Excursions in Computing Science: Book 11d. Forces and Invariants Part I. Electrostatics and Electromagnetism

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August 2, 2024

I. Prefatory Notes

1. Central Forces. What I will be discussing in this Book is a fair amount of physics with, as yet, no theory based, as is general relativity, on simple first principles. So either we must put up with many assertions with "only" observation and experimental authority as motivation, or else we must start with vague plausibilities which will ultimately not pan out. I prefer the latter.

So let's make two assumptions about the action of forces. First we'll talk merely of "influence" and we'll suppose that this influence is neither absorbed, interrupted nor augmented as it travels through space. Second we'll assume that geometry is Euclidean, so that the surface of a sphere is $4\pi r^2$ and the circumference of a circle is $2\pi r$, where r is the radius in each case.

Then the "influence" per unit area on a sphere concentric with the source of the influence $\propto 1/r^2$ (or, in 2D, per unit length on a circle ditto $\propto 1/r$).

If the "influence" is a force, it has direction and must be a vector

$$\begin{aligned} F &\propto \begin{pmatrix} x \\ y \\ z \end{pmatrix} r^{-3} \qquad \text{in 3D} \\ F &\propto \begin{pmatrix} x \\ y \end{pmatrix} r^{-2} \qquad \text{in 2D} \end{aligned}$$

It is easier to express this as a "potential"

$$\begin{aligned} F &= \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} P \propto \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \frac{1}{r} \qquad \text{in 3D} \\ F &= \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} P \propto \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \ln r \qquad \text{in 2D} \end{aligned}$$

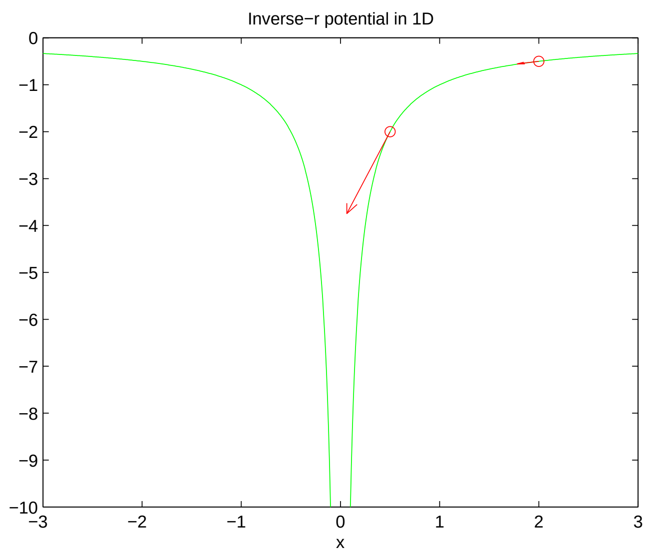
The neat thing about a potential is that it is a scalar, and the contribution from several sources is just the sum of the separate potentials

$$P = P_1 + P_2 + \cdots \propto \frac{1}{r_1} + \frac{1}{r_2} + \cdots$$

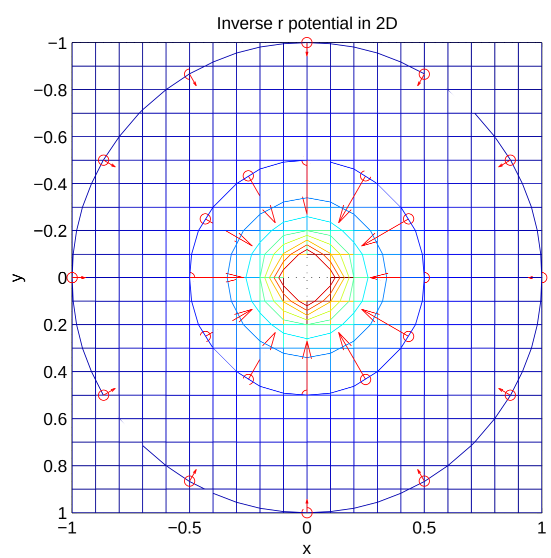
(instead of having to add up vectors).

Potentials give us a straightforward way of visualizing a force field. We can draw the potential as a simple function of the underlying space and then imagine what would happen to a small ball placed anywhere on the surface. Under the imagined influence of everyday gravity (constant in magnitude and direction), the direction and acceleration with which the ball rolls is the force at the location of the ball.

Here is the picture for a $-1/r$ potential in one dimension.

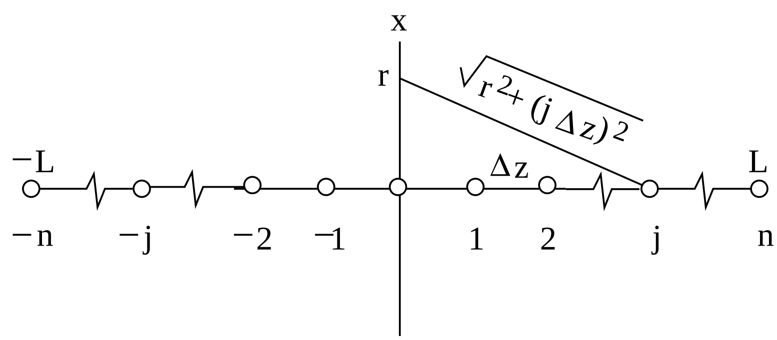


And, from above, in two dimensions



Using the potential to calculate the effect of multiple charges, or of a distribution of charges, is generally easier (obviously) than doing it with the vector force field. We must beware, though, of the subtlety that it is not the value of the potential which is of use but its slope.

For example, here is a line of charges whose potential at $(x, y, z) = (r, 0, 0)$ we need to find.



We'll suppose the charge at position j is Q_j and we'll consider all charges to be equal: $Q_j = Q_0, j = -n : n$. The distance from position j to $(r, 0, 0)$ is $r_j = \sqrt{r^2 + (j\Delta z)^2}$.

So the potential energy of a test charge q at $(r, 0, 0)$ is

$$P = -\sum_{j=-n}^{n} \frac{E_C q Q_j}{r_j} = -E_C q Q_0 \sum_{j=-n}^{n} \frac{1}{\sqrt{r^2 + (j\Delta z)^2}}$$

where E_C is the "Coulomb" constant relating qQ/r to energy.

It will be better to consider the charge per unit length

$$\lambda = \frac{(2n+1)Q_0}{2L} \approx \frac{nQ_0}{L} = \frac{Q_0}{\Delta z}$$

So the potential energy of q at $(r, 0, 0)$ now is

$$P = -E_C q \lambda \sum_{-L}^{L} \frac{\Delta z}{\sqrt{r^2 + (j\Delta z)^2}}$$

The sum in this is just

$$\begin{aligned}
\text{antislope}_{z=-L:L} \frac{1}{\sqrt{r^2 + z^2}} &= 2 \text{ antislope}_{z=0:L} \frac{1}{\sqrt{r^2 + z^2}} \\
&= 2 \ln \left(z + \sqrt{z^2 + r^2} \right)_0^L \\
&= 2 \left(\ln \left(L + \sqrt{L^2 + r^2} - \ln(r) \right) \right)
\end{aligned}$$

which we can check by taking $\text{slope}_z(z + \sqrt{z^2 + r^2})$ (or look up in a table of integrals).

Here comes the subtlety: this result can be simplified if L is big enough (and hence so is n). For large L, the L^2 dominates the r^2 in the square root, and so the first part of the above is essentially independent of r.

Thus the slope of that first part, for large L, is essentially zero

$$\text{slope}_r \ln(L + \sqrt{L^2 + r^2}) \approx 0 \qquad \text{large } L$$

So, even though the first part becomes arbitrarily large as L grows, we can forget about it[1], leaving us only with

$$P = -E_C q \lambda(-2 \ln r) = 2E_C q \lambda \ln r$$

[1]This is the first instance where a friend rubbed my nose in an insight, rather than a program I wrote. Thanks, Wolf!

Now if we make L infinite, we have a line of charge density λ per unit length and the position $z = 0$ is no longer special in $(r, 0, 0)$:

For *any* point $(r, 0, z)$ the potential is the above.

Finally, there is nothing special about the x-axis in this discussion. So we have the potential due to the infinite line of charge at any point (x, y, z), where $\sqrt{x^2 + y^2} = r$:

$$P = 2E_C q\lambda \ln \sqrt{x^2 + y^2}$$

2. Gravity vs. Electricity. Both are central forces if the source is a point,

$$F = -\frac{G_N m_1 m_2}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{in 3D}$$

using Newton's gravitational constant $G_N = 66.7\text{pJm}^2/\text{kg}^2$;

$$F = -\frac{E_C q_1 q_2}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{in 3D}$$

using Coulomb's electric constant $E_C = 9\text{GJm}^2/\text{coul}^2$, where Coulomb's constant is allowed by the definition of the unit charge, a "coulomb", to equal exactly 9×10^9.

