

Is there a "direction" which, while it cannot be general, is useful? Let's try $\eta = \delta$, the Dirac delta-function, defined to give

$$\int_{-\infty}^{\infty} dx\delta(x) = 0$$

and

$$\int_{-\infty}^{\infty} dxf(x)\delta(x) = f(0)$$

or

$$\int_{-\infty}^{\infty} dxf(x)\delta(x-y) = f(y)$$

This is the continuous analog of the Kronecker delta, $\delta_{jk} = 1$ if $j = k$ but 0 otherwise:

$$\begin{aligned} \sum_j \delta_{jk} &= 1 \\ \sum_j f_j\delta_{jk} &= f_k \end{aligned}$$

Then, using different independent variables x and y,

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon}$$

This is analogous to the discrete case of independent variables

$$\text{slope}_{x_j} x_k = \delta_{jk} \qquad \text{slope}_{f(x_j)} f(x_k) = \delta(x_j - x_k)$$

So if $F[f] = \int_{-\infty}^{\infty} f(x)dx$ its slope is

$$\int_{-\infty}^{\infty} \delta(x-y)dx = 1$$

and the slope of $F[f] = \int_{-\infty}^{\infty} f(x)^n dx$ is

$$\int_{-\infty}^{\infty} nf(x)^{n-1}\delta(x-y)dx = nf(y)^{n-1}$$

A slope which will be important for us is (the limits on the integral are $\pm\infty$, left out for visual simplicity)

$$\begin{aligned}
\text{slope}_{J(y)} \quad \exp & \left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
& \text{slope}_{J(y)}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
& \int dx_1 dx_2 \ \text{slope}_{J(y)}(J(x_1)\Delta_F(x_1-x_2)J(x_2)) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
& \int dx_1 dx_2(\delta(x_1-y)\Delta_F(x_1-x_2)J(x_2) + J(x_1)\Delta_F(x_1-x_2)\delta(x_2-y)) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
& \left(\int dx_2(\Delta_F(y-x_2)J(x_2) + \int dx_1 J(x_1)\Delta_F(x_1-y))\right) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)\left(\int dx\Delta_F(y-x)J(x) + \int dx J(x)\Delta_F(x-y)\right) \\
= & -\frac{i}{2}\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) 2\int dx\Delta_F(y-x)J(x)
\end{aligned}$$

I've written out in detail the process of taking functional slopes just to show that it is formally the same as taking ordinary slopes: integrals are transparent to taking slopes, slopes of products are sums, etc. We make two assumptions: whatever $\Delta_F()$ is (it is defined in Note 37), it is independent of $J()$, and $\Delta_F(-x) = \Delta_F(x)$.

We are especially going to be interested in this result when $J \to 0$: it is 0.

But the second slope does not go to zero.

$$\begin{aligned}
\text{slope}_{J(y_1)}\text{slope}_{J(y_2)} \quad \exp & \left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right) \\
= & \ \text{slope}_{J(y_1)}\left(-i\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)\int dx\Delta_F(y_2-x)J(x)\right) \\
= & -\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)\left(\int dx\Delta_F(y_2-x)J(x)\right)^2 \\
& -i\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)\text{slope}_{J(y_1)}\int dx\Delta_F(y_2-x)J(x)
\end{aligned}$$

$$= \quad -\exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)$$
$$\left(\left(\int dx \Delta_F(y_2-x)J(x)\right)^2 + i\Delta_F(y_2-y_1)\right)$$

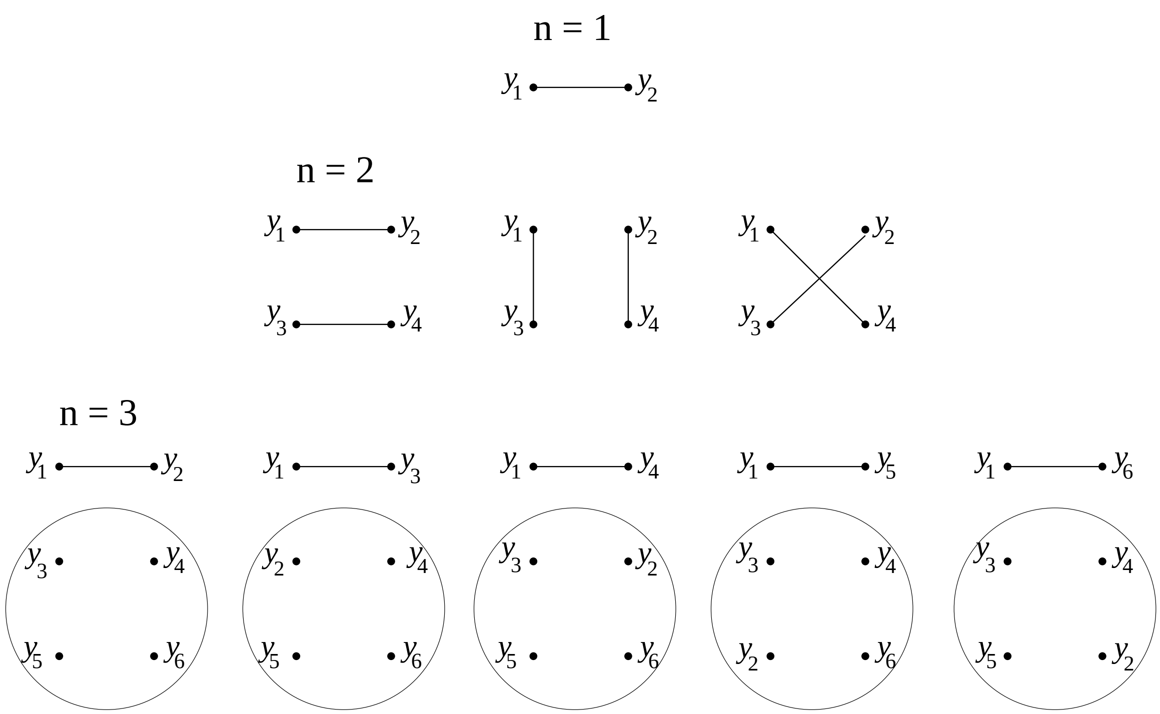
This becomes $-i\Delta_F(y_2-y_1)$ when $J \to 0$.
It should be evident that slope^n of this function goes to 0 for any odd n. So the next interesting power of slopes is the fourth.

$$\text{slope}_{J(y_1)}\text{slope}_{J(y_2)}\text{slope}_{J(y_3)}\text{slope}_{J(y_4)} \exp\left(-\frac{i}{2}\int dx_1 dx_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)\right)$$

$$\overrightarrow{J()\to 0} \quad \begin{array}{l} -\Delta_F(y_1-y_2)\Delta_F(y_3-y_4) \\ -\Delta_F(y_1-y_3)\Delta_F(y_2-y_4) \\ -\Delta_F(y_1-y_4)\Delta_F(y_2-y_3) \end{array}$$

(Working this out is an Excursion.)

There is a pattern here which saves us the labour of working out higher orders of slope. It is given by the possible pairwise connections of $2n$ points, which works out to $(2n-1)!! = (2n-1)(2n-3)\cdots 3{\cdot}1$, the *odd factorial*. Here are all the graphs for $n = 1, 2, 3$. Note how the first ($n = 1$) captures the slope^2 result $\Delta_F(y_1-y_2)$, and the next three ($n = 2$) capture the slope^4 results above.



The last five ($n = 3$) represent fifteen graphs: the circled subgraphs become each of the three $n = 2$ graphs.

It is now easy to construct graphs for higher n and from them write down the corresponding results for the slopes of order $2n$. It is also easy to see how the number of graphs is $(2n-1)!!$.

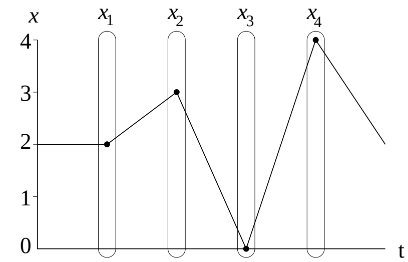
33. Gaussian integrals. The previous Note suggests that there is no general mathematics which will help us sum up over all possible paths the values associated with each path—i.e., find the

antislope (sum) of all possible functionals (values from path).

We'll have to specialize. Fortunately the amplitudes are exponentials of actions and the actions are sums—well, integrals, which is formally the same thing.

Furthermore, the contributions to the action involve squares—both of velocities in the kinetic energy, and, for the harmonic oscillators that describe fields, of positions.

Why are these fortunate? Let's look at one of the 625 possible paths the program of Note 31 was written to sum over. This is the path 2,3,0,4.



But each of these numbers is only one selection from the sum over all possible values of x. So we show these ranges and label them, respectively, x_1 (which has value 2 for this example), x_2 (value 3), x_3 (value 0) and x_4 (value 4).

To sum a functional $F[f]$ over all possible paths (not just the discrete values $x = 0, 1, 2, 3, 4$) we must express $F[f]$ as an integral of f over all possible values of x_1, x_2, x_3, x_4

$$F[f] = \int_{\infty}^{\infty} dx_1 \int_{\infty}^{\infty} dx_2 \int_{\infty}^{\infty} dx_3 \int_{\infty}^{\infty} dx_4 f(x_1, x_2, x_3, x_4)$$

Now if f happened to be an exponential of a sum, say,

$$f = e^{-a(x_1^2+x_2^2+x_3^2+x_4^2)}$$

this becomes a product

$$\begin{aligned} F &= \int_{\infty}^{\infty} dx_1 e^{-ax_1^2} \int_{\infty}^{\infty} dx_2 e^{-ax_2^2} \int_{\infty}^{\infty} dx_3 e^{-ax_3^2} \int_{\infty}^{\infty} dx_4 e^{-ax_4^2} \\ &= \left(\int_{\infty}^{\infty} dx e^{-ax^2} \right)^4 \end{aligned}$$

and this is the Gaussian integral we first figured out how to solve in Note 6 of Book 9c (Part I) (here using the notation of the Excursion *Notation* of that Book).

$$F = \left(\sqrt{\frac{\pi}{a}} \right)^4$$

Of course, we will have to do this integral N times and ultimately to take the limit $N \longrightarrow \infty$ so that the time steps are continuous just like the x-steps.

The problem is that there is no limit. Well, that's just what happened in ordinary integration, before we "normalized" it by multiplying by Δx, when we found there is no limit in the sequence of series

$$\sum_{j=a:1:b} f_j\ , \qquad \sum_{j=a:0.5:b} f_j\ , \qquad \ldots$$

where $j = a : s : b$ means that j runs from a to b in steps of size s.

The analog of weighting by Δx in the integral, viz

$$\int_a^b dx f(x) =^{\lim}_{\Delta x\to 0} \sum_{j=a:\Delta x:b} \Delta x f_j$$

is to weight each of the Gaussian integrals by, of course,

$$\frac{1}{A} = \sqrt{\frac{a}{\pi}}$$

Thus

$$\int \cdots \int \int \frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_N}{A} e^{-a(x_1^2+x_2^2+\cdots x_N^2)} = 1$$

Now we can do a path integral a little closer to the physics. Let's look at

$$\frac{1}{A^n}\int dx_1 dx_2 \cdots dx_{n-1} \exp\left(-\sum_{j=0}^{n-1} a(x_{j+1}-x_j)^2\right)$$

with

$$a = \frac{\pi}{A^2} = \frac{m}{2i\hbar\Delta t}$$

so that

$$a(\Delta x)^2 = -\frac{i}{\hbar}\frac{m}{2}\left(\frac{\Delta x}{\Delta t}\right)^2 \Delta t = -\frac{i}{\hbar}\frac{p^2}{2m}\Delta t$$

which is $-i/\hbar$ times the kinetic energy part of the Lagrangian times Δt, and which would integrate over time to give the action for a particle free of forces caused by any potential energy.

Note that x_0 and x_n will be the fixed endpoints of all the paths, so that there are $n-1$ integrations.

Now each variable, x_j, is tangled with its neighbour, because of the $(x_{j+1} - x_j)^2$.

