Excursions in Computing Science:
Book 8d. Rocket Science.
Part II Orbits.

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I. Prefatory Notes

Part I Propulsion.

1. The rocket equation.
2. Specific impulse.
3. Fuels.
4. Multistage rockets.
5. Thrust.
6. Photon sails.
7. Solar wind.

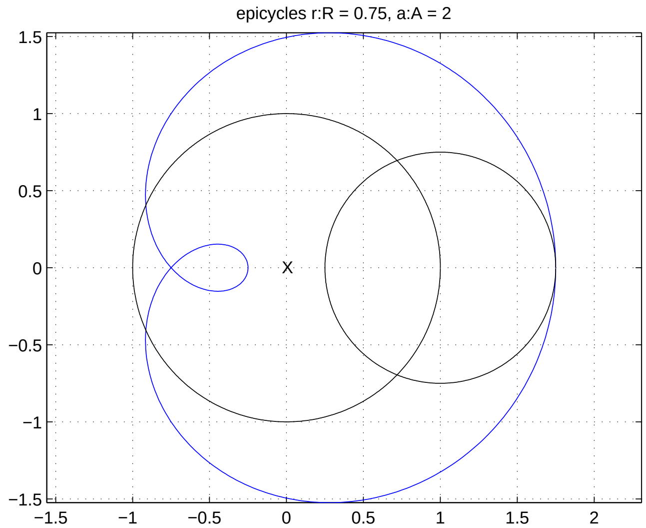
8. Ideology. Once it was believed that the planets orbit in circles around the Earth. This should be obvious: the Earth is the centre of the universe because everything falls towards it; and circles are the perfect form so anything that is not imperfect, which is to say, not on Earth, must move in circles.

Unfortunately it contradicted observation. If you watch, say, Mars, night after night, you'll see that it moves across the sky from one night to the same time next night in a west-to-east direction. Except occasionally it backtracks. (Mars is easy to spot as a reddish planet.)

The Moon is OK. The Sun is almost OK, except it rises and sets much further north in the summer than in the winter (south if you're in the southern hemisphere, whose winter is at the same time as summer in the north).

So for the planets at least, to preserve the notions of the central Earth and the perfect circle, the people who liked to watch them decided that they followed circles *on top of* the circular orbits: *epicycles*.

Can you see how the blue orbit appears to backtrack when viewed from position **X**? The epicycle is 3/4 the size of the main circle and is turning twice as fast.



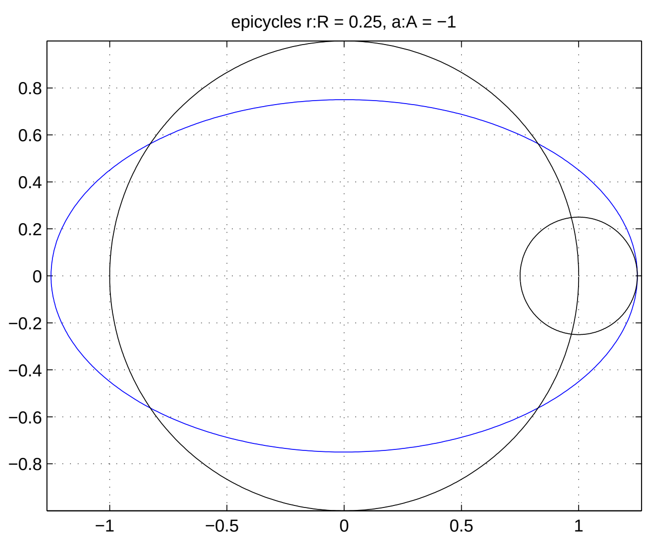
We can make a wonderful variety of patterns from epicycles of different sizes and different speeds. But even these were not enough to fit all the data and the astronomers had to resort to epicycles on epicycles.

Finally Kepler said "enough!" and changed the rules. He said:

- the planets move in *ellipses*; and
- they go around the *Sun* not the Earth.

So we're going to have to study ellipses as well as circles.

A curious thing happens when the epicycle is, say, a quarter of the main circle and turns at the same rate but *backwards*.