To compare gravitational with electric forces we'll imagine, first, two 1-kg masses (each still considered to be at a point) 1 meter apart.

$$\text{PE}_{\text{grav}} = \frac{G_N m_1 m_2}{r} = 66.7\text{pJ} = 0.42\text{GeV}$$

(using 1 eV $= 0.16 \times 10^{-18}$J $= 0.16$ attoJ).

Next, imagine two 1-kg agglomerations of pure "protonium", an impossible substance made entirely of protons magically prevented from repelling itself. Each agglomeration is considered to be acting as a point charge, and the two are also 1 meter apart.

$$\text{PE}_{\text{elec}} = \frac{E_C q_1 q_2}{r} = 83.6 \times 10^{24}\text{J}$$

using proton mass 1.66×10^{-27}kg and proton charge 0.16×10^{-18} coulombs so $q_1 = q_2 = 96$Mcoul.

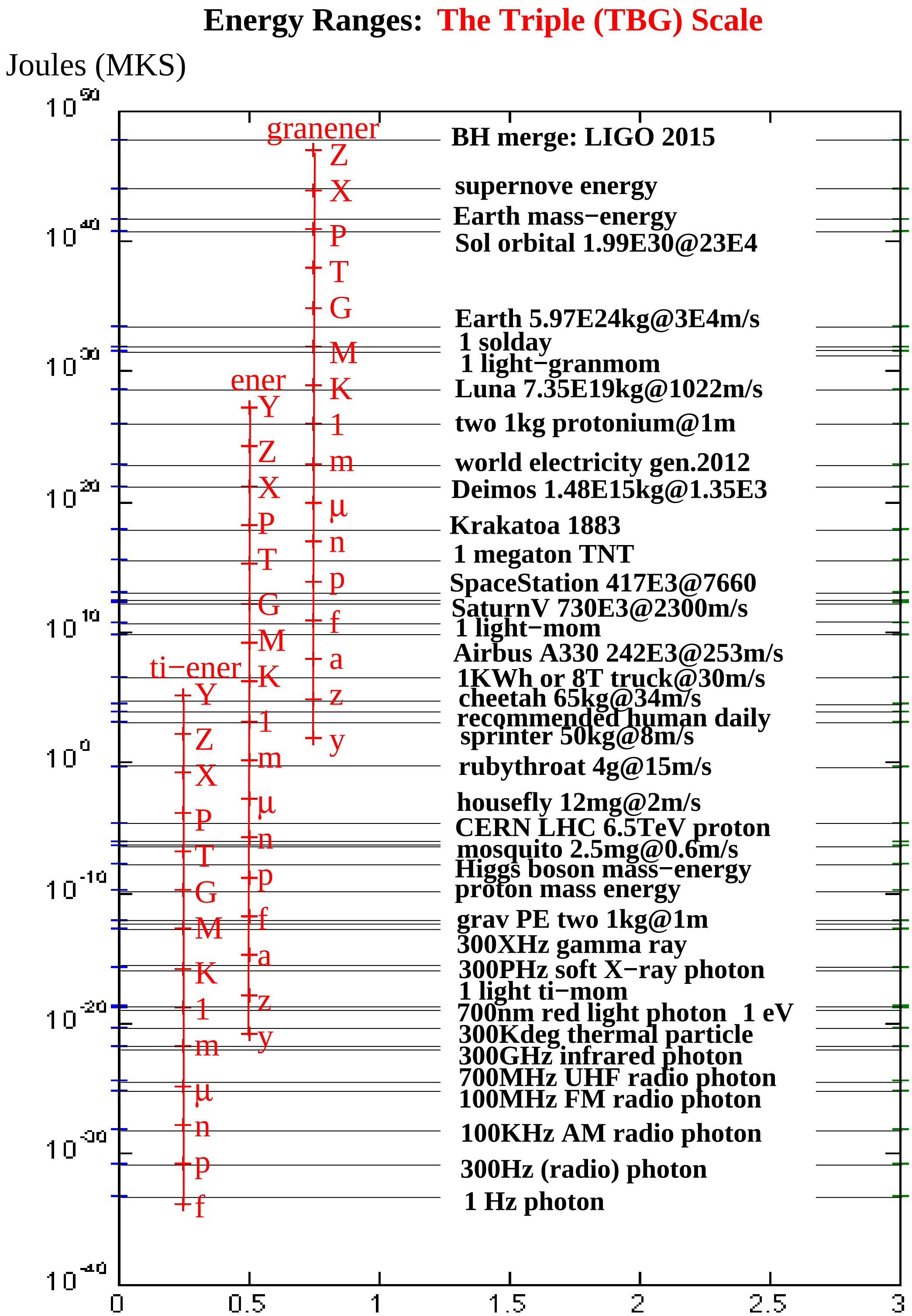
The electrostatic example has 1.25×10^{36} times the energy of the gravitational example.

And the signs of the forces are opposite: gravity attracts, but like electric charges repel.

3. Energy and momentum scales. To grasp such large differences in energies we need some illustration of the range of energies encountered in our universe. We'll step through energies in multiples of 1000, as we did in Week ii for lengths and times.

But the range of lengths we considered in Week ii fit on a single scale from femtometer to nonameter (and the corresponding times from yoctoseconds to exaseconds). Energies are related to masses, hence volumes, and potentially have the *cube* of the ranges of length or time, So to avoid new vocabulary (Notes 8 and 9 of Week ii extend the prefixes but not very satisfactorily and certainly not conventionally) I've introduced a triple scale, each using only the prefixes from yocta to Yotta.

Here are the examples. I'll explain the new triple scale below. It would take too long to describe all the details of each example: many of them are from
https://en.wikipedia.org/wiki/Orders_of_magnitude_(energy)
(last accessed 16/4/15); others are from



https://www.speedofanimals.com/animals/ or from special searchs, often leading to wikipedia pages. (I've used them verbatim without further checks.) The rubric, e.g., 1.99E30@23E4 means that the mass is 1.99×10^{30}kg and the speed is 23×10^4 m/s = 230 Km/s. From this, kinetic energies can be found using $mv^2/2$. For photons the energies are $\hbar\omega = hf$ for frequency f (angular frequency ω).

The triple scales (T for "ti", B for "base", G for "gran") are each based on four definitions (because of the four physical measures, L for length, M for mass, T for time and Q for electric charge—others will be needed when we come to the weak and strong forces).

The base scale, in the middle, attempts to be a human scale (but different from the Standard International scale of meter-kilogram-second-coulomb for variety). For energy I chose a sprinter of 50kg doing 8m/s (well, 65kg and 7m/s would do) which is 1.6 KJoules in MKS. For time, the usual MKS second. For distance, 8 MKS meters, so the sprinter's speed is 1. And for charge the standard MKSQ Coulomb.

The scale needs names for these units, too. I chose suggestive abbreviations from the English. Here is a table. I derived it from a program

`mezh2base(mezh)`

where `mezh` is a row vector of four components giving the powers of L, M, T and Q that make the physical measures of each unit.