The dx_1 integral would be

$$I_1 = \int dx_1 e^{-a((x_2-x_1)^2+(x_1-x_0)^2)}$$

We can isolate x_1 into a completed square, leaving a leftover term in x_0 and x_2, which we take as constants in this step.

$$\begin{aligned}(x_2-x_1)^2+(x_1-x_0)^2 &= 2x_1^2 - 4x_1\frac{x_2+x_0}{2} + x_2^2 + x_0^2\\ 2\left(x_1 - \frac{x_2+x_0}{2}\right)^2 &= 2x_1^2 - 4x_1\frac{x_2+x_0}{2} + 2\left(\frac{x_2+x_0}{2}\right)^2\end{aligned}$$

and the difference

$$x_2^2 + x_0^2 - 2\left(\frac{x_2+x_0}{2}\right)^2 = \frac{(x_2-x_0)^2}{2}$$

so, putting $y = x_1 + (x_2 + x_0)/2$,

$$\begin{aligned}I_1 &= \int dy e^{-2ay^2} e^{-a(x_2-x_0)^2/2}\\ &= \sqrt{\frac{\pi}{2a}} e^{-a(x_2-x_0)^2/2}\end{aligned}$$

Moving on to the second integral

$$I_2 = \int dx_2 e^{-a((x_3-x_2)^2+(x_2-x))^2/2}$$

we can similarly complete the square to get

$$y = x_2 - \frac{2}{3}\left(x_3 + \frac{x_2}{2}\right)$$

and the difference (as above)

$$\frac{1}{3}(x_3 - x_0)^2$$

So

$$\begin{aligned} I_2 &= \int dy e^{-3ay^2/2} e^{-a(x_3-x_0)^2/3} \\ &= \sqrt{\frac{2\pi}{3a}} e^{-a(x_3-x_0)^2/3} \end{aligned}$$

and the combination of the two steps is

$$\sqrt{\frac{2\pi}{3a}}\sqrt{\frac{\pi}{2a}} e^{-a(x_3-x_0)^2/3} = \frac{1}{\sqrt{3}}\frac{\pi}{a} e^{-a(x_3-x_0)^2/3}$$

After $n-1$ steps we get the combined result for the integral

$$\frac{1}{\sqrt{n}}\left(\frac{\pi}{a}\right)^{(n-1)/2} e^{-a(x_n-x_0)^2/n}$$

which when multiplied by $1/A^n = (a/\pi)^{n/2}$ gives

$$\begin{aligned} \sqrt{\frac{a}{\pi n}} e^{-a(x_n-x_0)^2/n} &= \sqrt{\frac{m}{2i\pi\hbar n\Delta t}} e^{im(x_n-x_0)^2/(2\hbar n\Delta t)} \\ &= \sqrt{\frac{m}{2i\pi\hbar(t_n-t_0)}} e^{im(x_n-x_0)^2/(2\hbar(t_n-t_0))} \end{aligned}$$

using $n\Delta t = t_n - t_0$.

Since this is independent of n it is unchanged in the limit $n \longrightarrow \infty$, and is a finite value for the whole path integral

$$\lim_{n\to\infty} \frac{1}{A^n} \int dx_1 dx_2 \cdots dx_{n-1} \exp\left(-\sum_{j=0}^{n-1} a(x_{j+1}-x_j)^2\right)$$

Now let's do something fancier, leading up to "Feynman diagrams".

$$\begin{aligned} Z(\ell) &= \int dx e^{-gx^2 - fx^3 + \ell x} \\ &= \int dx \left(1 - fx^3 + \frac{1}{2!}(fx^3)^2 - \cdots\right) e^{-g(x^2-(\ell/g)x)} \\ &= \int dx \left(1 - fx^3 + \frac{1}{2!}(fx^3)^2 - \cdots\right) e^{-g(x-(\ell/2g))^2} e^{\ell^2/(4g)} \end{aligned}$$

where I've expanded e^{-fx^3} in the second line and completed the square in the third.
Now note (see Excursion *Even moments*)

$$x^n e^{-gx^2-fx^3+\ell x} = \partial_\ell^{(n)} e^{-gx^2-fx^3+\ell x}$$

so we can replace

$$1 - fx^3 + \frac{1}{2!}(fx^3)^2 - \cdots$$

by

$$1 - f\partial_\ell^{(3)} + \frac{1}{2!}(f\partial_\ell^{(3)})^2 - \cdots = e^{-f\partial_\ell^{(3)}}$$

and we have

$$\begin{aligned} Z(\ell) &= \int dx e^{-f\partial_\ell^{(3)}} e^{-g(x-(\ell/2g))^2} e^{\ell^2/(4g)} \\ &= e^{-f\partial_\ell^{(3)}} e^{\ell^2/(4g)} \int dx e^{-g(x-(\ell/2g))^2} \\ &= e^{-f\partial_\ell^{(3)}} e^{\ell^2/(4g)} \sqrt{\frac{\pi}{g}} \end{aligned}$$

(Of course, replacing x^3 by the operator $\partial_\ell^{(3)}$ requires us to keep the function of ℓ to its right.)
We'll go on to work out some of these terms, but first let's see which ones are possible.
We explore the relationships among powers of g (ignoring the $\sqrt{\pi/g}$ factor), f and ℓ. We note that powers of f correspond to powers of $\partial_\ell^{(3)}$ in the expansion of $e^{-f\partial_\ell^{(3)}}$, and that powers of $1/g$ correspond to powers of ℓ^2 in the expansion of $e^{\ell^2/(4g)}$.

Here's a table showing what happens to powers of ℓ after operating with powers of $\partial_\ell^{(3)}$.

		V_f	0	1	2	3
			f^0	f^1	f^2	f^3
E			$\partial_\ell^{(0)}$	$\partial_\ell^{(3)}$	$\partial_\ell^{(6)}$	$\partial_\ell^{(9)}$
0	g^0	ℓ^0	ℓ^0	—	—	—
1	g^{-1}	ℓ^2	ℓ^2	—	—	—
2	g^{-2}	ℓ^4	ℓ^4	ℓ^1	—	—
3	g^{-3}	ℓ^6	ℓ^6	ℓ^3	ℓ^0	—
4	g^{-4}	ℓ^8	ℓ^8	ℓ^5	ℓ^2	—
5	g^{-5}	ℓ^{10}	ℓ^{10}	ℓ^7	ℓ^4	ℓ^1

Check the pattern: if we call the power of ℓ in the body of the table V_ℓ, then

$$V_\ell + 3V_f = 2E$$

We finish this exercise by finding the actual coefficients in the two cases $f\ell^2$ and $f^2\ell^4$ (apart, still, from the $\sqrt{\pi/g}$).
$V_f = 1, E = 3$

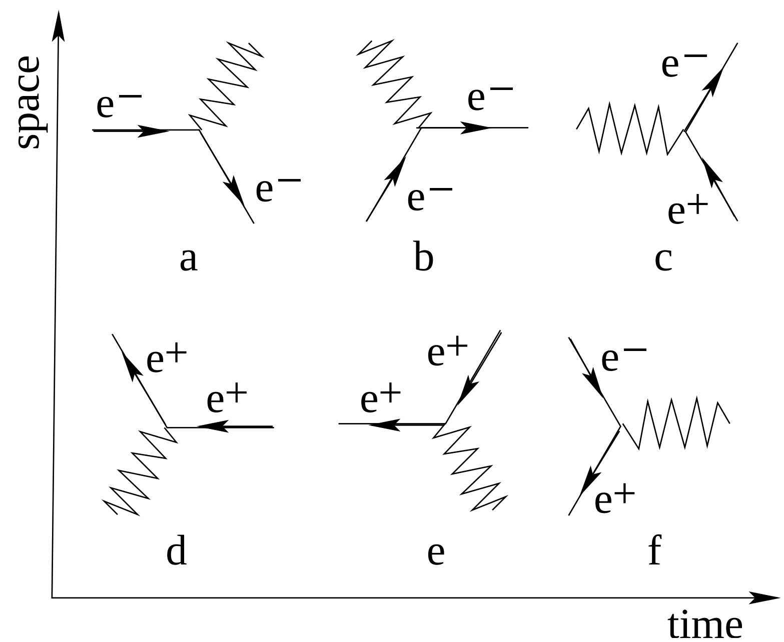
$$\begin{aligned} e^{-f\partial_\ell^{(3)}} e^{\ell^2/(4g)} &= (\cdots - f\partial_\ell^{(3)} + \cdots)\left(\cdots + \frac{1}{3!}\left(\frac{\ell^2}{4g}\right)^3 + \cdots\right) \\ &= -\frac{f}{3!(4g)^3}\partial_\ell^{(3)}\ell^6 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{f}{3!(4g)^3}(6\cdot 5\cdot 4)\ell^3 \\
&= -\frac{6!}{(3!)^2(4g)^3} f\ell^3 \\
&= -20\frac{f\ell^3}{(4g)^3}
\end{aligned}
$$

$V_f = 2, E = 5$

$$
\begin{aligned}
e^{-f\partial_\ell^{(3)}} e^{\ell^2/(4g)} &= \left(\cdots + \frac{1}{2!}f^2\partial_\ell^{(6)} + \cdots\right)\left(\cdots + \frac{1}{5!}\left(\frac{\ell^2}{4g}\right)^5 + \cdots\right) \\
&= -\frac{f^2}{2!5!(4g)^5}\partial_\ell^{(6)}\ell^{10} \\
&= -\frac{10!}{2!4!5!(4g)^5} f^2\ell^4 \\
&= 630\frac{f^2\ell^4}{(4g)^5}
\end{aligned}
$$

34. Diagrams and QED. Particle interactions are readily represented as diagrams. Here are six interactions found in QED (quantum electrodynamics) all repreented by the same diagram.



A fermion (electron, positron) is represented as a solid line with the arrow on the particle (electron) going forward in time and that on the antiparticle (positron) going backward in time—explicitly using Feynman's ideas quoted at the end of Note 31.

The wiggly line is the photon (massless boson).

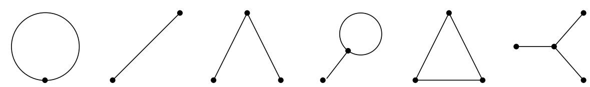
(The diagrams observe the direction of time but not the speed of light, or else no line could exceed 45 degrees. The above diagrams are simply 60-degree rotations of each other, which is cute but not essential.)

Diagram (a) shows an electron giving off a photon and thereby being deflected. Diagram (b) shows a photon striking an electron and so deflecting it (Compton effect). Diagram (c) shows a photon creating a positron-electron pair (pair creation). Diagram (d) is the complement to (b) but with a positron. Diagram (e) is similarly the complement to (a). Diagram (f) shows electron and positron annihilating to create a photon—the opposite of (c).

So it is worth our while to study simple diagrams a little. Let's start generally. Leonhard Euler related vertices V, edges E and faces F for any[1] diagram

$$V - E + F = 1$$

Let's see.



V 1	V 2	V 3	V 2	V 3	V 4
E 1	E 1	E 2	E 2	E 3	E 3
F 1	F 0	F 0	F 1	F 1	F 0

We can prove this relationship by induction. We suppose it is true for some existing figure, then we add something— vertex and an edge, or just an edge—and see that it is still true.

$$\begin{aligned}(V, E, F) + \text{vertex} + \text{edge} &= (V+1, E+1, F)\\ (V, E, F) + \text{edge} &= (V, E+1, F+1)\end{aligned}$$

where there is an extra face because the new edge has connected two existing vertices (or an existing vertex with itself).

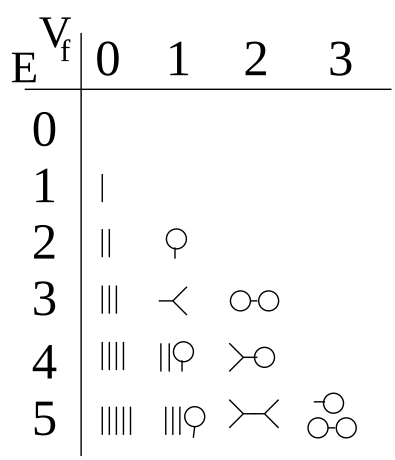
That's the induction step. Any of the diagrams shown above could serve as the start step, or just the diagram consisting of a sole vertex, $V = 1$.

Now let's specialize the kinds of vertices allowed. We'll permit two types: leaves—vertices terminating only one edge, counted by V_ℓ; and interior vertices with fixed "fanout", f, counted by V_f.

Since each edge is terminated by two vertices, and the f-vertices terminate f edges each, it should be apparent that

$$V_\ell + fV_f = 2E$$

We can make a table of possibilities when $f = 3$.



[1]but see the Excursions

We have seen this table before—in Note 33. It doesn't there have the diagrams, but the exponents of ℓ in the body of that table equal the number of leaf vertices in this one.

So the diagrams directly represent Gaussian integrals and can help us find which ones are important to calculate.