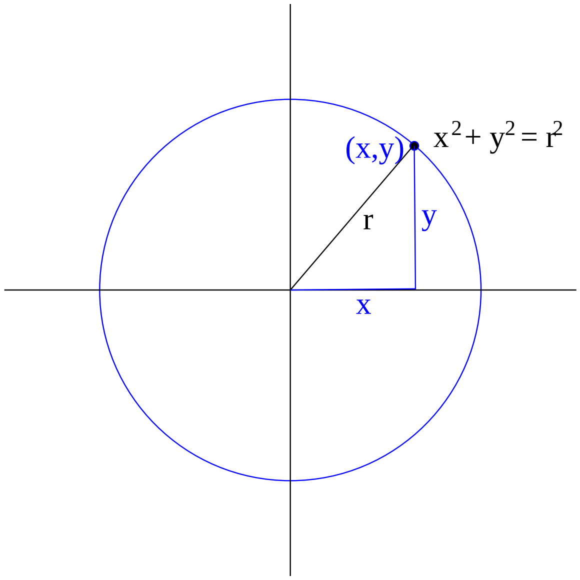


But this is not the way we'll study ellipses. [1]

There was a nasty fight over Kepler's new ideas, but now nobody uses epicycles. Was this the victory of mere convenience over High Principle, or were Kepler's precedessors wrong?

[1] A little trigonometry gives, for this epicycle of radius r on the main circle of radius R, $x = (R + r)\cos\theta$ and

9. Circles and ellipses. Here is a circle on the "x–y plane".



In this representation, (x, y) gives the two numbers needed for any point on the blue circle. x is the horizontal distance, positive to the right, and y is the vertical distance, positive upwards. We are not free to use any numbers. Once we have specified x, say, the value of y is almost pinned down. This constraint is given by the "equation of the circle"

$$x^2 + y^2 = r^2$$

r is the *radius* of the circle, which is the distance of any point on it from the centre, which here is the origin, where the two axes meet.

Here are some possible values for x and y if the circle has radius $r = 5$.

x	y
5	0
4	3
3	4
0	5
−3	4
−4	3
−5	0
−4	−3
−3	−4
0	−5
3	−4
4	−3

Find the positions of these points on the circle and check that they obey the equation of the circle. For example, $(x, y) = (-3, 4)$

$$x^2 + y^2 = (-3)^2 + 4^2 = (-3) \times (-3) + 4 \times 4 = 9 + 16 = 25 = 5 \times 5$$

$y = (R - r)\sin\theta$, from which

$$\frac{x^2}{(R+r)^2} + \frac{y^2}{(R-r)^2} = 1$$

which you can compare to the ellipse equation in Note 9.

From the point of view of this math, an ellipse is a *generalization* of the circle.

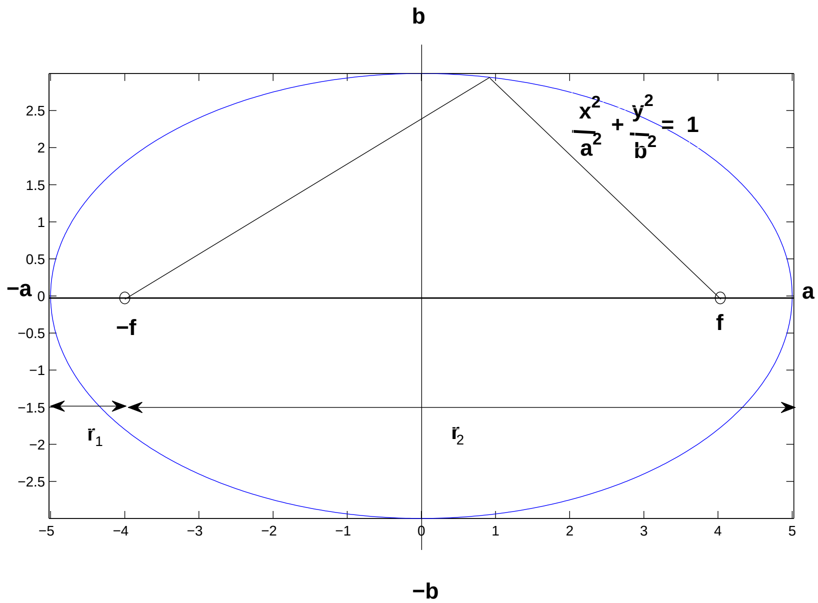
If we divide both sides of the equation of the circle by r^2 we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

The *generalization* is to let the denominator be different for each fraction

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Here is what this looks like.