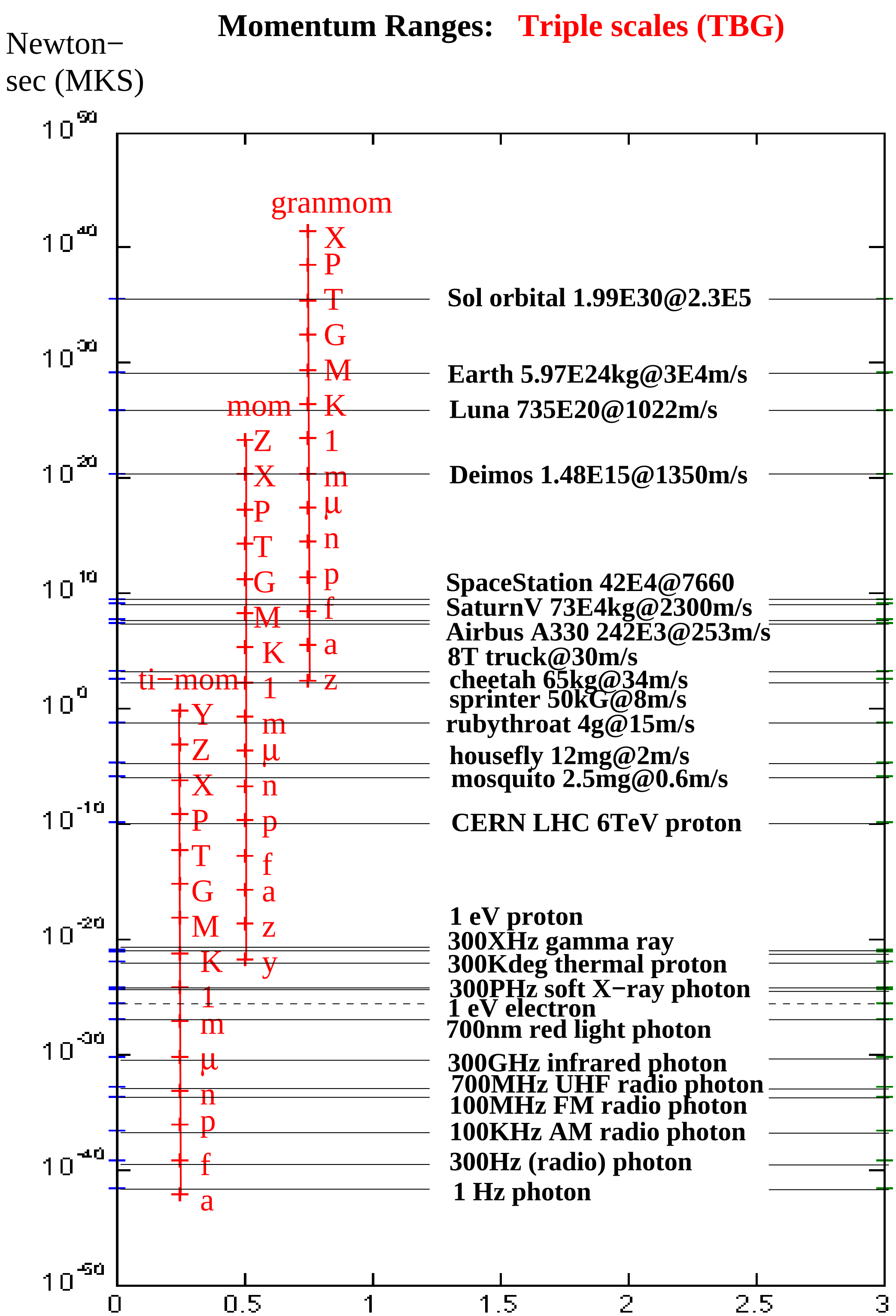
unit of	name	mezh				MKSQ
length	*len*	1	0	0	0	8
mass	*pond*	0	1	0	0	
time	*sec*	0	0	1	0	1
charge	*char*	0	0	0	1	1
energy	*ener*	2	1	−2	0	1600
momentum	*mom*	1	1	−1	0	
force	*push*	1	1	−2	0	
action	*act*	2	1	−1	0	
speed	*vel*	1	0	−1	0	
pressure	*presh*	−1	1	−2	0	
voltage	*epush*	2	1	−1	1	
capacitance	*capa*	2	1	0	−2	
inductance	*induc*	2	1	0	−1	
magfield	*magfield*	2	1	0	−1	
	magflux	0	1	0	−1	
current	*curr*	0	0	−1	1	
	currden	−2	0	−1	1	

(This is a work in progress.) The MKSQ equivalents are given only for the defining units, length, time, charge and energy. All the rest can be calculated from these, having found the MKSQ equivalents for length, time, mass and charge.

That's the base scale. For smaller quantities I define a new scale, with all names prefixed "ti-" (pronounced "tee"). Thus the energy scale is in units of ti-ener. The four definitions are

- ti-ener is 1 electron-Volt, or 0.16 attoJoules;
- ti-act is Planck's constant $h = 0.66 \times 10^{-33}$ Joule-sec or 4.14 eV-fs (so time is 1 ti-sec = 4.14 fs);
- length is ti-len = 1 nm;
- charge is the charge on the proton, 0.16 atto-Coulomb;

and the rest follow.


Momentum Ranges: Triple scales (TBG)
Newton−sec (MKS)
granmom
mom
ti−mom
Sol orbital 1.99E30@2.3E5
Earth 5.97E24kg@3E4m/s
Luna 735E20@1022m/s
Deimos 1.48E15@1350m/s
SpaceStation 42E4@7660
SaturnV 73E4kg@2300m/s
Airbus A330 242E3@253m/s
8T truck@30m/s
cheetah 65kg@34m/s
sprinter 50kG@8m/s
rubythroat 4g@15m/s
housefly 12mg@2m/s
mosquito 2.5mg@0.6m/s
CERN LHC 6TeV proton
1 eV proton
300XHz gamma ray
300Kdeg thermal proton
300PHz soft X−ray photon
1 eV electron
700nm red light photon
300GHz infrared photon
700MHz UHF radio photon
100MHz FM radio photon
100KHz AM radio photon
300Hz (radio) photon
1 Hz photon

For larger quantities, the "gran" scale is defined:

- 1 gran-ener is the energy given in Note 2 for two 1-kg protonium spheres 1 meter apart, or 83.6YJ;
- length is 1 light year = 9.46Pm;
- time is 1 year = 31.5Ms (so lightspeed is 1);
- and charge is one of those spheres of protonium or 96MCoulombs;

and the rest follow.

With the above definitions and discussion, we can give triple scales for all the quantities in the above table. I've shown the energy scales. The other quantity of fundamental interest is momentum.

The examples are a subset of those for energy. Here momenta are calculated either as mv for objects with mass, or as $h/\ell = hf/c$ where ℓ is the wavelength (or f is the frequency and c is lightspeed). In this chart there is a dashed line. This gives a momentum we'll be considering shortly.

4. Divergence, gradient and $\vec{\text{div}}$ $\vec{\text{grad}}$. An important property of the central forces we discussed in Note 1 is that their divergence is zero.

$$\begin{aligned}
\vec{\text{div}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= (\partial_x \; \partial_y \; \partial_z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{r^2} \\
&= \frac{3}{r^3} - 3\frac{x^2+y^2+z^2}{r^5} \\
&= 0
\end{aligned}$$

This further makes precise the discussion in Note 1 about "influences". In particular, it assumes that the charge causing the force is isolated at the origin and that there are no sources of charge, or sinks of charge (which could be charges of the opposite sign) in the space considered to include the field.

It makes precise the phrase "absorbed, interrupted or augmented" (i.e., sinks or sources) in Note 1.

The introduction of a field, its divergence and this way of thinking makes a major depature from the original concept of action at a distance, the former framework of Newton and Coulomb.

Then charges q and Q somehow interacted directly. Now a "field" due to source charge Q fills all the space around it, and the test charge q responds not to Q but to the field in its *immediate locality*.

So we must now see what happens if this immediate locality also contains charges.

We'll look at the central source charge, Q, but now not as a point but as a sphere, of radius R, of uniform *charge density* $\rho = Q/(4\pi R^3/3)$.

We must know two things, proved in Book 11c, Note 7. First, to a test charge *outside* the sphere, the source charge Q acts exactly as if it were all at the point at the centre of the sphere.

Second, to a test charge *inside* the sphere at a distance $r < R$ from the centre, the source charges further from the centre than r have effects which all cancel.

From these two we can conclude that the force on the test charge at radius r is due to the partial source charge $(4/3)\pi r^3\rho$ acting as if it were at the centre. So the force is

$$E_C q \frac{4}{3}\pi r^3 \rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{r^3} = E_C q \frac{4}{3}\pi \rho \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The divergence of *this* is

$$E_C q \frac{4}{3}\pi\rho\, \vec{\text{div}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = E_C q \frac{4}{3}\pi\rho\, (\partial_x\ \partial_y\ \partial_z) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= 4\pi E_C q\rho$$

We now have two results for the force $\vec{F}$.

$$\vec{\text{div}} \cdot \vec{F} = 0 \quad \text{in empty space}$$
$$\vec{\text{div}} \cdot \vec{F} = 4\pi E_C q\rho \quad \text{in space with charge density}\rho$$

The first is a special case: in empty space $\rho = 0$.

So the field law of electrostatics is

$$\vec{\text{div}} \cdot \vec{F} = 4\pi E_C q\rho$$

This is a purely local law, as we said earler. It describes the force *here* depending on the charge density *here*. If $\rho = 0$ here then $\vec{\text{div}} \cdot \vec{F} = 0$ here. Otherwise $\vec{\text{div}} \cdot \vec{F} = 4\pi E_C q\rho$ here.

If we prefer to work with potential energy, P, than with forces, $\vec{F}$, the field law undergoes a subtle change.

$$\vec{\text{div}} \cdot \vec{\text{grad}} P = 4\pi E_C q\rho$$

or

$$(\partial_x\ \partial_y\ \partial_z) \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} P = 4\pi E_C q\rho$$

Both the *divergence* and the *gradient* are vector slope operations, applied in different ways.