Note that one of these ($E = 3$, $V_f = 1$) is the QED diagram ($V_\ell = 3$) we started this Note with. Another ($E = 5$, $V_f = 2$, $V_\ell = 4$) is a combination of two of them and might represent an electron and a positron annihilating into a photon which then creates a new electron-positron pair.

Not every diagram in the above table makes sense physically. The diagrams that do help us calculate the corresponding integrals.

So far, though, the integrals are only suggestive. But we can see how we might put into the Lagrangian a cubic term which would give rise to the diagrams we started this Note with. If we consider the fermion wavefunction ψ of Note 27 (Part IV) to be a field, and the electromagnetic potential momentum A of Notes 5 (Part I), 13 (Part III), etc. to be the photon field, then a term $\bar{\psi}\psi A$ might be considered, say, to annihilate a positron and an electron and create a photon, or any other of the six processes given by the 3-way diagrams involving two fermions and a photon.

(Note, though, that the exponent is not the Lagrangian but the action, which is the time-integral of the Lagrangian or, for fields, the timespace integral of the Lagrangian density. That's where the functional slopes of Note 32 come in.)

Let's see if we can take part of this step. We must combine the photon field, with Lagrangian (I discuss it a little more in Note 36)

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the electron/positron field, with Lagrangian (Note 27 of Part IV)

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i \not{\partial} - m)\psi$$

The current, j_μ, must be a current of electrons (or positrons) and so must be described by the Dirac field. In Note 27 we had both aspects of the Dirac equation (with $\bar{\psi} = \psi^\dagger d_0$ and $\not{\partial} = \partial_\mu d^\mu$)

$$\begin{aligned} (i \not{\partial} - m)\psi &= 0 \\ (i \not{\partial} + m)\bar{\psi} &= 0 \end{aligned}$$

Pre- and post-multiplying by $\bar{\psi}$ and ψ respectively, then adding, cancels the m and gives

$$0 = i(\bar{\psi} \not{\partial}\psi + \psi \not{\partial}\bar{\psi}) = i\partial_\mu(\bar{\psi}d^\mu\psi)$$

This is a continuity equation and implies a current (see Note 29 of Part IV)

$$j^\mu = \bar{\psi}d^\mu\psi$$

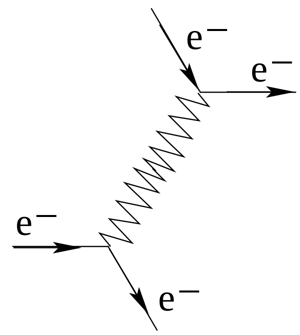
So the combined Lagrangian now has a cubic term like the x^3 term leading to the 3-way Feynman diagrams of this Note.

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i \not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}d^\mu\psi A_\mu$$

QED stands for quantum electrodynamics, the relativistic quantum field theory of electromagnetism, which is everything apart from nuclear physics and gravitation. QED Feynman diagrams, in the hands of professional physicists, provide the guide to the unprecedentedly accurate calculations that marked the breakthrough of quantum field theory.

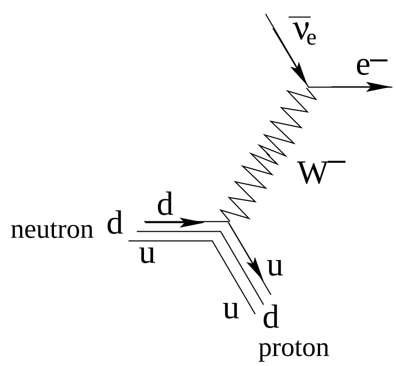
Apart from looking at “propagators” in Notes 36 and 37, we won’t attempt any of these calculations. They require much work and physical intuition based on close familiarity with thousands of particle experiments from many enormous accelerators.

However, we conclude this Note with a (qualitative) look at the Coulomb interaction of electrostatics from the point of view of quantum field theory. Here is the diagram.



35. Chirality and electroweak. The U(1) faze field of QED can be absorbed into a more general U(1)×SU(2) faze theory of the electromagnetic and weak forces. The kind of phenomenon we’d like to explain is “beta decay” or common radioactivity. Rutherford distinguished three types of radioactive emissions: alpha particles, which are helium nucleii of two protons and two neutrons; beta particles, which turned out to be electrons emitted by neutrons turning into protons; and gamma radiation, which turned out to be high-energy photons now called gamma rays.

Here is the field theory diagram for beta decay, which this Note will discuss.



Neutrons and protons consist of up-quarks and down-quarks, three each. A neutron becomes a proton by one of its d-quarks becoming a u-quark by emitting a W^- boson which then decays into an electron and an antineutrino. Note that, unlike the photon at the end of the previous Note, which is neutral, the W^- boson carries a (negative) charge, in order that electric charge be conserved at each vertex.

The neutrino is an elusive particle—no charge, almost no mass—which Pauli in desperation conjectured to account for missing momentum, and Fermi later named “little neutral one”. Neutrinos now play a decisive role in challenging the Standard Model of particle physics.

But we must start with the consideration, suggested by T D Lee and C N Yang (the Yang of WYMH) and checked experimentally by C S Wu and E Ambler, all in 1956, that the weak interaction does not conserve parity.

In the first paragraph, “U(1)×SU(2)” means that the Lagrangian will contain a sum of terms from both groups. Following Note 17 (Part III)

$$\mathcal{D}_\mu = \partial_\mu - ig_1\frac{Y}{2}B_\mu - ig_2\frac{f_j}{2}W_\mu^j$$

which we will expand into a Lagrangian below.

Also in the first paragraph. "parity" means symmetry between right hand and left hand, i.e., in mirror reflection. We'll take it as experimentally determined that all neutrinos are left-handed.[2] Conversely, all *anti*neutrinos are right-handed.

What do we mean by "handedness"? Since fermions such as electrons and neutrinos have spin 1/2 and so two directions of spin, we can ask if the spin is in the direction of motion of the particle, or in the opposite direction. (Use the right-hand rule: fingers curve along with spin, thumb gives "direction" of spin.) We call the particle right-handed if its spin and momentum point in the same direction, left-handed if the directions are opposite.

The Pauli matrices, $f_j, j = 1, 2, 3$ (from Note 19, Part IV), are each double the 2-by-2 representation of the generators of the rotation group, and so can be taken as the 3D components of the spin. Thus $\vec{f} = (f_1, f_2, f_3)$ is a vector and its dot product with the momentum $\vec{p}$ will be positive or negative if the particle is right-handed or left-handed respectively. If both $\vec{f}$ and $\vec{p}$ are normalized

$$\vec{f}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

is called the *helicity* and has the values

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

if diagonalized. This contains the two possible values, ± 1, of the helicity.

Recall that this matrix is a reflection, f, and that it produces two projections (see Note 19)

$$P_R = \frac{1}{2}(I+F) = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \qquad P_R = \frac{1}{2}(I-F) = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$$

$$P_R^2 = P_R \qquad P_L^2 = P_L \qquad P_R + P_L = I \qquad P_R P_L = 0$$

Unfortunately, helicity is not Lorentz-invariant: an observer moving faster than the particle sees its momentum, but not its spin, reversed, and so will disagree about the helicity with an observer who is slower than the particle.

So we need to find a matrix

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

or analogous to it, in our discussions of right-handed and left-handed particles, but which is Lorentz-invariant. The property whose values ± 1 are distinguished by this is called the *chirality* of the particle.

The whole situation is relativistic so we would seem to need a quantity related to the Dirac equation. It should be a reflection so that we can make a projection from it and so it should square to I. There are at least four reflections in the algebra, but each has its own meaning. The product of all four basic matrices does not square to I (see Note 22, Part IV)

$$(d_{0123})^2 = d_{01230123} = -d_{123123} = d_{2323} = d_{33} = -I$$

[2]The Standard Model infers from the left-handedness of all neutrinos that they have no mass. But recent experiments say they must have mass. The Standard Model has not been adjusted to accommodate right-handed neutrinos.

but id_{0123} squares to I.

However,

$$\begin{array}{rcl} id_{0123} & = & i(f_3 \overleftarrow{\times} I)(f_{31} \overleftarrow{\times} f_1)(f_{31} \overleftarrow{\times} f_2)(f_{31} \overleftarrow{\times} f_3) \\ & = & i f_{3313131} \overleftarrow{\times} f_{123} \\ & = & -i f_1 \overleftarrow{\times} f_{123} \\ & = & -i \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \overleftarrow{\times} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ & = & -i \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \overleftarrow{\times} \begin{pmatrix} i & \\ & i \end{pmatrix} \\ & = & \begin{pmatrix} & I \\ I & \end{pmatrix} \end{array}$$

is not diagonal, although at least it has no i.

Is there another representation, γ^μ, with $(\gamma^0)^2 = I$ and $(\gamma^j)^2 = -I$ and which anticommute, for the 4-by-4 matrices in 4D? We try (the *chiral* representation)

$$\begin{array}{ccc} \gamma^0 & \gamma^j & \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \\ f_1 \overleftarrow{\times} I & -i f_2 \overleftarrow{\times} f_j & f_3 \overleftarrow{\times} I \\ \begin{pmatrix} & I \\ I & \end{pmatrix} & \begin{pmatrix} & -f_j \\ f_j & \end{pmatrix} & \begin{pmatrix} I & \\ & -I \end{pmatrix} \end{array}$$

Let's solve Dirac's equation for free (plane wave) particles using this chiral representation.

$$\begin{array}{rcl} 0 & = & (i\gamma^\mu \partial_\mu - m)\psi \\ & = & (i\gamma^\mu \partial_\mu - m)u e^{-ip_\mu x^\mu} \\ & = & (\gamma^\mu p_\mu - m)u e^{-ip_\mu x^\mu} \\ & = & (\gamma^0 p_0 - \gamma^j p_j - m)\psi \\ & = & \begin{pmatrix} -m & p_0 + \vec{f}\cdot\vec{p} \\ p_0 - \vec{f}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \end{array}$$

where $\vec{f}$ is the vector (f_1, f_2, f_3) of 2-by-2 Pauli matrices, and ψ_R and ψ_L are the 2-component vectors (spinors) with labels R and L, of no particular significance but just to distinguish them.

But we notice the two opposite helicities, $\pm\vec{f}\cdot\vec{p}$, in the form, so it is plausible to consider R to mean right-handed and L to mean left-handed.

So we take γ^5 to distinguish right from left handed chirality and use the two projections based on it to map the field $\psi = (\psi_R, \psi_L)$ into its chiral components.

$$\begin{array}{rclc} P_R & = & \frac{1}{2}(I + \gamma^5) = \begin{pmatrix} I & \\ & 0 \end{pmatrix} & P_R\psi = \psi_R \\ P_L & = & \frac{1}{2}(I - \gamma^5) = \begin{pmatrix} 0 & \\ & I \end{pmatrix} & P_L\psi = \psi_L \end{array}$$

Because γ^5 anticommutes with $\gamma^\mu, \mu = 0, 1, 2, 3,$ we have

$$P_R\gamma^\mu = \frac{1}{2}(I + \gamma^5)\gamma^\mu = \gamma^\mu \frac{1}{2}(I - \gamma^5) = \gamma^\mu P_L$$

and similarly

$$P_L\gamma^\mu = \gamma^\mu P_R$$

and because $\bar{\psi} = \psi^\dagger\gamma^0$ (Note 27 in Part IV)

$$\bar{\psi}P_R = \psi^\dagger\gamma^0 P_R = \psi^\dagger P_L\gamma^0 = (P_L\psi)^\dagger\gamma^0 = \psi_L^\dagger\gamma^0 = \bar{\psi}_L$$

and similarly

$$\bar{\psi}P_L = \bar{\psi}_R$$

The fermion Lagrangian of Note 34 includes current and mass terms

$$\bar{\psi}\gamma^\mu\psi A_\mu \qquad \text{and} \qquad \bar{\psi}\psi m$$

respectively. Here I have switched entirely from the d^μ representation of Note 34 to the γ^μ representation of this Note: the formal properties are all that matter, and they are the same for d^μ and γ^μ. Using projection properties $P_R + P_L = I$ and $P_R P_L = 0$, let's see what the coefficients of A_μ and of m become under chiral projection.

Current

$$\begin{aligned}
\bar{\psi}\gamma^\mu\psi &= \bar{\psi}(P_R + P_L)\gamma^\mu(P_R + P_L)\psi \\
&= \bar{\psi}P_R\gamma^\mu P_R\psi + \bar{\psi}P_R\gamma^\mu P_L\psi + \bar{\psi}P_L\gamma^\mu P_R\psi + \bar{\psi}P_L\gamma^\mu P_L\psi \\
&= \bar{\psi}\gamma^\mu P_L P_R\psi + \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}\gamma^\mu P_R P_L\psi \\
&= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R
\end{aligned}$$

So the current coefficient couples fields of the same chiralities.