You should check that when $x = a$, y must be 0, and that when $y = b$, x must be 0. And the same for the negative values $x = -a$ or $y = -b$. Thus $2a$ is the width of the ellipse, in this example its *major axis*, and $2b$ is the height, its *minor axis*.

To fix our ideas, this ellipse is drawn for $a = 5$ and $b = 3$. It could be tall and skinny instead of short and fat. And when $a = b$ we have the special case of a circle. That is the sense in which the ellipse generalizes the circle.

Instead of having a single centre the ellipse has two focus points called *foci*. The constant is not a single radius from the centre to the curve but the *sum of the two distances* from the foci to the curve.

These two distances are shown for an arbitrary point.

If that point were, specially, the rightmost point of the ellipse, $(x, y) = (a, 0)$, the sum of the two distances is $2a$: look carefully.

We can consider instead the two (equal) distances to the top point of the ellipse $(x, y) = (0, b)$. Then there is a triangle of sides f and b and hypotenuse a. That hypotenuse comes from the distances being equal and summing to $2a$.

So

$$f^2 + b^2 = a^2$$

and you can check that this gives $f = 4$ in the example where $a = 5$ and $b = 3$.

The two other quantities shown are two radii I'll need for circular orbits in Note 15. They satisfy

$$r_1 + r_2 = 2a$$

which is easy to see, as is

$$r_2 - r_1 = 2f$$

So

$$b^2 = \left(\frac{r_1 + r_2}{2}\right)^2 - \left(\frac{r_1 - r_2}{2}\right)^2 = r_1 r_2$$

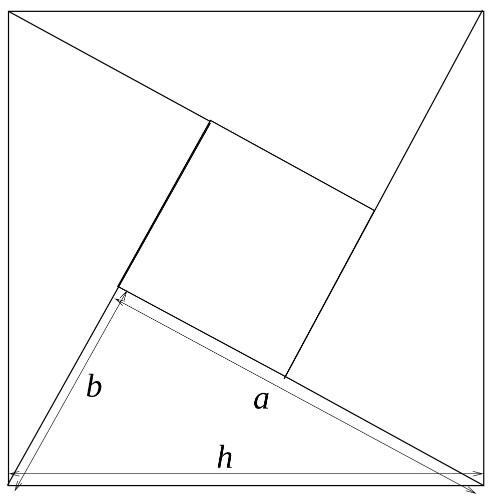
and

$$f^2 = a^2 - b^2 = (a - b)(a + b)$$

10. Pythagoras. I've freely used "Pythagoras' theorem" in the previous Note, that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides. And I've used algebra, which has letters standing for numbers.

This Note backtracks to explain these. If you had no difficulties, by all means skip it.

A right-angled triangle has one angle of 90 degrees. Its *hypotenuse* is the side opposite that right-angle, which is the longest side. Here is the proof of the theorem. In the diagram the hypotenuse of the lower right-angled triangle is labelled h.



This depends on knowing that the area of the big square is h^2 and figuring out that the side of the small square is $a - b$ and so its area is $(a - b)^2$. And on knowing that the area of a triangle with perpendicular sides a and b can be doubled to make the area ab of a rectangle.

The result is that

$$h^2 = 4\frac{ab}{2} + (a - b)^2 = a^2 + b^2$$

because $(a - b)^2 = (a - b) \times (a - b) = a^2 - 2ab + b^2$:

$\times$	a	$-b$
a	a^2	$-ab$
$-b$	$-ab$	b^2

In this "times table" I've left out the + signs in $a + (-b) = a - b$ along the top and down the left side. So we must remember to put them back in again once we've done the multiplications.