Because of this it is conventional to write both as the same symbol, $\bigtriangledown$. Thus

$$\vec{\text{div}} \cdot \vec{\text{grad}} P = \bigtriangledown \cdot \bigtriangledown P = (\partial_x^2\ \partial_y^2\ \partial_z^2) P$$

is written

$$\bigtriangledown^2 P$$

and the field laws of electrostatics and gravity are respectively

$$\bigtriangledown^2 P_E = 4\pi E_C q\rho_E$$
$$\bigtriangledown^2 P_G = 4\pi G_N m\rho_G$$

5. Electrodynamics departs from gravitation. The critical difference between electricity and gravity is that electricity comes in two kinds: likes repel while unlikes attract. The exact balance between positive and negative charges cancels out almost all of the stupendous difference between electrostatic and gravitational energies revealed in Note 2.

But the two-kindedness of electricity does permit *currents*, as we shall see, and currents give us all the benefits we derive from electricity—whether by the friction which generates heat and light or by the magnetic effects which run transformers, generators and motors. Electrostatic effects, whether lightning strikes or sparks in dry, winter-heated houses, are of little practical use.

How do we get a force from an electrically neutral wire carrying a current? The current consists of light, negatively-charged electrons moving within the wire through the stationary background of positively-charged heavy nuclei (and most of their orbiting electrons) that make up the wire.

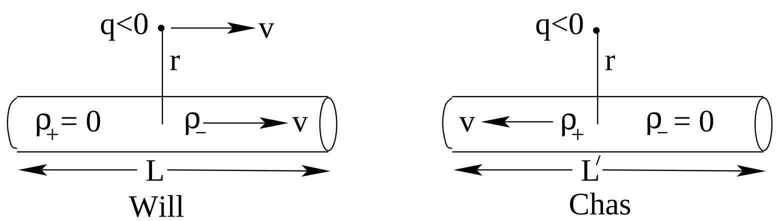
We consider a free electron outside of the wire moving parallel to and at the same velocity as the current electrons in the wire.

Since both the current electrons and the test electron are moving and the nuclei are stationary, we have a situation which needs special relativity for a full description. Even though none of these speeds is very high, the miniscule relativistic effects multiplied by the tremendous Coulomb energies produces a significant result.

For example, the speed of the electrons causing a 1 Ampere current in copper wire of 1 mm diameter (gauge AWG 18: standard house wiring) is 90 microns/sec.: 1 Ampere is 1 Coulomb/sec $= 6.24 \times 10^{18}$ electrons/sec; copper has 84.4×10^{27} electrons/m^3; AWG 18 wire has cross-section 0.82 mm^2; so the electron speed is

$$\frac{6.24 \times 10^{18}}{84.5 \times 10^{27} \times 0.82 \times 10^{-6}} = 90 \times 10^{-6} \text{ m/s}$$

Here is the relativistic argument, from the points of view of Will on the wire (Will sees the nuclei as stationary) and of Chas on the free charge (Chas sees the external electron as stationary).



Because of the Lorentz contraction, the length, L', of the piece of wire that Chas sees is shorter than the length, L, that Will sees.

$$L' = L\sqrt{1-v^2}$$

(We'll work with units, such as gran units in Note 3, where lightspeed $c = 1$. We can use fizzmezh (physical measures) arguments to put c back in later if we wish.)

The total nuclear charge of this finite piece of wire is a scalar, Q: Q cannot depend on speed or else, for example, heating the wire up, which would speed up the electrons more than the nuclei, would charge the wire, an effect never observed. So, if σ is the cross-section area of the wire (the same for both Will and Chas)

$$\rho L\sigma = Q = \rho' L'\sigma$$

so

$$\rho'_+ = \rho_+/\sqrt{1-v^2}$$

(I've labelled the nuclear charge density, ρ_+, with a + sign.)

For the electron charge density, ρ_-, we must switch our viewpoints: electrons are stationary in Chas' (primed) frame and moving in Will's (unprimed) frame. So

$$\rho_- = \rho'_-/\sqrt{1-v^2}$$

So while Will sees a neutral wire

$$\rho = \rho_+ + \rho_- = 0$$

Chas sees a wire with a net charge density

$$\begin{aligned}
\rho &= \rho'_+ + \rho'_- \\
&= \frac{\rho_+}{\sqrt{1-v^2}} + \rho_-\sqrt{1-v^2} \\
&= \rho_+\left(\frac{1}{\sqrt{1-v^2}} - \sqrt{1-v^2}\right) \\
&= \rho_+\frac{v^2}{\sqrt{1-v^2}}
\end{aligned}$$

(Putting back c this is

$$\rho_+ \frac{v^2/c^2}{\sqrt{1-v^2/c^2}})$$

Thus, in Chas' frame, the wire is no longer neutral but has a net positive charge. So it attracts the negative free electron. The potential energy for the attraction is (see Note 1)

$$\begin{aligned} P' &= 2E_C q \frac{Q}{L'} \ln r/a \\ &= 2E_C q \pi a^2 \rho' \ln r/a \\ &= \frac{2E_C q \pi a^2 \rho_+ v^2}{\sqrt{1-v^2}} \ln r/a \\ &= \frac{2E_C q I v}{\sqrt{1-v^2}} \ln r/a \end{aligned}$$

where a is the radius of the wire, $\pi a^2 = \sigma$ is its area of cross-section, the linear charge density $\lambda' = Q/L'$ relates to the volume charge density ρ' by $\lambda' = \sigma\rho'$ and the current in the wire $I = \rho_+\sigma v$.

That is, doing the fizzmezh, the potential energy is

$$\frac{2E_C q I v/c^2}{\sqrt{1-v^2/c^2}} \ln r/a$$

and the ratio E_C/c^2 is, compared with gravity in MKsQ units, a very large number, $10^{36}/(0.3 \times 10^9)^2$.

This sizeable energy gives *magnetism*. We notice that it is proportional to the current in the wire times the speed v of the free particle. The following picture shows how large these energies can get.

This was a nice copper tube until a bolt of lightning passed through it in New South Wales in 1905: the currents within the tube attracted each other, collapsing the tube in on itself. (And if you hear of "Z-pinch fusion", this is the sort of effect being exploited to extract fusion energy from a plasma.)

We've calculated the potential energy, P', in Chas' frame. We must convert it back to Will's frame. Here we see the importance of using potential energy rather than force in our calculations. From Week 7a, Notes 4 and 9, we saw that energentum—the 4-vector comprising energy and momentum—transforms relativistically exactly like timespace: with the Lorentz transformation.

So converting back to Will's frame requires us to extend the potential energy P' to a 4-vector by including a "potential momentuum" M'. In Will's frame these will be P and M respectively.

We can work in two dimensions of timespace since the directions perpendicular to the motion are not affected. Here's the Lorentz transformation

$$\begin{pmatrix} P' \\ M' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} P \\ M \end{pmatrix}$$

with $\gamma = 1/\sqrt{1-v^2}$.

Because the wire is neutral in Will's frame, $P = 0$, so we have

$$\begin{pmatrix} P' \\ M' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ M \end{pmatrix} = \gamma \begin{pmatrix} -vM \\ M \end{pmatrix}$$

So

$$\frac{-vM}{\sqrt{1-v^2}} = P' = \frac{2E_C qIv}{\sqrt{1-v^2}} \ln r/a$$

and thus

$$M = -2E_C qI \ln r/a$$

or

$$M = \frac{-2E_C qI}{c^2} \ln r/a$$

and we have found a "potential momentum" associated with the current in Will's frame, even though there is no potential energy.

Electromagnetism is described by a P-M 4-vector. Gravity, as we found in Book 11c (Part II), Notes 24 and 30, is described by a metric tensor which is a 4-by-4 matrix. In both cases, the second slope of this quantity equals another quantity which is a density: the stress-energy tensor in the case of gravity; and, so far, a charge density in the case of electrostatics.

We're going to have to add a current density to the charge density in order to get a 4-vector which equals the second slope of the P-M 4-vector.

The final form for electromagnetism will connect 4-vectors P_{EM} and ρ_{EM}

$$\Box^2 \vec{P_{EM}} = 4\pi E_C q \vec{\rho_{EM}}$$

where we'll find the 4-dimensional $\Box^2$ operator extends the 3-dimensional $\bigtriangledown^2$ operator.

We can figure out what this new operator $\Box^2$ should look like by noting that it generalizes

$$\bigtriangledown^2 = \bigtriangledown \cdot \bigtriangledown = (\partial_x\ \partial_y\ \partial_z) \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

from 3D to 4D (including time).