Given that neutrinos are left-handed only, this leaves only one term

$$\begin{aligned}
\bar{\psi}\gamma^\mu\psi &= \bar{\psi}_L\gamma^\mu\psi_L \\
&= \bar{\psi}P_R\gamma^\mu P_L\psi \\
&= \frac{1}{4}\bar{\psi}(1+\gamma^5)\gamma^\mu(1-\gamma^5)\psi \\
&= \frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma^5)\psi
\end{aligned}$$

which has two pieces: Lorentz 4-vector $\bar{\psi}\gamma^\mu\psi$ and Lorents pseudo- (or axial) vector $\bar{\psi}\gamma^\mu\gamma^5\psi$. The theory of beta-decay, which is a theory of the weak force, is sometimes referred to as V-A, or vector-axial, theory.

Mass

$$\begin{aligned}
\bar{\psi}\psi &= \bar{\psi}(P_R + P_L)\psi \\
&= \bar{\psi}(P_R^2 + P_L^2)\psi \\
&= \bar{\psi}P_R P_R\psi + \bar{\psi}P_L P_L\psi \\
&= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L
\end{aligned}$$

So the mass coefficient couples fields of different chiralities.

But neutrinos have only one chirality. So it is concluded that neutrinos are massless. (The observation that neutrinos do have mass thus contradicts the assumption of the Standard Model that there are only left-handed neutrinos. The right-handed neutrinos have not been detected yet.)

We are now ready to write the Lagrangian for the combined electromagnetic and weak forces. This combines U(1) and SU(2) parts and, as in Note 17 (Part III), we won't assume the U(1) field is the electromagnetic field, A_μ, because we're going to have to mix things up.

The following discussion gives in hindsight a long process of trial and error by some very smart people—Fermi, Feynman, Salam, Weinberg—and will have some twists and turns in it which I'll try to justify as we go along.

From Note 17 (Part III) the U(1) and SU(2) Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} = \frac{1}{4}\sum_{j=1}^{3} W^j_{\mu\nu}W^{j\mu\nu}$$

(summing over repeated indices) where the covariant slope

$$\mathcal{D}_\mu = \partial_\mu - ig_1\frac{Y}{2}B_\mu - ig_2\frac{f_j}{2}W^j_\mu$$

gives the interactions of forces B_μ and W^j_μ with the fermion fields, and, from Note 15 (Part III), the parts for the four boson fields, B_μ and W^j_μ, use

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ W^j_{\mu\nu} &= \partial_\mu W^j_\nu - \partial_\nu W^j_\mu - ig_2[W^j_\mu, W^j_\nu]_- \end{aligned}$$

(B_μ is commutative ("Abelian") but W^j_μ is not).

We will look at the interaction part of this Lagrangian, piece by piece: ψ represents the fields ν_L, e_L and e_R for left-handed neutrinos and electrons and right-handed electrons.

First, all the U(1) pieces together:

$$\mathcal{L} = \cdots + \frac{g_1}{2}(Y_L(\bar{\nu}_L\gamma^\mu\nu_L + \bar{e}_L\gamma^\mu e_L) + Y_R\bar{e}_R\gamma^\mu e_R)B_\mu + \cdots$$

and we use two different Y for left- and right-handed parts, in case they turn out not to be the same.

Second, all the SU(2) (only left-handed) pieces together:

$$\begin{aligned} \mathcal{L} &= \cdots + \frac{g_2}{2}(\bar{\nu}_L, \bar{e}_L)\begin{pmatrix} W^3_\mu & W^1_\mu - iW^2_\mu \\ W^1_\mu + iW^2_\mu & -W^3_\mu \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \cdots \\ &= \cdots + \frac{g_2}{2}(\bar{\nu}_L, \bar{e}_L)\begin{pmatrix} W^0_\mu & -\sqrt{2}W^+_\mu \\ -\sqrt{2}W^-_\mu & -W^0_\mu \end{pmatrix}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \cdots \end{aligned}$$

where the matrix of $W^j_\mu, j = 1, 2, 3$, comes from the sum $f_j W^j_\mu$ with f_j being the above reflection matrices, and where, to keep the expressions to single symbols, we've defined

$$\begin{aligned} W^\pm_\mu &= -W^1_\mu \pm W^2_\mu \\ W^0_\mu &= W^3_\mu \end{aligned}$$

We will be comparing these parts of the Lagrangian with the interaction of the electromagnetic field and fermions

$$\mathcal{L}_{\text{EM}} = \cdots - e(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R)A_\mu + \cdots$$

where e is the charge on the proton, and you should be able to avoid confusing e with the fields e_L and e_R because the latter are always subscripted.

Let's look first at the neutrino interactions from both U(1) and SU(2) parts.

$$\bar{\nu}_L \nu_L$$

$$\begin{aligned}
\frac{g_1}{2} Y_L B_\mu + \frac{g_2}{2} W_\mu^0 &= \frac{1}{2}(g_1 Y_L, g_2) \begin{pmatrix} B_\mu \\ W_\mu^0 \end{pmatrix} \\
&= \frac{1}{2}(g_1 Y_L, g_2) \frac{1}{\sqrt{D}} \begin{pmatrix} g_2 & g_1 Y_L \\ -g_1 Y_L & g_2 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\
&= \frac{1}{2\sqrt{D}} (0, (g_1 Y_L)^2 + g_2^2) \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\
&= \left(0, \frac{\sqrt{D}}{2}\right) \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}
\end{aligned}$$

This chain of equations is our first twist-and-turn. I said we would not assume the U(1) field B_μ to be the electromagnetic field A_μ, but A_μ must appear eventually, and furthermore, for neutrino interactions, must have no effect. The way we do this is to suppose another field, Z_μ, such that B_μ and W_μ^0 together are a linear combination of A_μ and Z_μ. That linear combination is given by the matrix above, carefully chosen so that, in the neutrino case, the coefficient of A_μ becomes 0. The matrix is chosen to be orthogonal—its inverse is its transpose—and its determinant is $D = (g_1 Y_L)^2 + g_2^2$.

All these considerations are indpendent of, and the same for, each $\mu = 0, 1, 2, 3$.

The result gives a force between neutrinos mediated by the neutral boson field Z_μ.

Now we must look at electron interactions, to bring in the electromagnetic force. In parallel:

$$\bar{e}_L e_L$$

$$\begin{aligned}
&\frac{g_1}{2} Y_L B_\mu - \frac{g_2}{2} W_\mu^0 \\
&= \frac{1}{2}(g_1 Y_L, -g_2) \begin{pmatrix} B_\mu \\ W_\mu^0 \end{pmatrix} \\
&= \frac{1}{2}(g_1 Y_L, -g_2) \frac{1}{\sqrt{D}} \begin{pmatrix} g_2 & g_1 Y_L \\ -g_1 Y_L & g_2 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\
&= \frac{1}{2\sqrt{D}} (2 g_1 g_2 Y_L, (g_1 Y_L)^2 - g_2^2)) \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}
\end{aligned}$$

$$\bar{e}_R e_R$$

$$\begin{aligned}
&\frac{g_1}{2} Y_R B_\mu \\
&= \frac{1}{2}(g_1 Y_R, 0) \begin{pmatrix} B_\mu \\ W_\mu^0 \end{pmatrix} \\
&= \frac{1}{2}(g_1 Y_R, 0) \frac{1}{\sqrt{D}} \begin{pmatrix} g_2 & g_1 Y_L \\ -g_1 Y_L & g_2 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\
&= \frac{1}{2\sqrt{D}} (g_1 g_2 Y_R, g_1^2 Y_L Y_R) \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}
\end{aligned}$$

Comparing both of these with $\mathcal{L}_{EM}$

$$(-e, 0) \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}$$

we must have

$$\begin{aligned}
Y_R &= 2 Y_L \\
-e &= \frac{g_2 (g_1 Y_L)}{\sqrt{(g_1 Y_L)^2 + g_2^2}}
\end{aligned}$$

Next twist-and-turn: we set $Y_L = -1$. We can do this because Y_L appears only multiplied by g_1, and g_1 is the constant parameter giving the interaction strength: we don't need a parameter Y_L too.

So

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} = g_1 \cos\theta_W = g_2 \sin\theta_W$$

That's the next twist-and-turn: θ_W is called the electroweak mixing angle or the Weinberg angle, and gives the relative strength, g_1 vs. g_2, of the electromagnetic and weak forces. You'll see the angle if you draw a right triangle of base g_2 and height g_1. We'll also use rearrangements (abbreviating $\cos(\theta_W)$ as c_W and $\sin(\theta_W)$ as s_W):

$$\begin{aligned} g_1 &= e/c_W & \qquad g_2 &= e/s_W \\ \frac{g_1^2}{\sqrt{D}} &= g_1 s_W = e\frac{s_W}{c_W} & \qquad \frac{g_2^2}{\sqrt{D}} &= g_2 c_W = e\frac{c_W}{s_W} \end{aligned}$$

Back to $\bar{e}_L e_L$ and $\bar{e}_R e_R$, in parallel:

$\bar{e}_L e_L$

$$\begin{aligned} \frac{1}{2\sqrt{D}} & (- \quad 2g_1 g_2, g_1^2 - g_2^2)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\ &= -e\left(1, \frac{c_{2W}}{s_{2W}}\right)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\ &= -e\left(1, \frac{1}{c_W s_W}\left(\frac{1}{2} - s_W^2\right)\right)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \end{aligned}$$

$\bar{e}_R e_R$

$$\begin{aligned} \frac{1}{2\sqrt{D}} & (- \quad 2g_1 g_2, 2g_1^2)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\ &= -e\left(1, \frac{s_W}{c_W}\right)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \\ &= -e\left(1, \frac{-s_w^2}{c_W s_W}\right)\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \end{aligned}$$

using $c_{2W} = \cos(2\theta_W) = c_W^2 - s_W^2$ and $s_{2W} = \sin(2\theta_W) = 2c_W s_W$ and other games with cos and sin.

The last of these two can be combined into an expression which generalizes to u and d quarks, of charges 2/3 and −1/3 respectively:

$$\frac{e}{\cos\theta_W \sin\theta_W}\left(\frac{1}{2}F_3 - Q_f \sin^2\theta_W\right)$$

where Q_f is electric charge of the fermion, in units of e, and F_3 is the upper or lower eigenvalue of the f_z reflection matrix:

	singlets			doublets			
	e_R	u_R	d_R	e_L	ν_L	u_L	d_L
F_3	0	0	0	−1	1	1	−1
Q_f	−1	2/3	−1/3	−1	0	2/3	−1/3

The Q_f can be related to the "hypercharge" of Note 34 (Part IV) of Book 8c.

The electroweak theory for quarks parallels the above discussion exactly so I won't elaborate on it.

Finally we look at the cross terms coupling neutrinos and electrons.

$$\bar{\nu}_L e_l : -\frac{g_2}{\sqrt{2}}W_\mu^+ \qquad \bar{e}_L \nu_l : -\frac{g_2}{\sqrt{2}}W_\mu^-$$

These lead to the beta-decay diagram we started this Note with.

We have progressed from the boson fields we originally constructed our Lagrangian with, $B_\mu, W_\mu^1, W_\mu^2, W_\mu^2$, to the linearly related physical force fields, A_μ and Z_μ, which are neutral, and W_μ^+ and W_μ^-, which are charged force carriers.

We have used diagrams as intuitive pictures of particle interactions. But quantum field theory thinks in terms not of particles but of fields. The diagrams are also guides to the terms in the functional integral of the action, which is itself an integral of the Lagrangian density over all timespace. We can label leaves, vertices and edges of a diagram to relate it to the integrals to be calculated. The leaves are labelled with the field functions for the ultimate incoming and outgoing "particles". The internal vertices, of fanout f, are labelled with the interaction terms we've been describing for QED and electroweak theories in the last two Notes: f is given by the number of interacting fields and the label is the corresponding term omitting the fields themselves. For internal edges the labels are the propagators to be discussed in the next two Notes.

36. Green's functions. An important form of slope equation is

$$\mathcal{D}_x\phi(x) = J(x)$$

where $\mathcal{D}_x$ is a slope *operator* such as $\mathcal{D}_x = \partial_x$ or $\mathcal{D}_x = \partial_x^2 + m^2$, etc.