The two $-ab$ on the top-right to bottom-left diagonal sum to $-2ab$ and that's as far as we can go towards the answer.

We're still doing arithmetic with letters in all this, so if that is not clear here is what it means.

We're using letters to stand for *any* numbers. The easiest way to get comfortable with this is, when you see a letter, *invent* a number it might stand for. The try it again with a different number, and repeat until you can see that it is going to work for any number.

Here's an example. For $(a-b)^2 = a^2 - 2ab + b^2$ let's try 5 for a and 3 for b. Then, working both sides in parallel

$$\begin{aligned} (a-b)^2 &=? \quad a^2 - 2ab + b^2 \\ (5-3)^2 &=? \quad 5^2 - 2 \times 5 \times 3 + 3^2 \\ 2^2 &=? \quad 5^2 - 2 \times 5 \times 3 + 3^2 \\ 4 &=? \quad 25 - 30 + 9 \\ 4 &= \quad 4 \end{aligned}$$

we get the same thing, which encourages us to believe the original equation in letters.

We'll probably have to do it again, using different numbers, until tedium induces us to accept the original equation.

Note that we can put two letters side-by-side to indicate multiplication ($ab = a \times b$) which we cannot do with numbers ($2 \times 5 \neq 25$).

Note that different letters say that the numbers we use for them *can* be different but need not be. I could have used 5 for both a and b,

11. Circular orbits. As it happens most of the planets have essentially circular orbits. For planets, the important part of Kepler's innovation was to shift the centre from the Earth to the Sun.

Circular orbits are easier to work with than elliptical and a good starting point for us.

Like a ball on a rope which you are swinging around your head, the planet is kept a fixed distance from the sun by a force which attracts it towards the sun with an acceleration

$$\frac{GM}{r^2}$$

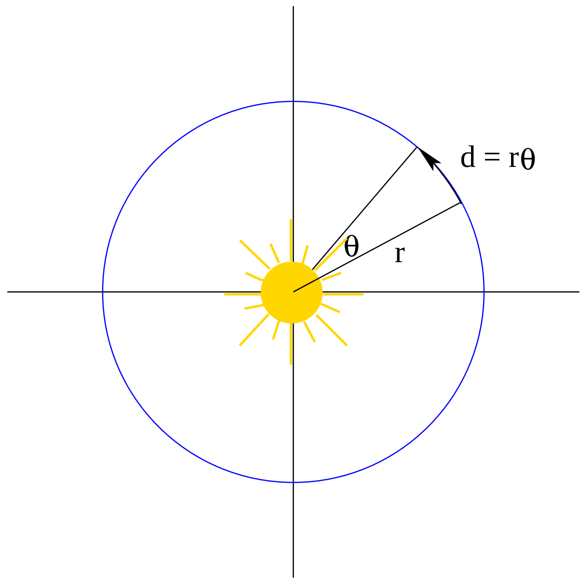
This formula is due to Newton: r is the distance from the Sun, M is the mass of the Sun, and G is Newton's *gravitational constant*. We'll assume we know all these numbers, and will come back soon to see how, and what they are.

Otherwise the planet would keep going in a straight line, that is, with a tendency to accelerate *away* from the Sun.

For it to follow the circular orbit, these two accelerations must balance.

We need to know something about angles to figure out the acceleration away from the centre (*centrifugal*: centre-fleeing).