Notes 2 and 3 of Book 11c (Part I) tell us

$$\begin{pmatrix} \partial_{t'} \\ \partial_{x'} \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}$$

because that matrix is the transpose of the inverse of the timespace transformation. Thus these slopes have the same kind of invariant that we saw in Note 4 of Week 7a

$$\partial_{t'}^2 - \partial_{x'}^2 = \partial_t^2 - \partial_x^2$$

or, in 4D,

$$\partial_{t'}^2 - \bigtriangledown'^2 = \partial_t^2 - \bigtriangledown^2$$

We'll turn this around for direct comparison with the 3D (potential) invariants $\bigtriangledown^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$, and define

$$\Box^2 = \bigtriangledown^2 - \partial_t^2$$

or, after fizzmesh,

$$\Box^2 = \bigtriangledown^2 - \frac{1}{c^2}\partial_t^2$$

Our electromagnetic equation

$$\Box^2 \vec{P_{EM}} = 4\pi E_C q \vec{\rho_{EM}}$$

can be broken down into two, one relating the one component each of potential energy P and charge density ρ

$$\bigtriangledown^2 P - \frac{1}{c^2}\partial_t^2 P = 4\pi E_C q\rho$$

and one relating the three components each of the potential momentum $\vec{M}$ and the current density $\vec{j}$ (again, fizzmezh dictates j/c^2)

$$\bigtriangledown^2 \vec{M} - \frac{1}{c^2}\partial_t^2 \vec{M} = \frac{4\pi E_C q}{c^2}\vec{j}$$

We'll show next how these to equations expand to the four "Maxwell's equations" in conventional notation.

6. Invariants, cross-products and convention. An ordinary matrix has several "invariants" under rotations, reflections and more general transformations.

Here is the 2-dimensional rotation transformation.

$$\left(\begin{array}{cc} c & s \\ -s & c \end{array}\right)\left(\begin{array}{cc} a & d \\ b & e \end{array}\right)\left(\begin{array}{cc} c & -s \\ s & c \end{array}\right) = \left(\begin{array}{cc} c^2a + cs(b+d) + s^2c & c^2d + cs(e-a) - s^2b \\ c^2b + cs(e-a) - s^2d & s^2a - cs(b+d) + c^2e \end{array}\right)$$

From this we can see two things. First, the *trace* (the sum of the diagonal elements) is invariant.

$$(c^2a + cs(b+d) + s^2c) + (s^2a - cs(b+d) + c^2e) = a + e$$

Second, so is the difference of the off-diagonal elements

$$(c^2d + cs(e-a) - s^2b) - (c^2b + cs(e-a) - s^2d) = d - b$$

(A third invariant is the determinant, but this algebra is a tedious way to see it. It's better to see that, if we diagonalize the matrix—which we can always do by a suitable rotation if the matrix is symmetric—the determinant is the product of the resulting diagonal elements (just as the trace is their sum) and so must be independent of the coordinate system expressing the matrix. Determinants are too complicated to interest us here.)

Here is a 2-dimensional reflection transformation ($y \leftrightarrow -y$)

$$\left(\begin{array}{cc} 1 & \\ & -1 \end{array}\right)\left(\begin{array}{cc} a & d \\ b & e \end{array}\right)\left(\begin{array}{cc} 1 & \\ & -1 \end{array}\right) = \left(\begin{array}{cc} a & -d \\ -b & e \end{array}\right)$$

We can see the invariance of the trace (and of the determinant) but the difference $d - b$ changes sign" this is only "pseudo-invariant" because of that negative effect of mirrors. But note that mirrors reflect right hands into left hands.

Although the above is only one reflection, it can be rotated into any other reflection, so the invariance of trace and $b - d$ under rotation means that what we found for $y \leftrightarrow -y$ holds for any reflection.

The two invariants above hold for any size of matrix (in any number of dimensions) as the argument about diagonalizing can be extended to show. The pseudo-invariant also holds, in that the difference between any element and its transposed element is invariant under rotations, and under reflection changes sign only,

Now consider a matrix made up of all pairs of combinations of the components of two vectors. In 2D

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} (v_x \; v_y) = \begin{pmatrix} u_x v_x & u_x v_y \\ u_y v_x & u_y v_y \end{pmatrix}$$

The trace is just the dot product of the two vectors—and is invariant, which explains why the dot product is so useful.

The pseudo-invariant

$$u_x v_y - u_y v_x$$

is also important, apart from its inconvenient all-handedness. It is called the *cross product* of the two vectors.

In 3D the dot product (the trace) is still an invariant scalar. But the cross-product has three components.

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} (v_x \; v_y \; v_z) = \begin{pmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{pmatrix}$$

These can be written as a vector—a "pseudo-vector"—at the peril of forgetting that it changes handedness—i.e., direction—under reflection.

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}$$

Note that the convention is to run the indices cyclically through the first of the two terms—yz, zx, xy—and to identify the components with the complement of their indices—no x in yz, no y in zx, no z in xy.

In four dimensions the dot product and trace still produce a scalar, but there are six components in the "cross product". The conventions that work out by serendipity in 3D fail in any other number of dimensions. For this reason, I have not developed the electromagnetic equations in the conventional way, which uses cross-products for the magnetic part.

Since Note 5 (and Notes 1 and 4 before it) deal with vectors of slope operators, here is the 3D matrix

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} (v_x \; v_y \; v_z) = \begin{pmatrix} \partial_x v_x & \partial_x v_y & \partial_x v_z \\ \partial_y v_x & \partial_y v_y & \partial_y v_z \\ \partial_z v_x & \partial_z v_y & \partial_z v_z \end{pmatrix}$$

The trace is the divergence: its invariance, again, is what makes it useful.

The cross product is the *curl*, and is also useful but at the risk of antihandedness.

We can write variously

$$\begin{array}{cc} \text{div } \vec{f} & \text{curl } \vec{f} \\ \vec{\text{div}} \cdot \vec{f} & \vec{\text{curl}} \times \vec{f} \\ \vec{\bigtriangledown} \cdot \vec{f} & \vec{\bigtriangledown} \times \vec{f} \end{array}$$

Unfortunately, the cross product, as well as being quasi-invariant under coordinate changes, also defies our normal mathematical experience.

For one thing, it anticommutes

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

which you can see immediately by swapping $\vec{u}$ and $\vec{v}$ in

$$\vec{v} \times \vec{u} = \begin{pmatrix} v_y u_z - v_z u_y \\ v_z u_x - v_x u_z \\ v_x u_y - v_y u_x \end{pmatrix}$$

Even worse, it is not associative

$$\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$$

It is best to show this by example. But we first need five paragraphs of important and elsewhere useful preliminaries.

It is handy to know that $\vec{u} \times \vec{v}$, written as a vector, is perpendicular to both $\vec{u}$ and $\vec{v}$

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0 = \vec{v} \cdot (\vec{u} \times \vec{v})$$

and has magnitude $\mid u \mid\mid v \mid \sin\theta$ where θ is the angle between $\vec{u}$ and $\vec{v}$.

You can see that $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ by writing out all the components of the sum and seeing that they all cancel. And you can use anticommutativity and swapping $\vec{u}$ and $\vec{v}$ to get the second perpendicularity from the first one.

Since "perpendicular" is ambiguous as to direction, we invoke the "right hand rule", which brings in the handedness of the cross product. Use the fingers of your right hand to "rotate" the $\vec{u}$ into the $\vec{v}$: your thumb goes in the direction of $\vec{u} \times \vec{v}$.

To work out the magnitude, it is easiest to confine $\vec{u}$ and $\vec{v}$ to the x-y plane and then follow this up using invariance under rotation to see it for any other coordinate system with $\vec{u}$ and $\vec{v}$ no longer restricted to the x-y plane. From the definition of $\vec{u} \times \vec{v}$ now only the z component is nonzero

$$(\vec{u} \times \vec{v})_z = u_x v_y - u_y v_x$$

But the sine of the angle between $\vec{u}$ and $\vec{v}$ is, from the sine of the difference between the direction of $\vec{u}$ and the direction of $\vec{v}$, i.e.,

$$\begin{aligned} \sin(\theta_v - \theta_u) &= \cos\theta_u \sin\theta_v - \sin\theta_u \cos\theta_v \\ &= \frac{u_x}{\mid u \mid}\frac{v_y}{\mid v \mid} - \frac{u_y}{\mid u \mid}\frac{v_x}{\mid v \mid} \end{aligned}$$

So

$$u_x v_y - u_y v_x = \mid u \mid\mid v \mid \sin(\theta_v - \theta_u) = \mid u \mid\mid v \mid \sin\theta$$

End of preliminaries.