This might remind us of a matrix equation

$$Av = J$$

which has the solution

$$v = A^{-1}J$$

provided that the inverse, A^{-1}, can be found. That inverse has the property $A^{-1}A = I$ or

$$A^{-1}A = (\delta_{jk})$$

using, for the identity matrix I, the Kronecker delta $\delta_{jk} = 1$ *if* $j = k$ *else* 0.

The "inverse" of a slope operator is called its Green's function, G, and has the analogous property

$$\mathcal{D}_x G(x) = \delta(x)$$

or

$$\mathcal{D}_x G(x - x') = \delta(x - x')$$

using the Dirac delta function (see Note 32).

Thus

$$\phi(x) = \int dx' G(x - x')J(x')$$

because

$$\begin{aligned}
\mathcal{D}_x\phi(x) &= \mathcal{D}_x \int dx' G(x - x')J(x') \\
&= \int dx' \mathcal{D}_x G(x - x')J(x') \\
&= \int dx' \delta(x - x')J(x') \\
&= J(x)
\end{aligned}$$

We can modify this to include initial conditions $\phi_0(x)$ if this is a solution to the "homogenous equation" $\mathcal{D}_x\phi_0(x) = 0$:

$$\phi(x) = \phi_0(x) + \int dx' G(x - x')J(x')$$

(What is the analogous matrix solution?)

So far, this is an impractical formalism. How do we figure out whar $G()$ is?

Try the Fourier transform

$$G(x - x') = \int \frac{dk}{2\pi} e^{-ik(x-x')}G(k)$$

(See Notes 1 and 2 of Week 9, and the Excursion *Continuous FT* in Week 9: in particular note that

$$\frac{1}{L}\sum_k e^{-i(j-j')k} = \delta_{jj'}$$

extends to

$$\int \frac{dk}{2\pi} e^{-i(j-j')k} = \delta(j-j')$$

where I've absorbed the $2\pi/L$ of the exponent into k or into the js and where the discrete length L becomes 2π in the continuous case).

Then

$$\begin{aligned} \mathcal{D}_x G(x-x') &= \mathcal{D}_x \int \frac{dk}{2\pi} e^{-ik(x-x')} G(k) \\ &= \int \frac{dk}{2\pi} \mathcal{D}_x e^{-ik(x-x')} G(k) \end{aligned}$$

on one side, and

$$\delta(x-x') = \int \frac{dk}{2\pi} e^{-ik(x-x')}$$

on the other side.

Thus

$$\mathcal{D}_x e^{-ik(x-x')} G(k) = e^{-ik(x-x')}$$

or

$$G(k) = \frac{e^{-ik(x-x')}}{\mathcal{D}_x e^{-ik(x-x')}}$$

Let's see with $\mathcal{D}_x = \partial_x^2 + m^2$, for instance.

$$\begin{aligned} \partial_x^2 e^{-ik(x-x')} &= \partial_x(-ike^{-ik(x-x')}) \\ &= -k^2 e^{-ik(x-x')} \end{aligned}$$

So, for $\mathcal{D}_x = \partial_x^2 + m^2$,

$$G(k) = \frac{1}{-k^2 + m^2}$$

A small change of direction for the above discussion: physicists find it more convenient to have

$$G(k) = \frac{1}{k^2 - m^2}$$

so we go back to the beginning and redefine

$$\mathcal{D}_x G(x-x') = -\delta(x-x')$$

All this extends to the Klein-Gordon equation in four (Minkowski) dimensions.

$$(\not\partial_x \cdot \not\partial_x + m^2)\phi(x) = J(x)$$

so

$$(\not\partial_x \cdot \not\partial_x + m^2) G(x-x') = -\delta(x-x')$$

where x now includes t, x, y, z and the subscript x on $\not\partial_x \cdot \not\partial_x = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ indicates slopes with respect to t, x, y, z rather than t', x', y', z'.

Since

$$\begin{aligned} \not\partial_x \cdot \not\partial_x e^{-i\not{k}\cdot\not{x}} &= (\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2) e^{-i(k_t t - k_x x - k - yy - k_z z)} \\ &= (-k_t^2 + k_x^2 + k_y^2 + k_z^2) e^{-i(k_t t - k_x x - k - yy - k_z z)} \\ &= (-k_t^2 + \bcancel{k} \cdot \bcancel{k}) e^{-i\not{k}\cdot\not{x}} \end{aligned}$$

thus

$$G(k) = \frac{1}{k_t^2 - (\vec{k} \cdot \vec{k} + m^2)} = \frac{1}{k_t^2 - E^2}$$

using $E^2 - p^2 = m^2$ from special relativity (with $c = 1$ and $\hbar = 1$ in $E^2 - (\hbar k c)^2 = (mc^2)^2$).
We want to find (now in four dimensions)

$$\begin{aligned} G(x - x') &= \int \frac{d^4k}{(2\pi)^4} e^{-\vec{k} \cdot (\vec{x} - \vec{x}')} G(k) \\ &= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk}{2\pi} e^{-\vec{k} \cdot (\vec{x} - \vec{x}')} G(k) \end{aligned}$$

Since

$$\frac{1}{k_t - E} - \frac{1}{k_t + E} = \frac{k_t + E - (k_t - E)}{k_t^2 - E^2} = \frac{2E}{k_t^2 - E^2}$$

$G(x - x')$ can be linearized to (I'm showing E explicitly as a function, E_k, of k)

$$\frac{1}{2E_k}\left(\frac{1}{k_t - E_k} - \frac{1}{k_t + E_k}\right)$$

Given

$$\int \frac{dk}{2\pi} e^{-ikt} = \delta(t)$$

what can we make of

$$\int \frac{dk}{2\pi} \frac{e^{-ikt}}{k} \quad ?$$

Let's find the slope with respect to t of the second.

$$\begin{aligned} \partial_t \int \frac{dk}{2\pi} \frac{e^{-ikt}}{k} &= \int \frac{dk}{2\pi k} \partial_t e^{-ikt} \\ &= -i \int \frac{dk}{2\pi} e^{-ikt} \\ &= -i\delta(t) \end{aligned}$$

What is $\delta(t)$ the slope of? Try the *Heaviside step function*

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

(and, if needed, we can include $\theta(0) = 1/2$ so that $\theta(t) + \theta(-t) = 1$).
Then

$$\int \frac{dk}{2\pi} \frac{e^{-ikt}}{k} = -i\theta(t)$$

and

$$\begin{aligned} \int \frac{dk_t}{2\pi} \frac{1}{2E_k} \frac{1}{k_t - E_k} e^{-ik_t(t-t')} &= \frac{1}{2E_k} \int \frac{dq}{2\pi q} e^{-(q+E_k)(t-t')} \\ &= -i\theta(t - t') \frac{e^{-E_k(t-t')}}{2E_k} \end{aligned}$$

while

$$
\begin{aligned}
\int \frac{dk_t}{2\pi}\frac{1}{2E_k}\frac{1}{k_t+E_k}e^{-ik_t(t-t')} &= -\frac{1}{2E_k}\int \frac{dk_t}{2\pi}\frac{1}{-k_t-E_k}e^{-ik_t(t-t')} \\
&= -\frac{1}{2E_k}\int \frac{dq}{2\pi q}e^{i(q+E_k)(t-t')} \\
&= -\frac{1}{2E_k}\int \frac{dq}{2\pi q}e^{-i(q+E_k)(t'-t)} \\
&= i\theta(t'-t)\frac{e^{-iE_k(t'-t)}}{2E_k}
\end{aligned}
$$

In the above, first $q = k_t - E_k$, then, in the second term, $q = -k_t - E_k$. There are other ways to do this integral as written, but I have glossed over an important subtlety.

We cannot integrate the Green's function $1/(k_t^2 - E_k^2)$ because of the pole at $k_t = E_k$—and, for conservation of energentum $k_t = E_k$ for all particles that we can actually observe, even if not necessarily for "virtual" particles that pop in and out of existence during any physical process.

So Feynman added a small imaginary term, $i\varepsilon$, which we can slip into the E_k term: $E_k \longrightarrow E_k + i\varepsilon$. Thus we are integrating

$$
\frac{1}{k_t^2-(E_k-i\varepsilon)^2} = \frac{1}{2E_k}\left(\frac{1}{k_t-E_k+i\varepsilon}+\frac{1}{-(k_t-E_k)+i\varepsilon}\right)
$$

and we chose the sign of q to agree with that of $i\varepsilon$ both times because the steps leading to $\theta(t)$ must be refined to include the 2-number plane. (Why can we be sloppy about the exact form of ε?)

Now we complete the integration.

$$
\begin{aligned}
\int \frac{d^3k}{(2\pi)^3} &\int \frac{dk_t}{2\pi}e^{-ik_t(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}G(k) \\
&= \int \frac{d^3k}{(2\pi)^3 2E_k}\left(-i\theta(t-t')e^{-iE_k(t-t')}+i\theta(t'-t)e^{-iE_k(t'-t)}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= -i\int \frac{d^3k}{(2\pi)^3 2E_k}\theta(t-t')e^{-iE_k(t-t')}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&\quad -i\int \frac{d^3k}{(2\pi)^3 2E_k}\theta(t'-t)e^{-iE_k(t'-t)}e^{-i\vec{k}\cdot(\vec{x}'-\vec{x})} \\
&= -i\int \frac{d^3k}{(2\pi)^3 2E_k}\left(\theta(t-t')e^{-i\not k(\not x-\not x')}+\theta(t'-t)e^{i\not k(\not x-\not x')}\right)
\end{aligned}
$$

In the third step we changed the sign on k, the 3D integration variable, thus changing the sign on d^3k. In the fourth step we recombined time and space into 4D timespace.

Note that this expression is *time-ordered*: it $t > t'$ the first term applies; if $t < t'$ the second applies; and if $t = t'$ we use both, equally weighted by 1/2.

Or we can speak of particles going forward in time and antiparticles going backward.

This Green's function, extended into the 2D number plane for the Klein-Gordon equation, is in a form which we can relate, in the next Note, to *propagators*.

But first we look at two more examples of Green's functions.

The Dirac equation (see Note 27 in Part IV) for fermions is

$$
(i\not\partial - m)\psi(x) = J(x)
$$

and its Green's function must satisfy(in 4 dimensions)

$$(i\not{\partial} - m)G(x - x') = \delta^4(x - x')$$

Taking the Fourier transform and going through the above steps

$$(\not{k} - m)G(k) = 1$$

which we cannot invert immediately because k is a (4-by-4) matrix, not a number. But we can get a number to invert by multiplying both sides by $\not{k} + m$.

$$(k^2 - m^2)G(k) = \not{k} + m$$

So

$$G(k) = \frac{\not{k} + m}{k^2 - m^2}$$

or, to avoid poles,

$$G(k) = \frac{\not{k} + m}{k^2 - m^2 + i\varepsilon}$$

(We can be a little sloppy about where we put the ε since it ultimately goes to zero.)

This will become

$$(i\not{\partial} + m)G_{\mathrm{KG}}(x - x')$$

where G_{KG} is the Klein-Gordon Green's function derived above.

The third example of Green's functions is the photon field. Klein-Gordon gives fields of spin 0, Dirac gives fields of spin 1/2, and Maxwell gives fields of spin 1.

We'll pick up on Notes 5 (Part I) and 15 (Part III) where Maxwell's equations wound up as (see Excursion *Maxwell's tensor and equations*)

$$\partial_\mu F^{\mu\nu} = j^\nu$$

with

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Here we use "Heaviside-Lorentz" units in which the $4\pi E_C q$ are all absorbed into the charge/current density j^ν.

And we resort to co- and contra-variant sub-and super-scripts because the slash notation, which works well for vectors, seems unwieldy for the tensor $F^{\mu\nu}$. All we need remember is that moving μ or ν up or down changes the sign on any component for which either μ or ν is 0. (The sign changes twice if both are 0, but then $F^{00} = 0$, as do F^{11}, F^{22} and F^{33}, so there is no issue.) That is, in the Excursion, $E^j = -E_j$.

We can rearrange

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \partial^\mu g^{\nu\lambda} A_\lambda - \partial^\nu \partial_\mu A^\mu \\
&= \partial_\mu \partial^\mu g^{\nu\lambda} A_\lambda - \partial^\nu \partial^\lambda A_\lambda \\
&= (\partial_\mu \partial^\mu g^{\nu\lambda} - \partial^\nu \partial^\lambda) A_\lambda \\
&= j^\nu
\end{aligned}$$

so that the Green's function is the inverse of the operator on A_λ

$$(\partial_\mu \partial^\mu g^{\nu\lambda} - \partial^\nu \partial^\lambda) G_{\nu\alpha}(x - x') = g^\lambda_\alpha \delta(x - x')$$

But note that this is a matrix operator as well as a slope operator. And note that A_λ does not have four indepndent components, but can be restricted in various ways without changing the physics it describes.