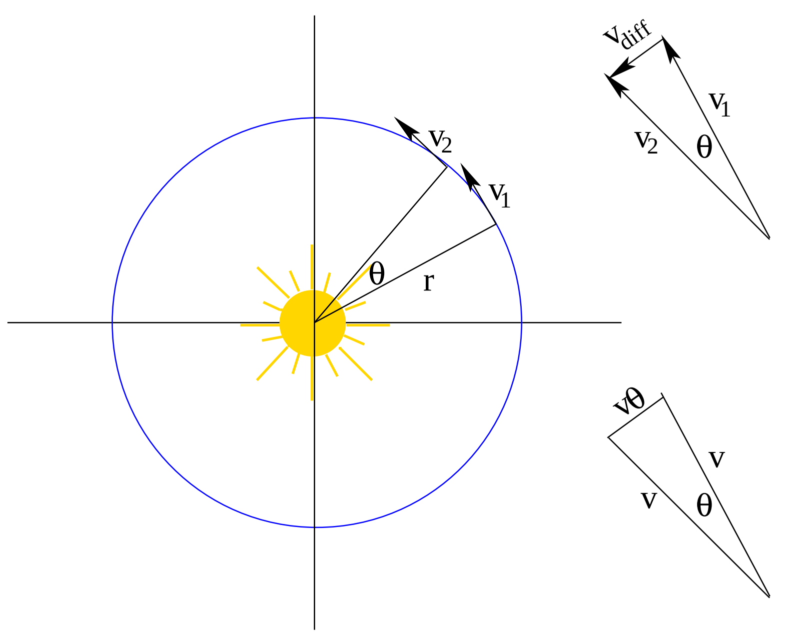
Instead of dividing the circumference of the circle—the orbit—into 360 *degrees* we'll use the measure of angle that is given by how far it takes us around the circumference, relative to the radius.



Thus an angle θ takes us $d = r\theta$ meters along the circumference, where the radius r is given in meters. (Or kilometers (1000 meters) for both, or even megameters (1000 kilometers), gigameters (1000 megameters) or terameters (1000 gigameters) which would be more suitable for planetary orbits)

(A full revolution, instead of being 360 degrees, would be $2\pi r/r = 2\pi$ "radians", which is the name of this way of measuring angles.)

Now let's look at the change in velocity of the orbiting planet which would be required to keep it in the circle.



For the planet to move a constant speed v in its orbit we can say that v is the *magnitude*—that is their length—of each of the arrows $\vec{v_1}$ and $\vec{v_2}$.

So in the lower diagram on the right, v plays the same role as r does for the circle on the left.

Thus the length of the change v_{diff} in velocity is

$$v\theta$$

in the same way that $r\theta$ was the length of the arc of the circle swept out by the same angle θ.

Now a change in velocity is an acceleration. We just must get the time t into the formula.

If we say that the *rate of change* of the angle is $\dot{\theta}$ then

$$\theta = \dot{\theta}t$$

But $\dot{\theta}$ leads to the rate of change of velocity, which is exactly the acceleration a that we need,

$$a = (\dot{v\theta}) = v\dot{\theta}$$

where the orbital speed v (the magnitude of the orbital velocity) is constant in time so all the change lies in the angle θ.

Also, velocity itself is the rate of change of distance travelled, d

$$v = \dot{d}$$

Putting all these together

$$\begin{aligned} d = vt &= \dot{d}t = r\dot{\theta}t \\ \dot{\theta} &= \frac{v}{r} \\ a &= v\dot{\theta} = \frac{v^2}{r} \end{aligned}$$

This would be the centrifugal acceleration if there were no Sun holding the planet in its orbit.

We balance the two opposing accelerations

$$\begin{aligned} \frac{GM}{r^2} &= \frac{v^2}{r} \\ v^2 &= \frac{GM}{r} \end{aligned}$$

12. Finding and using GM_{Earth}. So far G and M for the Sun, have not appeared separately but only as a product. The reason we have two quantities is so that we can keep G universal and vary M according to the centre of attraction. If we change that centre to the Earth, so that the Moon or any artificial satellites are the orbiting bodies, we can make a measurement which gives us the product GM_{Earth}.

The acceleration due to gravity at the surface of the Earth can be measured as about 9.8 meters per second per second, or even more approximately as 10 m/s^2.

This can be done, for example, with an inclined plane with top surface not flat but scooped into a wide circular groove so that a ball rolling down the plane can be made to oscillate from side to side. The oscillations give fixed time intervals and the distance of the ball from the top of the plane at the extreme of each oscillation will increase as $gt^2/2$ from which the acceleration g can be calculated.