Now we can see the counterexample to the associativity of $\vec{u} \times \vec{v} \times \vec{w}$. Let $\vec{u} = (1,0,0)$, $\vec{v} = (0,1,0)$ and $\vec{w} = (0,-1,0)$. Then

$$(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u}$$

but

$$\vec{u} \times (\vec{v} \times \vec{w}) = 0$$

(The magnitudes are 1; the angles are $\pi/2$ or π; use the right hand rule.)

We did not need all the above properties to figure out

$$\vec{u} \times (\vec{v} \times \vec{w}) = \vec{v}(\vec{u} \cdot \vec{w}) - (\vec{u} \cdot \vec{v})\vec{w}$$

and

$$(\vec{u} \times \vec{v}) \times \vec{w} = \vec{v}(\vec{u} \cdot \vec{w}) - \vec{u}(\vec{v} \cdot \vec{w})$$

(except not to be surprised that they're not the same, and to see that the right-hand sides have components in the right directions). We do it by brute force plus a small trick.

Consider the x-component of

$$\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \times \begin{pmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{pmatrix}$$

It is

$$\begin{aligned} u_y(v_x w_y - v_y w_x) - u_z(v_z w_x - v_x w_z) &= v_x(u_y w_y + u_z w_z) - (u_y v_y + u_z v_z) w_x \\ &= v_x(\vec{u} \cdot \vec{w}) - (\vec{u} \cdot \vec{v}) w_x \end{aligned}$$

where the last line comes from the trick of adding and subtracting $u_x v_x w_x$ to and from the line before.

A similar operation on the other two components gives the result for $\vec{u} \times (\vec{v} \times \vec{w})$.

The second result comes directy from this by anticommutativity and the substitution

$$\vec{u} \to \vec{w} \qquad \vec{v} \to \vec{u} \qquad \vec{w} \to \vec{v}$$

What about $\vec{\bigtriangledown} \times (\vec{\bigtriangledown} \times \vec{w})$? This is almost a direct specialization of the first result above, but we must be careful about the order: $\vec{\bigtriangledown}$ is an *operator* and must be written before its operand. Fortunately we've done this above, always writing the $\vec{w}$ last.

$$\vec{\bigtriangledown} \times (\vec{\bigtriangledown} \times \vec{w}) = \vec{\bigtriangledown}(\vec{\bigtriangledown} \cdot \vec{w}) - (\vec{\bigtriangledown} \cdot \vec{\bigtriangledown})\vec{w}$$

This is nice. It has a $(\vec{\bigtriangledown} \cdot \vec{\bigtriangledown})\vec{w} = \bigtriangledown^2 \vec{w}$ in it: just what appears in the pair of electromagnetic equations at the end of Note 5. If $\vec{w}$ happened to be the particle momentum $\vec{M}$ we might be able to go somewhere with this.

What about the first term? We're not going to worry about it just yet. Instead we'll make two further observations about dot and cross products, then we'll see if we can use the connection we've just sensed between $\vec{\bigtriangledown} \times \vec{\bigtriangledown} \times$ and $\bigtriangledown^2$, finally returning with a slightly different approach to terms such as the first.

The dot and cross products relate to each other in two important ways.

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = (v_x, v_y, v_z) \begin{pmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{pmatrix} = 0$$

because everything cancels—or because $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$ and so the dot product is zero.

And, because $\vec{v} \times \vec{v} = 0$ (the angle between $\vec{v}$ and itself is 0)

$$\vec{v} \times \vec{v}a = (\vec{v} \times \vec{v})a = 0$$

for any scalar a.

These results go directly over from $\vec{v}$ to $\vec{\bigtriangledown}$ because $\vec{\bigtriangledown}$ is a vector almost like $\vec{v}$. (Be careful: $\vec{\bigtriangledown}a \times \vec{\bigtriangledown}b \neq 0$ in general because $\vec{\bigtriangledown}a$ and $\vec{\bigtriangledown}b$ are different vectors—unlike $\vec{v}a \times \vec{v}b = (\vec{v} \times \vec{v})ab = 0$. But in $\vec{\bigtriangledown} \times \vec{\bigtriangledown}a$, $\vec{\bigtriangledown}$ is the same vector as $\vec{\bigtriangledown}$.)

(They go further. Conversely if $\vec{\bigtriangledown} \times \vec{v} = 0$ then there must be some a such that $\vec{v} = \vec{\bigtriangledown}a$; and if $\vec{\bigtriangledown} \cdot \vec{v} = 0$ then there must be some $\vec{w}$ such that $\vec{v} = \vec{\bigtriangledown} \times \vec{w}$. Look up the proof!)

The discussion of this Note is headed in the direction of electromagnetic orthodoxy. So to be conventional I'll make a final change to the results of Note 5. Instead of using potential energy and potential momentum, convention uses *specific* potential energy and *specific* potential momentum: i.e., without the test charge q.

Remembering that q was negative in Note 5, we change signs on the left hand sides of the two electromagnetic equations there. And we introduce conventional symbols and names, the *electric potential* $\phi = P/\mid q\mid$ and the *magnetic vector potential* $\vec{A} = \vec{P}/\mid q\mid$.

We'll make two related definitions:

$$\begin{array}{rrcl} \text{the } \textit{magnetic field} & \vec{B} & \stackrel{\text{def}}{=} & \vec{\bigtriangledown} \times \vec{A} \quad \text{and} \\ \text{the } \textit{electric field} & \vec{E} & \stackrel{\text{def}}{=} & -\vec{\bigtriangledown}\phi - \partial_t \vec{A} \end{array}$$

We see the analogy of the electric field with the gravitational field, which is also the gradient of a potential. But here we introduce also a time dependence because of the time dependences in the electromagnetic equations. The magnetic field will exploit our discoveries about $\vec{\bigtriangledown} \times \vec{\bigtriangledown} \times \cdots$

Here are the modified electromagnetic equations for the rest of the Note.

$$\begin{array}{rcll} \bigtriangledown^2\phi - \dfrac{1}{c^2}\partial_t^2\phi & = & -4\pi E_C\rho & \text{EM1} \\ \bigtriangledown^2\vec{A} - \dfrac{1}{c^2}\partial_t^2\vec{A} & = & -4\pi E_C\vec{j} & \text{EM2} \end{array}$$

We now transform these two equations into the four *Maxwell equations* which are the conventional way of writing classical electromagnetism.

There is one more preliminary. Because $\vec{B} = \vec{\bigtriangledown} \times \vec{A}$, the vector potential $\vec{A}$ can have any divergence whatsoever. If

$$\vec{B} = \vec{\bigtriangledown} \times \vec{A} = \vec{\bigtriangledown} \times \vec{A}'$$

then

$$\vec{\bigtriangledown} \cdot (\vec{A}' - \vec{A}) = 0$$

so

$$\vec{A}' - \vec{A} = \vec{\bigtriangledown}\psi$$

for any scalar field ψ, and so

$$\vec{\bigtriangledown}\vec{A}' = \vec{\bigtriangledown}\vec{A} + \bigtriangledown^2\psi$$

with an arbitrary difference between the divergences of two equally satisfactory vector potentials.

We take advantage of this to include a time dependence in $\vec{A}$ as well as in $\vec{E}$

$$\vec{\bigtriangledown} \cdot \vec{A} = -\frac{1}{c^2}\partial_t\phi$$

Now we can derive Maxwell's equations. First, from the definition of $\vec{B}$

$$\vec{\bigtriangledown} \cdot \vec{B} = 0 \qquad \text{Maxwell 3}$$

Second, from the definition of $\vec{E}$ and $\vec{B}$

$$\begin{array}{rcl} \vec{\bigtriangledown} \times \vec{E} & = & \vec{\bigtriangledown} \times \vec{\bigtriangledown}\phi - \vec{\bigtriangledown} \times \partial_t\vec{A} \\ & = & 0 - \partial_t\vec{\bigtriangledown} \times \vec{A} \\ & = & -\partial_t\vec{B} \end{array}$$

So

$$\vec{\bigtriangledown} \times \vec{E} + \partial_t\vec{B} = 0 \qquad \text{Maxwell 2}$$