So we will not be able to find the inverse of the operator

$$\partial_\mu \partial^\mu g^{\nu\lambda} - \partial^\nu \partial^\lambda$$

We must go back to the Lagrangian that gives rise to the equation

$$\partial_\mu F^{\mu\nu} = j^\nu$$

and modify it in some way that does not alter the physics but includes the constraints on A_λ so that the resulting equations *can* be inverted.

Since $\partial_\mu A^\mu = 0$ (the "Lorentz gauge") is one way—a Lorentz-invariant way, since $\partial_\mu A^\mu$ is a Lorentz scalar—of constraining A_λ, we'll try including it in the Lagrangian.

The resulting Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu - \frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

all of which we must justify by working out from it the equations of motion, using Euler-Lagrange

$$\partial_\mu \partial_{\partial_\mu A_\nu} \mathcal{L} = \partial_{A_\nu} \mathcal{L}$$

Let's try operating on each term of $\mathcal{L}$ with the appropriate side of Euler-Lagrange.

$$\begin{aligned}
\partial_{\partial_\mu A_\nu} \mathcal{L}^{(1)} &= -\frac{1}{4} \partial_{\partial_\mu A_\nu} F_{\rho\sigma} F^{\rho\sigma} \\
&= -\frac{1}{4} \partial_{\partial_\mu A_\nu} (\cdots + F_{\mu\nu} F^{\mu\nu} + \cdots + F_{\nu\mu} F^{\nu\mu} + \cdots) \\
&= -\frac{1}{4} (F^{\mu\nu} \pm F_{\mu\nu} - F^{\nu\mu} \mp F_{\nu\mu}) \\
&= -\frac{1}{4} (F^{\mu\nu} + F^{\mu\nu} + F^{\mu\nu} + F^{\mu\nu}) \\
&= -F^{\mu\nu}
\end{aligned}$$

where we must carefully use $\partial_{\partial_\mu A_\nu} F_{\mu\nu} = 1$, $\partial_{\partial_\mu A_\nu} F_{\nu\mu} = -1$ and then check signs again, depending on whether or not μ or ν are 0. (In the second line, the indices in $F_{\mu\nu} F^{\mu\nu}$ and $F_{\nu\mu} F^{\nu\mu}$ are *not* summed.)

So the first term is

$$-\partial_\mu F^{\mu\nu}$$

The second term is

$$\begin{aligned}
\partial_{A_\nu} \mathcal{L}^{(2)} &= -\partial_{A_\nu} j^\mu A_\mu \\
&= -\partial_{A_\nu} (\cdots + j^\nu A_\nu + \cdots) \\
&= -j^\nu
\end{aligned}$$

And the first two terms, without the correction of the third, give Maxwell's equations as we had originally

$$-\partial_\mu F^{\mu\nu} = -j^\nu$$

Now the new term

$$\begin{aligned}
\partial_\mu \partial_{\partial_\mu A_\nu} \mathcal{L}^{(3)} &= -\frac{1}{2\xi} \partial_\mu \partial_{\partial_\mu A_\nu} (\partial_\lambda A^\lambda)^2 \\
&= -\frac{1}{\xi} \partial_\mu (\partial_\lambda A^\lambda) \partial_{\partial_\mu A_\nu} (\partial_\lambda A^\lambda) \\
&= -\frac{1}{\xi} \partial_\nu (\partial_\lambda A^\lambda)(\pm 1) \\
&= -\frac{1}{\xi} \partial^\nu (\partial_\lambda A^\lambda) \\
&= -\frac{1}{\xi} \partial^\nu (\partial^\lambda A_\lambda)
\end{aligned}$$

where again we must watch signs in the third step when we narrow the sum over μ to the one $\mu = \nu$ that gives a nonzero slope; but raising the ν in the fourth step counteracts the sign issue.

Comparing this with Maxwell's equations without the "gauge-fixing" term in the Lagrangian

$$(g^{\nu\lambda} \partial_\mu \partial^\mu - \partial^\nu \partial^\lambda) A_\lambda = j^\nu$$

We now have

$$\left(g^{\nu\lambda} \partial_\mu \partial^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\nu \partial^\lambda \right) A_\lambda = j^\nu$$

The Green's function now exists:

$$\left(g^{\nu\lambda} \partial_\mu \partial^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\nu \partial^\lambda \right) G_{\nu\alpha}(x - x') = g^\lambda_\alpha \delta(x - x')$$

giving, via Fourier transform

$$-\left(g^{\nu\lambda} k^2 - \left(1 - \frac{1}{\xi}\right) k^\nu k^\lambda \right) G_{\nu\alpha}(k) = g^\lambda_\alpha$$

Since $G_{\nu\alpha}(k)$ is a matrix depending only on k we must have, for coefficients a and b to be determined

$$G_{\nu\alpha}(k) = a g_{\nu\alpha} + b k_\nu k_\alpha$$

So

$$\begin{aligned}
g^\lambda_\alpha &= -\left(g^{\nu\lambda} k^2 - \left(1 - \frac{1}{\xi}\right) k^\nu k^\lambda \right) (a g_{\nu\alpha} + b k_\nu k_\alpha) \\
&= -\left(a k^2 g^\lambda_\alpha - a \left(1 - \frac{1}{\xi}\right) k_\alpha k^\lambda + b k^2 k^\lambda k_\alpha - b \left(1 - \frac{1}{\xi}\right) k^2 k^\lambda k_\alpha \right) \\
&= -\left(a k^2 g^\lambda_\alpha - \left(a \left(1 - \frac{1}{\xi}\right) - \frac{b k^2}{\xi} \right) k^\lambda k_\alpha \right)
\end{aligned}$$

Equating coefficients of the matrices g^λ_α and $k^\lambda k_\alpha$

$$1 = -a k^2$$

so

$$a = -1/k^2$$

and

$$0 = a\left(1 - \frac{1}{\xi}\right) - \frac{bk^2}{\xi}$$

so

$$b = \frac{a\xi}{k^2}\left(1 - \frac{1}{\xi}\right) = \frac{1-\xi}{k^4}$$

So finally we have the Green's function for the spin-1 photon field $A_\lambda(k)$

$$G_{\nu\alpha}(k) = -\frac{1}{k^2}\left(g_{\nu\alpha} - (1-\xi)\frac{k_\nu k_\alpha}{k^2}\right)$$

(and we would replace the outside $1/k^2$ by $1/(k^2 + i\varepsilon)$ to avoid the singularity).

Note that $\xi = \infty$, which would eliminate the gauge-fixing term $\partial^\mu \partial^\lambda A_\lambda/\xi$, messes this up.

However, apart from that, ξ is an arbitrary parameter and we can set it to whatever we like. Feynman and 't Hooft chose $\xi = 1$, which removes a whole term from the Maxwell equations.

Recall from Note 17 that "gauge" is the conventional term for faze theory, which we have had to invoke to find the photon field.

37. Propagators. The Green's functions in the previous Note have physical interpretation as *propagators*. For example, compare the scalar field Green's function

$$iG(x - x') = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(\theta(t-t')e^{-i\not{k}(\not{x}-\not{x}')} + \theta(t'-t)e^{i\not{k}(\not{x}-\not{x}')}\right)$$

with the Fourier-transformed simple field of Note 24 (in Part IV)

$$\phi_k = \frac{1}{\sqrt{L^3}}\sum_k \frac{1}{\sqrt{2\omega_k}}\left(e^{ik\cdot\ell 2\pi/L}U_k + e^{-ik\cdot\ell 2\pi/L}D_k\right)$$

which we make continuous ($L \to 2\pi, \ell \to x$), recognize $\omega_k = E_k$ the energy, and redefine slightly so that instead of $1/\sqrt{(2\pi)^3 2E_k}$ for each direction of the Fourier transform we have $1/((2\pi)^3 2E_k)$ for the k-to-x transform but just 1 for the x-to-k transform.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_k}\left(e^{i\not{k}\;\not{x}}\;U_k + e^{-i\not{k}\;\not{x}}\;D_k\right)$$

To connect these we consider the expression

$$< 0 \mid \phi(x')\phi(x) \mid 0 >$$

where $< 0 \mid$ and $\mid 0 >$ represent the "vacuum: state—the state with no excitations. This expression can be interpreted (from right to left): start in vacuum ($\mid 0 >$), create a particle at x ($\phi(x)$), annihilate the particle at x' ($\phi(x')$), resulting again in vacuum ($< 0 \mid$).

The "disappear operator can do nothing to the vacuum

$$D_k \mid 0 >= 0 \mid 0 >$$

but the "uppear" operator can create a particle of momentum k

$$U_k \mid 0 >=\mid k >$$

a new way of writing $\mid 0\cdots 010\cdots 0>$ with the 1 in the kth position: this notation works if there is only one particle.

The conjugates are the reverse

$$\begin{aligned} <0 \mid U_k &= 0 \\ <0 \mid D_k &= <k \mid \end{aligned}$$

Thus

$$\phi(x) \mid 0 >= \int \frac{d^3k}{(2\pi)^3 2E_k} e^{i\mathbf{k}\cdot\mathbf{x}} \mid k >$$

and

$$<0 \mid \phi(x') = \int \frac{d^3k'}{(2\pi)^3 2E_{k'}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} < k' \mid$$

and we can put them together, remembering the orthonormality of momentum states

$$< k' \mid k >= \delta(k-k')$$

to get

$$<0 \mid \phi(x')\phi(x) \mid 0 >= \int \frac{d^3k}{(2\pi)^3 2E_k} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

Thus

$$iG(x-x') = \theta(t-t') < 0 \mid \phi(x)\phi(x') \mid 0 > + \theta(t'-t) < 0 \mid \phi(x')\phi(x) \mid 0 >$$

and we see the opposite directions of antiparticles, travelling backwards in time, from particles, travelling forwards in time, as shown in the diagrams at the beginning of Note 34.

So Green's functions are propagators, and this particular Green's function is called the *Feynman propagator*

$$G(x-x') = \Delta_F(x-x')$$

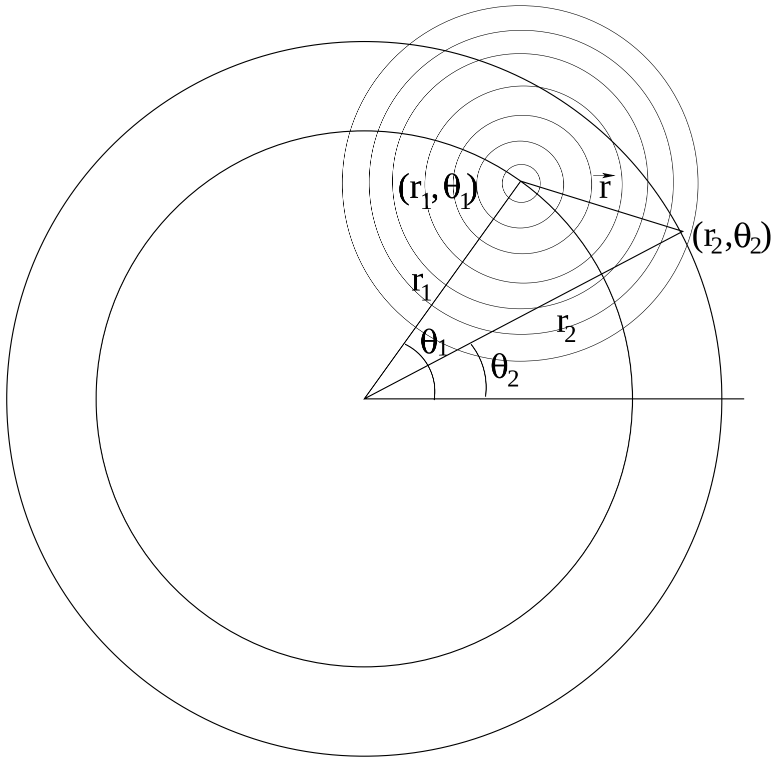
We saw in Note 36 that the Feynman propagator appears in both spin-0 and spin-1/2 fields. And $\Delta_F(x-x')$ appears, unidentified, in Note 32.

. . .