We also need to know how far we are from the centre of the Earth. The Earth's radius can be found to be about 6.4 megameters by measuring the angle of, say, some point on the Moon, simultaneously from two widely separated locations on Earth whose latitude and longitude are known.

Now we have the acceleration from Newton's formula

$$\begin{aligned} g &= \frac{GM_{\text{Earth}}}{r^2} \\ GM_{\text{Earth}} &= gr^2 = 10 \times (6.4_{10}6)^2 = 400_{10}12 \end{aligned}$$

to one significant figure.

We can use this to find out how far away the Moon is.

We had $v^2 = GM/r$ but this velocity v is the circumference of the Moon's orbit divided by its period T which is four weeks or 28 days.

$$\begin{aligned} \sqrt{\frac{GM}{r}} &= v = \frac{2\pi r}{T} \\ T^2 &= \frac{4\pi^2 r^3}{GM} \\ r^3 &= \frac{T^2 GM}{4\pi^2} \end{aligned}$$

The last two equations, giving T in terms of r and vice-versa, are expressions of another discovery of Kepler's, that the square of the period is proportional to the cube of the radius.

The third of the above equations, with M_{Earth}, gives the distance of the Moon.

$$r = \sqrt[3]{\frac{T^2 GM_{\text{Earth}}}{4\pi^2}} = \sqrt[3]{\frac{(28 \times 24 \times 3600)^2 \times 400_{10}12}{40}} = 400 \text{ megameters}$$

to one significant figure. (The calculation actually gives 388 megameters, and careful angular measurements from opposite sides of the Earth give 384 megameters.)

We've just made the connection between the Moon's orbit and an apple falling that Newton made when he was grounded from university by a plague and had to stay home and think for himself. That was some thinking, 450 years ago.

"If we want to know the distance to the Moon, why don't we just look it up?" Because what we're doing here is learning how to find out things nobody had found out before us. So we're putting ourselves in Newton's shoes.

13. Weighing Earth and Sun. We haven't separated G and M so far. But we're going to have to as soon as we get away from things orbiting the Earth and return to discussing planets.

A century after Newton, Cavendish finally did the painstaking work of measuring G. This required actually measuring the miniscule force between two masses in the laboratory. Since Newton told us $F = ma$ (force is mass times acceleration) Cavendish had to compare

$$\frac{GMm}{r^2}$$

with the offsetting force of twisting a very fine fibre from which the smaller mass m was suspended—actually, two of them, on the ends of a long arm near which two of the larger mass M were placed to attract them.

What we'll use for the result of Cavendish's and more modern measurements is

$$G = \frac{200}{3} \text{ newton meter}^2 \text{ per gigagram}^2$$

This means that two 1000-tonne masses placed 1 meter apart experience a force of 66.7 "newtons". One newton will accelerate a 1 kilogram mass at 1 meter per second^2, so the 200/3 newtons would accelerate one of the gigagram masses at 67 *microns* per second per second or about seven millionths of g, the acceleration due to gravity at the Earth's surface.

Needless to say, Cavendish suspended substantially smaller masses than a gigagram from his fibre.

Now we can weigh the Earth. Or at least get its mass. (Weight is a force, actually the force that a mass exerts under Earth's surface gravity acceleration of g.)

$$M_{\text{Earth}} = \frac{GM_{\text{Earth}}}{G} = \frac{400_{10}12}{(200/3)_{10}{-12}} = 6_{10}24 \text{ kilograms}$$

There is no official prefix for a number this big.

The mass of the Sun gets worse.

We can find GM_{Sun} from the orbital period of Earth together with its distance, 1 "astronomical unit" AU = 0.15 terameters, from the Sun.

Measuring the astronomical unit, the radius of Earth's orbit r_E, has been historically challenging. One way involves using radar to measure the distance d_{EV} between Earth and Venus, and then using the relative periods and the relationship between period and radius.[2]

We can rearrange the equation for the period-radius relationship

$$GM_{\text{Sun}} = 4\pi^2 \frac{r^3}{T^2} = 40 \frac{(0.15_{10}12)^3}{(365.25 \times 24 \times 3600)^2} = \frac{400}{3} 10^{18}$$

So

$$M_{\text{Sun}} = \frac{GM_{\text{Sun}}}{G} = \frac{(400/3)_{10}18}{(200/3)_{10}{-12}} = 2_{10}30 \text{ kilograms}$$

to one significant figure (and good to within 1% of the currently accepted value).