Third, from the assignment of $\vec{\bigtriangledown} \cdot \vec{A}$ and from EM 1

$$\bigtriangledown^2 \phi + \partial_t \vec{\bigtriangledown} \cdot \vec{A} = -4\pi E_C \rho$$

and from the definition of $\vec{E}$

$$\vec{\bigtriangledown} \cdot \vec{E} = 4\pi E_C \rho \qquad \text{Maxwell 1}$$

Fourth, from the definition of $\vec{B}$ and the assignment of $\vec{\bigtriangledown} \cdot \vec{A}$, EM 2, and the definition of $\vec{E}$

$$\begin{aligned} c^2 \vec{\bigtriangledown} \times \vec{B} &= c^2 \vec{\bigtriangledown} \times \vec{\bigtriangledown} \times \vec{A} \\ &= c^2 (\vec{\bigtriangledown}(\vec{\bigtriangledown} \cdot \vec{A}) - \bigtriangledown^2 \vec{A}) \\ &= -\vec{\bigtriangledown} \partial_t \phi - \partial_t^2 \vec{A} + 4\pi E_C \vec{j} \\ &= \partial_t (-\vec{\bigtriangledown} \phi - \partial_t \vec{A}) + 4\pi E_C \vec{j} \\ &= \partial_t \vec{E} + 4\pi E_C \vec{j} \end{aligned}$$

So

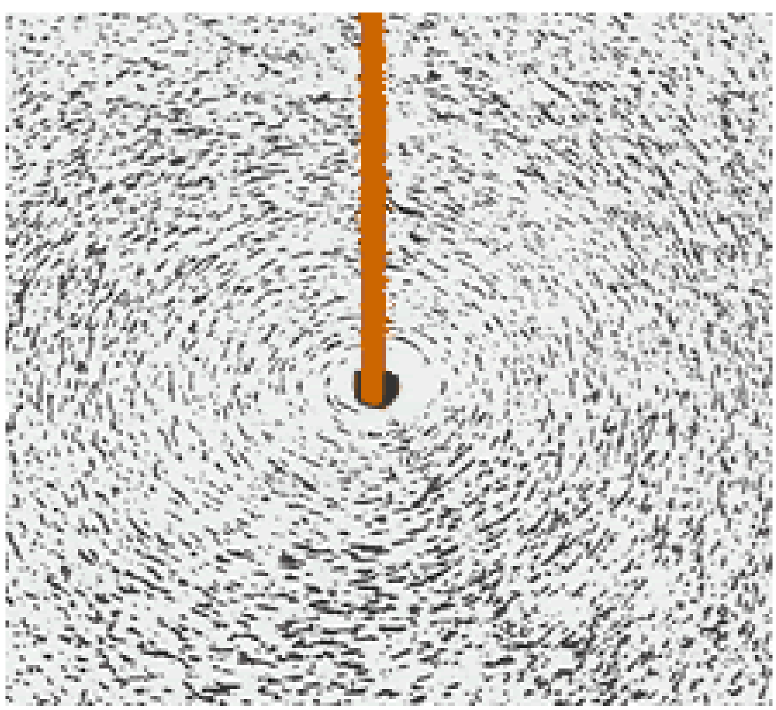
$$c^2 \vec{\bigtriangledown} \times \vec{B} = \partial_t \vec{E} + 4\pi E_C \vec{j} \qquad \text{Maxwell 4}$$

These are Maxwell's equations

$$\begin{aligned} \vec{\bigtriangledown} \cdot \vec{E} &= 4\pi E_C \rho \\ \vec{\bigtriangledown} \times \vec{E} + \partial_t \vec{B} &= 0 \\ \vec{\bigtriangledown} \cdot \vec{B} &= 0 \\ c^2 \vec{\bigtriangledown} \times \vec{B} &= \partial_t \vec{E} + 4\pi E_C \vec{j} \end{aligned}$$

Maxwell assembled these from the careful work of predecessors—except that, for balance, he added the time dependence of $\vec{E}$ in the last equation. This led to the wave equation and to Maxwell's prediction that light is an electromagnetic wave, and the derivation of the speed of light.

These equations introduce $\vec{B}$ as a counterpart to $\vec{E}$, with curl replacing divergence. We should try to visualize this new field. Here's a picture of iron filings scattered over a flat sheet and re-oriented by a current going through the wire [Kur12].



We can see the circular alignment of the filings around the wire. These circles are modelled by the curl in $\vec{B} = \vec{\triangledown} \times \vec{A}$. Of course, curl goes too far and insists on a *direction* which we do not observe. This handedness is cancelled by a second operation to determine the force on a moving test charge—it involves a cross product

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

7. Electromagnetic waves. In Note 5 we found the electromagnetic equation

$$\Box^2 \vec{P_{\rm EM}} = 4\pi E_C q \vec{\rho_{\rm EM}}$$

in the immediate presence of a charge-current density $\vec{\rho_{\rm EM}}$.

In the absence of any charge or current this becomes

$$\Box^2 \vec{P_{\rm EM}} = 0$$

and, as in Note 5, we can break it into two three-dimensional pieces, a scalar equation for the potential energy

$$\triangledown^2 P = \frac{1}{c^2}\partial_t^2 P$$

and a vector equation for the potential momentum

$$\triangledown^2 \vec{M} = \frac{1}{c^2}\partial_t^2 \vec{M}$$

These are four copies of the *wave equation*, which, as we shall see in the next Part, describes a wave in 3D space.

This wave is propagated with velocity c: lightspeed.

Hence Maxwell inferred that light itself is an electromagnetic wave.

The principle of relativity says that laws of physics are unaffected by uniform motion. We now, from the above, know that lightspeed is a law of physics. At the risk of circularity in our arguments in this Part we can now infer the postulate of special relativity, that lightspeed is independent of the motion of the observer. This is the historical chronology: Maxwell's electromagnetism came first, in 1864, and Einstein's special relativity followed in 1905.

Part II. Partial Slope Equations and Quantum Mechanics

8. Partial Slope Equations: Laplace's Equation.
9. The Wave Equation.
10. The Schrödinger Equation I: Physics.
11. The Schrödinger Equation II: Animating in 1D.
12. The Schrödinger Equation III: Animating in 2D.

Part III. Quantum Electromagnetism

13. The electromagnetic Schrödinger equation.
14. Simulating a charged wavepacket moving near a current.
15. Links with geometry.
16. Local action versus action-at-a-distance.
17. Other symmetries, other forces.

Part IV. Quantum Field Theory: Matrix Quantum Mechanics

18. Introduction to Quantum Fields.
19. Small matrices.
20. Tensor products.
21. Spin.
22. Vectors and spinors,
23. Multiple and independent systems.
24. A simple field.
25. The Yukawa potential.
26. Perturbation approximations.
27. Fermions.
28. Slopes and antislopes of 2D numbers, etc.
29. Charge conservation and antimatter.
30. Relativistic quantum field theory redux, so far.

Part V. Functional Integrals

31. Path amplitudes.
32. Functionals.
33. Gaussian integrals.
33. Gaussian integrals.
34. Diagrams and QED.
35. Chirality and electroweak.
36. Green's functions.
37. Propagators.
38. Quantum Computing.
39. Binary Fourier transform.
40. Quantum Fourier transform.
41. Finding periods.
42. Quantum key distribution.
43. No cloning.
44. Database search.
45. Detecting and correcting errors.

You've seen lots of ideas. Now *do* something with them!

1. To show that
$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \frac{1}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} r^{-3} \qquad \textit{in 3D}$$
and
$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \ln r = \begin{pmatrix} x \\ y \end{pmatrix} r^{-2} \qquad \textit{in 2D}$$
you need four things.
1. $\partial_r r^n = n r^{n-1}$ slopes of powers
2. $\partial_r \ln r = 1/r$ slope of logarithm
3. $\partial_x f(r) = \partial_r f(r) \partial_x r$ chain rule
4. $\partial_x r = \partial_x \sqrt{x^2 + y^2 + z^2} = x/r$
and similarly for $\partial_y r$ and $\partial_z r$ (except that in 2D there's no z).
a) Show these things.
b) Use them to show that the forces follow from the potentials as in Note 1.

2. Why might the potential due to the (infinitely) long charged wire be written
$$P = 2E_C q\lambda \ln(r/a)$$
where a might be the radius of the wire, just as well as
$$P = 2E_C q\lambda \ln(r)?$$

3. It is interesting to work with forces instead of potentials for the line of charges in Note 1. Using the diagram there we'll focus on the force at $(x, y, z) = (r, 0, 0)$.
Symmetry tells us that the z-components of the force cancel. So all we need calculate is the net x-component. The net force on a test charge q at $(r, 0, 0)$ is
$$\begin{aligned} F = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \sum_{j=-n}^{n} \frac{E_C q Q_j}{r_j^3} &= E_C q Q_0 \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \sum_{j=-n}^{n} \frac{1}{r_j^3} \\ &= E_C q\lambda \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \sum_{j=-n}^{n} \frac{\Delta z}{(r^2 + (j\Delta z)^2)^{3/2}} \end{aligned}$$
If you write a short program to do this sum you'll find
$$\text{antislope}_{z=0:\infty} \frac{1}{(r^2 + z^2)^{3/2}} = \frac{1}{r^2}$$
and so
$$F = 2E_C q\lambda \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \frac{1}{r^2} = 2E_C q\lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{r}$$
(The 2 comes from summing $-\infty$ to ∞ instead of 0 to ∞.)
For arbitrary x and y position $(x, y) = (c, s)r$ where c and s are cosine and sine respectively of the position of (x, y)
$$F = 2E_C q\lambda \begin{pmatrix} c \\ s \\ 0 \end{pmatrix} \frac{1}{r}$$
The next Excursion is a more powerful derivation of this.