In 1949 Feynman [Fey49] wrote a propagator (Green's function) equation

$$\psi(x_2,t_2) = \int d^3x_1 K(x_2,t_2;x_1,t_1)\psi(x_1,t_1)$$

The idea goes back as far as Christiaan Huygens who in 1678 proposed that every point of a wavefront of light can be considered as a source of a new spherical wave. Let's explore this briefly in the simplest case of a light wave propagating from a spherical front to a spherical front.



First, the amplitude for a lightwave a distance r from a source, of amplitude 1, at the origin is

$$\frac{e^{i\vec{k}\cdot\vec{r}}}{r}$$

where $\vec{r}$ is the vector (r, θ) which we'll write in Cartesian coordinates as (rc, rs) with $c = \cos\theta$, $s = \sin\theta$ as usual, and with magnitude r.

We'll take $k_x = k = k_y$ in $\vec{k} = (k_x, k_y)$: light is propagated isotropically (i.e., the same in all directions).

From this the intensity falls off, as it should, as

$$\frac{e^{i\vec{k}\cdot\vec{r}}}{r}\frac{e^{-i\vec{k}\cdot\vec{r}}}{r} = \frac{1}{r^2}$$

Now let's explore the propagation of a ray of light from (r_1c_1, r_1s_1) on wavefront of radius r_1, to (r_2c_2, r_2s_2) on what we know will in the end be a wavefront of radius r_2.

$$\begin{aligned}\vec{r} &= \vec{r_2} - \vec{r_1} \\ &= (r_2c_2, r_2s_2) - (r_1c_1, r_1s_1) \\ &= (r_2c_2 - r_1c_1, r_2s_2 - r_1s_1)\end{aligned}$$

and note that

$$\begin{aligned}r &= \sqrt{(r_2c_2 - r_1c_1)^2 + (r_2s_2 - r_1s_1)^2} \\ &= \sqrt{r_2^2 + r_1^2 - 2r_2r_1c_-}\end{aligned}$$

with $c_j = \cos\theta_j$, $s_j = \sin\theta_j$ and $c_- = \cos(\theta_2 - \theta_1) = c_2c_1 + s_2s_1$.

To find the total effect at (r_2, θ_2) of *all* the point sources on the wavefront of radius r_1, we must integrate over all θ_1 from 0 to 2π.

$$\int_0^{2\pi} d\theta_1 \frac{e^{i\vec{k}\cdot\vec{r}}}{r}\frac{e^{i\vec{k}\cdot\vec{r_1}}}{r_1}$$

I can't do this integral mathematically, but we know the answer from Huygen's principle

$$\frac{e^{i\vec{k}\cdot\vec{r_2}}}{r_2} = \int_0^{2\pi} d\theta_1 \frac{e^{i\vec{k}\cdot\vec{r}}}{r}\frac{e^{i\vec{k}\cdot\vec{r_1}}}{r_1}$$

Compare this with Feynman's 1949 propagator equation.

Note that the positions 1 and 2 are arbitrary.We can equally step from, say, position 0.

$$\psi(x_1, t_1) = \int d^3x_0 K(x_1, t_1; x_0, t_0)\psi(x_0, t_0)$$

So two or more propagators can be composed into a single propagator.

$$\psi(x_2, t_2) = \int\int d^3x_1 d^3x_0 K(x_2, t_2; x_1, t_1)K(x_1, t_1; x_0, t_0)\psi(x_0, t_0)$$

I have egregiously warped this discussion and the Huygens integral above is incorrect. See Excursion *Huygens 3D*.

38. Quantum Computing.
39. Binary Fourier transform.
40. Quantum Fourier transform.
41. Finding periods.
42. Quantum key distribution.
43. No cloning.
44. Database search.
45. Detecting and correcting errors.
46. Nonlocality: Einstein-Podolsky-Rosen.
47. Building a quantum computer.

II. **The Excursions**

You've seen lots of ideas. Now *do* something with them!

1. Look up Feynman's 1942 thesis referenced in Note 31 or his summary paper in *Reviews of Modern Physics* 1948 [Fey48], both available in [Bro05].

2. The constant-force Lagrangian calculation of Note 31 is an exercise in [FH65, p.28].
   If we know x_b and the time T when the particle reaches it, but not the initial velocity v_0, we can find v_0:

$$\begin{aligned} x_b &= \frac{1}{2}aT^2 + v_0T + x_a \\ v_0 &= \frac{x_b - x_a}{T} - \frac{1}{2}aT \end{aligned}$$

With these we can find the action (relative to mass m as in Note 31):

$$
\begin{aligned}
\frac{S}{m} &= \int_0^T dt \frac{L}{m} \\
&= \int_0^T dt \left(\frac{1}{2}(at+v_0)^2 + \frac{1}{2}a^2t^2 + av_0t + ax_a \right) \\
&= \frac{1}{3}a^2T^3 + av_0T^2 + \frac{1}{2}v_0^2T^2 + ax_aT \\
&= \frac{1}{3}a^2T^3 + a(x_b - x_a)T - \frac{1}{2}a^2T^3 + \frac{1}{2}\left(\frac{x_b - x_a}{T} - \frac{1}{2}aT \right)^2 T + ax_aT \\
&= -\frac{1}{24}a^2T^3 + \frac{1}{2}a(x_b - x_a)T + ax_aT + \frac{(x_b - x_a)^2}{2T}
\end{aligned}
$$

This now gives the phase change along the classical path—and the Euler-Lagrange equation, which told us $\dot{x} = a$, guarantees that this phase change is the minimum of all possible paths. The amplitude that the classical particle follows this path is thus $\exp(-iS/\hbar)$, and the probability is the square of the amplitude.

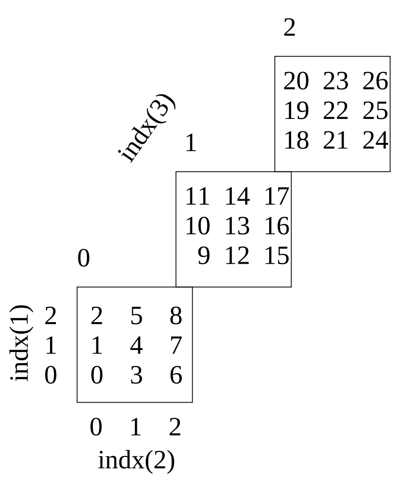
3. **Programming path integrals.** It would be easy to write four nested loops, 1 to 5 each, to generate all 625 paths shown in Note 31. But how would we write it in general so that we can change both the number of loops $n-1$ ($n = T/$`tstep`, the overall time T for the particle to travel from x_a to x_b, divided by the size of each time step), and the number of discrete x-positions, `nx`?
We need to be able to extract a set of indices, giving a path, from the variable k of a single loop

```
for k = 1:nx^(n-1)
```

Write a program

```
indx = j2indxn(k-1,n-1,nx)
```

which gives the 1-by-($n-1$) array, `indx`, of indices for an $n-1$-dimensional array of `nx`-by-`nx`-by-`nx`-by-$\cdots$ elements generalizing the following 3-by-3-by-3 example ($k-1$ runs from 0 to 26).



Thus `j2indxn(10,3,3)` is [1,0,1] and `j2indxn(17,3,3)` is [2,2,1].
Each step in the k-loop gives one whole path from x_a to x_b so an inner loop

```
for j = 1:n-1
```

can add up the phases given by the Lagrangian/$\hbar$ and calculate the amplitude for that path. The outer, k-loop, sums the amplitude for the total over all paths.

This Excursion is purely didactic. It cannot be used to find the total amplitude because only a small, discrete set of paths is explored. But it motivates the analytical calculations that follow.

4. My sources for the discussion of functionals in Note 32 are Berciu [Ber11], Ryder [Ryd85, §5.4] and Straub [Str04, p.13]. Although we cannot find a slope of a functional which is independent of "direction", we can find *extrema* of functionals: we don't need the "slope" but only the numerator of this slope, which we must set to zero. Thus, for one of our examples,

$$0 = F[f + \delta f] - F[f] = \int_0^1 dx(6f(x) - x)\delta f(x)$$

which, if true for any "direction", $\delta f(x)$, implies

$$6f(x) = x$$

or $f(x) = x/6$. Berciu explores the trial-and-error approach to solving this problem, and goes on to discuss its application to deriving the Euler-Lagrange equations (see Note 37 of book 8c (Part IV)) and other applications.
The idea of using the delta-function as a specific direction is in Ryder, and Straub gives the second- and fourth-order slopes of Note 32. Instead of taking Straub's advice and working through the fourth-order slope I have proceeded to higher orders. You might work out the fourth-order slope, though.
Zee [Zee10, p.13] says of the connection between the "odd factorial" and the diagrams at the end of Note 32 (although in the different context of Gaussian integrals), "This clever observation, due to Gian Carlo Wick, is known as Wick's theorem in the field theory literature." Was this historically a refinement of Freeman Dyson's work reconciling Feynman's intuitive diagrams with the field theory work of Schwinger and Tomonaga?

5. Feynman [FH65, §4.1] works out the path integral normalization discussed in Note 33, for non-relativistic quantum mechanics, to get

$$A = \sqrt{\frac{2\pi i\hbar\epsilon}{m}}$$

He also derives Schrödinger's equation from the path integral

$$K(b,a) =^{\lim}_{\epsilon\to 0} \frac{1}{A}\int\cdots\int\int \frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_N}{A}e^{iS[b,a]}$$

where S is the action

$$S[b.a] = \int_{t_a}^{t_b} dtL(\dot{x}, x, t)$$

using the Lagrangian

$$L = \frac{m\dot{x}}{2} - V(x,t)$$

6. **Matrix Gaussians.** We can generalize the multi-variable integral at the beginning of Note 33 to include variable coefficients a_k. I'll follow Zee [Zee10, p.14] by including a factor 1/2 to make it tidier to complete the square. First, only quadratic terms:

$$\int dx_1 dx_2 e^{-\frac{1}{2}(a_1^2x_1^2 + a_2^2x_2^2)} = \sqrt{\frac{2\pi}{a_1}}\sqrt{\frac{2\pi}{a_2}} = \sqrt{\frac{(2\pi)^2}{\prod_k a_k}}$$

If we think of a_1 and a_2 as the elements of a diagonalized (2-by-2) matrix A, this becomes

$$\int dx_1 dx_2 e^{-\frac{1}{2}x\cdot A\cdot x} = \sqrt{\frac{(2\pi)^2}{\det A}}$$

and in this formulation it no longer matters if A is diagonal; it need only be diagonalizable, which is the case if A is symmetric or hermitian. Apart from the expicit mention of dx_1dx_2 this formulation is also independent of the size of the matrix: if we wrote $\int d^n x$, A could be n-by-n.
Now, include linear terms and complete the squares.

$$\int dx_1 dx_2 e^{-\frac{1}{2}(a_1^2x_1^2+a_2^2x_2^2)+J_1x_1+J_2x_2} = \sqrt{\frac{2\pi}{a_1}}\sqrt{\frac{2\pi}{a_1}}e^{J_1^2/(2a_1)}e^{J_2^2/(2a_2)} = \sqrt{\frac{(2\pi)^2}{\prod_k a_k}}\exp\sum_k J_k^2/(2a_k)$$

This becomes, in matrix terms,

$$\int dx_1 dx_2 e^{-\frac{1}{2}x\cdot A\cdot x+J\cdot x} = \sqrt{\frac{(2\pi)^2}{\det A}}e^{\frac{1}{2}J\cdot A^{-1}\cdot J}$$

since A^{-1}, if diagonal, has just the elements $1/a_k$.
Compare the denominator, $\sqrt{\det A}$, with the product, A^n, of weighting factors in Note 33 (also called A but there just a number).