14. The Solar planets. Eventually, nine planets were discovered (including the now demoted Pluto), their periods observed and their orbital radii deduced.

Planet	Period, T		Radius, r			Speed
	Gsecs	Days 2,3 sig fig	Tmeters	AU 1 sig fig	light minutes	Km/s
Mercury	0.0076	88	0.058	0.4	3	47.7
Venus	0.0194	225	0.108	0.7	6	35.1
Earth	0.0316	365	0.150	1	8	29.8
Mars	0.0593	687	0.227	1.5	13	24.2
Jupiter	0.372	4310	0.778	5	43	13.1
Saturn	0.931	10800	1.427	10	79	9.6
Uranus	2.65	30700	2.87	20	160	6.5
Neptune	5.21	60300	4.50	30	251	5.4
Pluto	7.83	90600	5.90	40	329	4.7

The table gives the period in gigaseconds and then, approximately to at most 3 significant figures, in days. It gives the radius in terameters, in astronomical units to one significant figure, and in light minutes. The data in the Gsecs and Tmeters columns all satisfy Kepler's relationship $r^3/T^2 = 3.36$ to within 2 percent.

I've included the imprecise alternatives because those numbers are fairly easy to memorize should you want to do so. And I've included the radii in light minutes so you can work out the ranges of communication delays between question and answer. From Earth to Mars, for instance, there is a

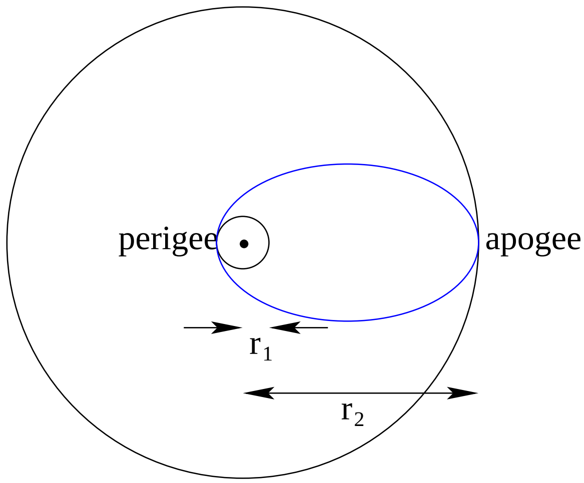
[2] For instance, if the radar measurement is made when the planets are aligned, $r_E = r_V + d_{EV}$ if they are on the same side of the Sun, $r_E = d_{EV} - r_V$ if they are on opposite sides; or if Earth-Sun-Venus forms a right-angled triangle $r_E^2 = d_{EV}^2 - r_V^2$. Any of these can be combined with $r_V^3 = (T_v/T_E)^2 r_E^3$ to give a solvable equation for r_E.

delay of at least five minutes when the planets are closest in their orbits, and of at most 21 minutes when they are farthest apart.

I've also calculated and included orbital speeds of each planet in kilometers per second, for comparison with ΔV calculations in Part I.

15. Transfer orbits. The first calculation of orbital rocketry we can discuss is transfer from a circular orbit of radius r_1 to another circular orbit of radius r_2.

The most fuel-efficient way of doing this is to ignite the rocket long enough to change from the r_1 circular orbit to an elliptical *transfer orbit*, coast up to r_2, then do another burn to change to the r_2 orbit.



The diagram shows that an increase in velocity at "perigee", in the same direction that you are already going in the r_1 circular orbit, will give you a centrifugal acceleration, taking you away from the r_1 circular orbit.

Kepler says that the new orbit must be an ellipse. It cannot be a circle because you were at radius r_1 when you started the rocket, which has made you go too fast to stay in that r_1 circle.

Similarly at "apogee" when you have reached radius r_2 on the elliptical orbit, a rocket burn in the direction you are already going will increase your velocity and give you a new orbit.