4. **Gauss' law.** If we can find a surface surrounding a "source" charge such that the force at the surface due to the charge is perpendicular to the surface everywhere and has the same magnitude everywhere we may be able to use it to figure out the force dependence on the position of a test charge q relative to the source charge Q.
   This works for simple shapes. For example we can go from a point charge (spherical surface) to an infinite line of charge (cylindrical surface).
   We consider the sum of the normal (perpendicular) force over the whole surface. (This makes more precise the argument at the beginning of Note 1 about the "influence" of the source charge.)
   For the sphere and the point charge Q

$$\begin{aligned} \text{sum } F_n &= \text{area} \times F_n \\ &= 4\pi r^2 \frac{E_C q Q}{r^2} \\ &= 4\pi E_C q Q \end{aligned}$$

   The basis for moving from the point charge Q to a charge Q stretched into a line of length L is to say that the sum over the new, cylindrical, surface is the same.

$$4\pi E_C q Q = \text{sum } F_n = 2\pi r L F_n$$

   So

$$F_n = \frac{4\pi E_C q Q}{2\pi r L} = \frac{2E_C q}{r}\frac{Q}{L}$$

   We've assumed that the areas at the ends of the cylinder, πr^2 each, don't count because we intend to go to an infinitely long line of charge, so the cylinder has no ends. But we must replace Q/L by the charge density λ because L is now infinite.
   This is the result of the previous Excursion, $F = 2F_C q\lambda/r$, or, since the force is directed, adjusted to the direction $(c, s, 0)^T$,

5. Show that the 3D divergence of $(x, y, z)/r^3$ is zero as stated in Note 4.

6. Show that the 2D divergence of $(x, y)/r^2$ is zero.

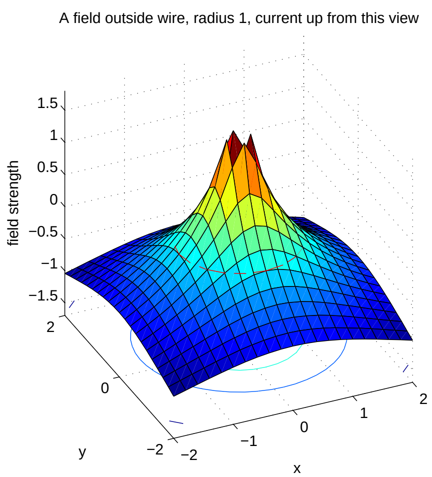
7. The field point of view in Note 4 is uncomfortably asymmetrical. The source charge causes a field but the test charge apparently does not. Discuss.

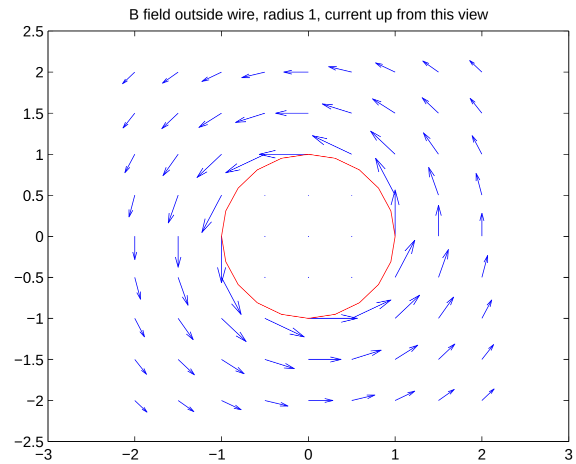
8. Feynman [FLS64] starts from Maxwell's equations (Note 6 above) and explores their use in multitudinous applications in detail. He also makes a couple of remarks which indicate that it is better to start with relativity than with Maxwell: he points out [FLS64, p.13-10] that Faraday's lines of force do not transform properly to moving frameworks, and he says [FLS64, p.13-12], of course, that the handedness implicit in the curl operator is not physical. His volume on electromagnetism, although mostly resolutely classical, gives all the clues I've needed to develop the subject using relativity and, later, quantum physics.

9. **Visualizing magnetic fields.** The $\vec{E}$ and $\vec{B}$ fields of Note 6 are the conventional representation of electromagnetism despite their difficulties, especially those of the magnetic field $\vec{B}$. For electrostatics the $\vec{E}$ field is already familiar from Newtonian gravitation (although its dependence also on a time-varying magnetic field adds new twists).
   In this Excursion we calculate various static magnetic $\vec{A}$ and $\vec{B}$ fields produced by different configurations of fixed currents.
   The calculations are based on the fields generated by a current in a wire (of radius 1):

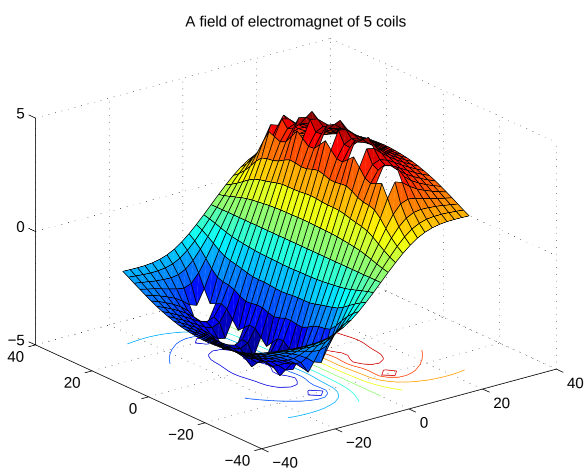
$$\begin{aligned} \vec{A} &\propto -\ln(r)(0, 0, 1) \\ \vec{B} &\propto \vec{\text{curl}}\vec{A} = (-y, x, 0)/r^2 \end{aligned}$$

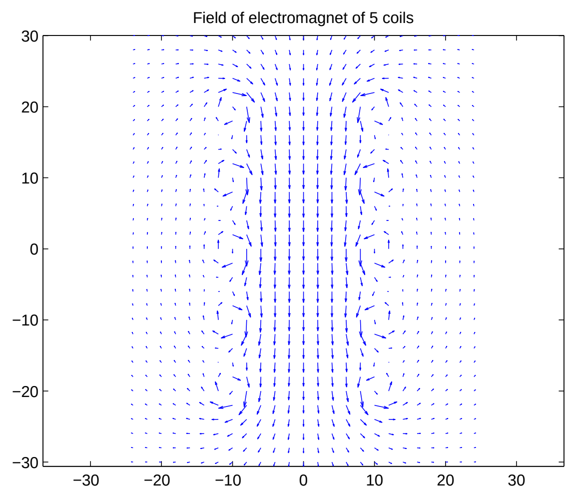
Write programs `wireSection2A` and `wireSection2B` to generate the plots





Next extend this to a single loop of wire with programs `loopSection2Aplot` and `loopSection2Bplot`. Note that the fields are additive: just add the two scalar results for A or the two vector results for $\vec{B}$. (In 2D A is scalar.)
Finally extend the loop calculation to five loops stacked on top of each other. You'll need to revisit your loop programs a) to include a parameter for the vertical position of each loop and b) to refrain from plotting until the field contribution from all five loops are added up: programs `coilSection2A` and `coilSection2B`.





This last ("coil") is called a "solenoid". Notice that the $\vec{B}$ field almost disappears outside the solenoid but is very strong inside.
The inner field can be reinforced by giving the electromagnet a core of iron or other material whose ferromagnetism (Note 34 of Book 9c, Part IV) reinforces the inner field.
In the limit of an infinite solenoid the $\vec{B}$ field disappears altogether outside.

10. Look up James Clerk Maxwell (1831–1879). What is he famous for besides electromagnetism?

11. Any part of the Prefatory Notes that needs working through.

References

[FLS64] R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume II.* Addison-Wesley, 1964.

[Kur12] Ron Kurtus. Moving electrical charges create magnetic field. URL http://www.school-for-champions.com/science/magnetic_field_moving_charges.htm Last accessed 16/4/26., 2012.