7. The $(x_{j+1}-x_j)^2$ calculation of Note 33 was first made by Feynman in 1948 [Fey48].

8. The derivation of the free (kinetic energy only) path integral in Note 33 is given by [Str04, pp.8,9].

9. **Even moments.** The "odd factorial" of Note 32 can also be generated by the Gaussian integrals of Note 33.
Let's find

$$\int dx x^{2n} e^{-ax^2}$$

by finding

$$\begin{aligned}
\partial_a^{(n)}\int dx e^{-ax^2} &= \partial_a^{(n)}\sqrt{\frac{\pi}{a}} \\
\partial_a\int dx e^{-ax^2} &= \int dx\partial_a e^{-ax^2} = \int dx(-x^2)e^{-ax^2} \\
\partial_a\left(\frac{\pi}{a}\right)^{1/2} &= \frac{1}{2}\left(\frac{\pi}{a}\right)^{-1/2}\partial_a\frac{\pi}{a} = \frac{1}{2}\left(\frac{\pi}{a}\right)^{-1/2}\left(-\frac{\pi}{a^2}\right) = -\frac{1}{2a}\sqrt{\frac{\pi}{a}}
\end{aligned}$$

Then

$$\begin{aligned}
\partial_a^{(2)}\int dx e^{-ax^2} &= \int dx(-x^2)^2 e^{-ax^2} \\
\partial_a^{(2)}\left(\frac{\pi}{a}\right)^{1/2} &= -\frac{1}{2}\partial_a\left(\frac{\pi}{a^3}\right)^{1/2} = \frac{3}{4}\left(\frac{\pi}{a^3}\right)^{-1/2}\frac{\pi}{a^4} = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}
\end{aligned}$$

and so on

$$(-)^n\int dx x^{2n}e^{-ax^2} = (-)^n\frac{(2n-1)\cdots 3\cdot 1}{(2a)^n}\sqrt{\frac{\pi}{a}}$$

So

$$\int dx x^{2n} e^{-ax^2} = \frac{(2n-1)!!}{(2a)^n}\sqrt{\frac{\pi}{a}}$$

What happens if a is replaced by $a/2$?
What happens if we insert an extra x?

$$\int_{-\infty}^{\infty} dx x x^{2n} e^{-ax^2} = ?$$

(I've shown the infinite limits explicitly as a hint.)
Zee [Zee10, p.13] uses this approach to calculate even moments $< x^{2n} >$.

10. Zee [Zee10, pp.42,3] discusses a $Z(\ell)$ differing from that of Note 33 in that he explores $-fx^4$ in the exponent instead of $-fx^3$. Work out the f-g-ℓ table for Zee's problem and show that $V_\ell + 4V_f = 2E$.

11. Zee [Zee10, p.45] shows that we can expand first in powers of ℓ then in powers of f and get the same coefficients by using the results of Excursion *Even moments*.

12. Does the Euler relationship of Note 34 apply to disconnected diagrams? How would you modify it, given the number, P, of pieces?

13. The Euler relationship of Note 34 pertains to diagrams on a plane, without counting the rest of the plane as itself a face. How would it change if the diagram were on the surface of a sphere? A torus?

14. **Current from faze invariance.** Show that the current, j^μ, in the Maxwell Lagrangian of Note 34

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

can be obtained by fazeing ($\partial_\mu \to D_\mu = \partial_\mu + (iq/\hbar)A_\mu$ as in Note 16) the Dirac Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{Dirac}} &= \bar{\psi}(i \not{D} - m)\psi \\
&= \bar{\psi}(id^\mu\partial_\mu - \frac{q}{\hbar}d^\mu A_\mu - m)\psi \\
&= \bar{\psi}(id^\mu\partial_\mu - m)\psi - \frac{q}{\hbar}\bar{\psi}d^\mu A_\mu\psi
\end{aligned}$$

15. Make the E-V_f table for $f = 4$, as Note 34 does for $f = 3$. Compare it with your $-fx^4$ table in the Excursions for Note 33,

16. Show that γ^μ, $\mu = 0, 1, 2, 3$, in Note 35, anticommute.

17. In Note 19 (Part IV) we saw that any reflection F gives rise to orthogonal projections because $F^2 = I$. What other (2-by-2) matrices, which are not reflections, also give rise to two orthogonal projections? These should be avoided in our discussions of Note 35.

18. For Note 35, confirm that under Lorentz transformation

$\bar{\psi}\psi$	scalar
$\bar{\psi}\gamma^\mu\psi$	vector
$\bar{\psi}\left(\frac{i}{2}[\gamma^\mu, \gamma^\nu]_-\right)\psi$	tensor
$\bar{\psi}\gamma^5\gamma^\mu\psi$	pseudo-vector
$\bar{\psi}\gamma^5\psi$	pseudo-scalar

19. In Note 35 we found a chiral representation of the Dirac matrices in which the product of γ^μ, $\mu = 0, 1, 2, 3$, times i is
$$\gamma^5 = \begin{pmatrix} I & \\ & -I \end{pmatrix}$$
Play with reflection algebras in 2D and 3D for both Euclidean and Minkowski spaces to see if you can find representations in which the product of all the basic matrices is imaginary and diagonal. For example, 2D Euclidean:
$$\overset{e_1}{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}} \quad \overset{e_2}{\begin{pmatrix} & -i \\ i & \end{pmatrix}} \quad \overset{ie_{12}}{\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}} \quad \overset{P_R}{\begin{pmatrix} & \\ & 1 \end{pmatrix}} \quad \overset{P_L}{\begin{pmatrix} 1 & \\ & \end{pmatrix}}$$
In 3D the representations are unique: what happens?

20. The discussion of the electroweak unified theory in Note 35 follows closely [Kan93, pp.81–9]. Kane goes on to observe that the bosons in that discussion are faze bosons and so massless (see Excursion *Goldstone and Higgs mechanisms* in Part III): this makes the weak force appear much stronger relative to electromagnetism than it actually is; massive bosons require more energy to produce and so have smaller amplitudes (and probabilities) than massless bosons, weakening the force appropriately.

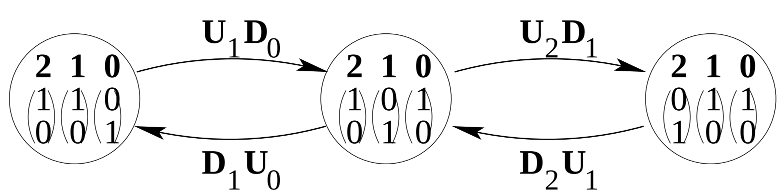
21. The discussion of Green's functions in Note 36 and of propagators in Note 37 is taken from [LP01, pp.40–45, 66–67, 143–151]. Look up other discussions, too.

22. **Huygens 3D**. Correct the math of Note 37 so that it describes a spherical, not a circular, wavefront of light.

23. Could the inverse A^{-1} of the matrix in Excursion *Matrix Gaussians* be a *discrete* propagator in the sense of Note 37?

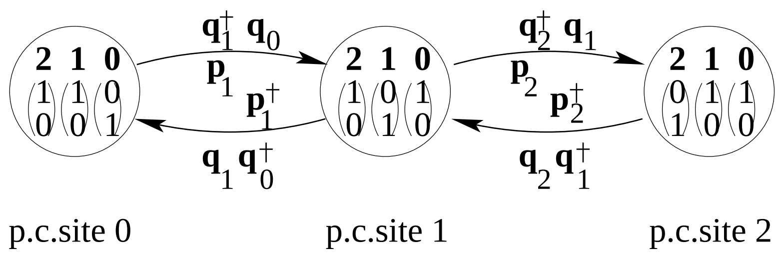
24. You might now appreciate and enjoy the history of QED in [Sch94].

25. **Feynman's program Hamiltonian.**
a) In [Fey99, Ch.6], pp.196ff. specifically, Feynman discusses a "program counter" which allows a quantum-mechanical system to sequence through a chain of "sites". Here is an example of three sites involving three bits each, labelled **0**, **1** and **2**, such that the correspondingly labelled Up and Down operators change a 0-bit to a 1-bit and vice-versa: each bit is a 2-D vector $(1,0)^T$ for 0 or $(0,1)^T$ for 1 (and they are combined by tensor product but we don't need to worry further than that the labelled operator affects only the correspondingly labelled bit).



Note that each site is characterized by exactly one 1-bit. We can consider the program-counter site to be labelled by the label of its 1-bit, hence the sites are 0, 1 and 2.
Clearly this scheme can be extended to any number of sites and hence, eventually, to a program any number of steps in length.
The transitions shown allow the system to move from any site to any other site. But we can suppose the program counter starts with bit 0 set to 1, and hence we start at site 0; and that

we wish to finish when the last bit is 1, hence at the last site. Like any quantum-mechanical system this one will follow all possible paths in all directions, but we know that, having started at 0, if we get at any time to the last site, then we will have executed all the steps of the program.
So it only remains to stick the program gates themselves somehow into this scheme.
The Up-Down notation may now get confusing, because the program gates themselves will ultimately be built up of Up and Down operators. So we'll switch to more conventional notation with q standing for D and its transpose $q^\dagger$ standing for U.
Then we can represent the program gates by p to distinguish them from the program-counter gates q. And, since program gates must all be reversible in quantum computing, and unitary so that $p^\dagger$ is the inverse of p, we can write a (two-step) program as



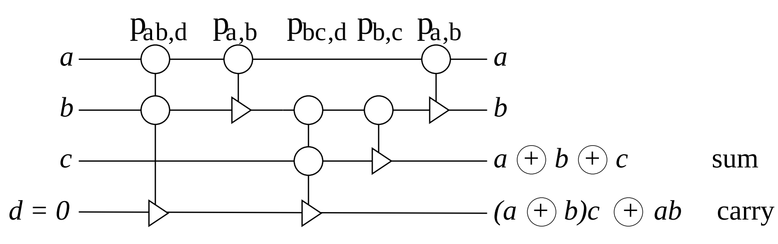
Everything on each lower, left-pointing, arrow is just the Hermitian conjugate of the corresponding upper, right-pointing, arrow. So a Hamiltonian which is the sum of all these pieces, is Hermitian.

$$H = q_1^\dagger q_0 p_1 + q_2^\dagger q_1 p_2 + \text{Hermitian conjugate}$$

(Note that we don't need to rearrange the order of the qs and ps because each operates on a different bit—the bits for the ps having not yet even been shown.)
Furthermore, any loop in a path followed by the time evolution of this quantum system does nothing. For example, the path from site 0 to site 1, back to site 0, on again to site 1 then finally to site 2 will perform program step p_1 then cancel it with step $p_1^\dagger$ then perform p_1 again and finally program step p_2: the net effect is p_1 followed by p_2 (i.e., p_2p_1 as right-to-left operators) never mind the loop en route.
We can use a Hamiltonian rather than a Lagrangian because we are not trying to be relativistic. Indeed, we want the time evolution which, for Hamiltonian H, is given by

$$e^{iHt} = 1 + iHt - \frac{H^2t^2}{2} - ..$$

These arbitrarily large powers of $H = \sum q_{j+1}^\dagger q_j p_{j+1} +$ H.c. generate all the possible paths we considered above. But the program counter sites automatically sort everything out into the sequential product of program operators $..p_j..p_2p_1$.
b) The program Feynman gives as an example is the reversible full adder (Week 10, Ex. *Matrix logic*, [Fey99, pp.190,195]) made up of CN (controlled-not) and CCN (controlled-controlled-not, or Toffoli) gates as follows.



reversible full−adder

The program is the sequence $p_{a,b}p_{b,c}p_{bc,d}p_{a,b}p_{ab,d}$ where the number of subscripts distinguishes CN from CCN gates and the subscripts give the affected (qu)bits.
Feynman goes on to express CN and CCN gates as ladder operators U and D which, to avoid subscripts, are better written in terms of the letters for the qubits, e.g., $\mathbf{a}^\dagger$ for U and $\mathbf{a}$ for D.
In Week 10, Ex. *Basic matrices for logic*, we expressed CN as

$$\begin{aligned}
\mathrm{CN}_{a,b} &= I_b \overleftarrow{\times} D_a U_a + X_b \overleftarrow{\times} U_a D_a \\
&= I_b \overleftarrow{\times} (I_a - U_a D_a) + (U_b + D_b) \overleftarrow{\times} U_a D_a \\
&= I_b I_a + (U_b + D_b - I_b)(U_a D_a) \\
&= I + (\mathbf{b}^\dagger + \mathbf{b} - I)\mathbf{a}^\dagger \mathbf{a}
\end{aligned}$$

where I've rearranged the order of the tensor products—which doesn't matter as long as it is done consistently—and finally just stopped writing them; and where the matrix forms of Ex. *Basic matrices for logic* in Week 10 justify the products and differences in the projections, $DU = I - UD$, and the sum in **not**, $X = U + D$.
A similar derivation gives (note that having the qubit labels permits us to rearrange the order of writing)

$$\mathrm{CCN}_{ab,c} = I + \mathbf{a}^\dagger \mathbf{a} \mathbf{b}^\dagger \mathbf{b}(\mathbf{c}^\dagger + \mathbf{c} - I)$$

The similarity of the two forms justifies the work we've just done.
So we can write out in detail, if labouriously, the program $p_{a,b}p_{b,c}p_{bc,d}p_{a,b}p_{ab,d}$, intersperse the program-control-site operators $q_0, \cdots, q_5$, write the Hamiltonian, and run the full-adder program.

26. Any part of the Prefatory Notes that needs working through.

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