That orbit is another ellipse, but if you do the burn right, that ellipse is the r_2 circular orbit you want.

The calculations we need are those to give the right ellipse for the transfer orbit, and to give correctly the r_2 circle.

An example we'll work through will take us from "low Earth orbit" at, say, the International Space Station, to "geosynchronous orbit", at which our orbital period is 24 hours so that we appear to remain stationary above a particular point on the Earth. For these two orbits

$$\frac{r_2}{r_1} = \frac{42164}{6371 + 408} = 6.2$$

Another orbital transfer, with almost the same geometry, would be from Earth's orbit around the sun to Saturn's orbit. (Then "perigee" becomes "perihelion" and "apogee" becomes "aphelion".)

$$\frac{r_2}{r_1} = \frac{9.5}{1} = 9.5$$

(The diagram is not to scale for either of these but shows $r_2/r_1 = 9$.)

We already know the orbital velocities for the circular orbits from Note 11

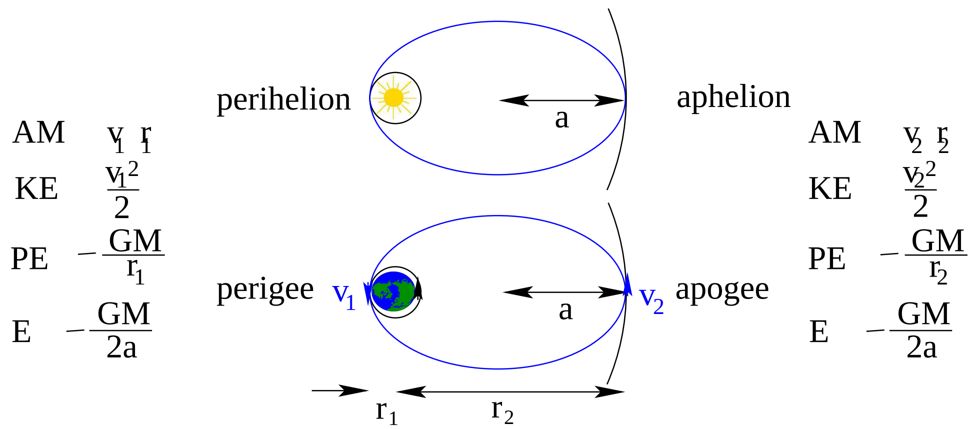
$$v = \sqrt{\frac{GM}{r}}$$

where $M = M_{\text{Earth}}$ (Note 12) for the transfer to GEO or $M = M_{\text{Sun}}$ (Note 13) for the transfer to Saturn.

16. Velocities in elliptical orbits. To get the ellipse right for the transfer orbit we must learn about *momentum* and *energy*.

These are both *conserved* quantities—no matter what happens, they are the same after as before. These conservations help our calculations a lot.

Let's expand our diagram to include these quantities: at the low point of the ellipse on the left and at the high point on the right.



- "AM" stands for *angular momentum* vr, which is conserved, so

$$v_1 r_1 = v_2 r_2$$

.

- "E" stands for *energy* and is the sum of *kinetic energy*, "KE", and *potential energy*, "PE". Total energy is conserved, so

$$\frac{v_1^2}{2} - \frac{GM}{r_1} = \frac{v_2^2}{2} - \frac{GM}{r_2}$$

(I haven't yet shown that the total energy can also be written $-GM/(2a)$. But it is the same on both sides.)

The above two equations enable us to calculate the two velocities v_1 at perigee (perihelion) and v_2 at apogee (aphelion).

I'll do this here and then come back to explain about momentum and energy.

From angular momentum conservation we can get v_2 in terms of v_1 or vice-versa.

$$v_1 = \frac{r_2}{r_1} v_2 \qquad\qquad v_2 = \frac{r_1}{r_2} v_1$$

We'll also use

$$r_1 + r_2 = 2a$$

from the diagram and Note 9.

Here is the discussion for v_1, starting with the conservation of energy.

$$\begin{aligned}
\frac{v_1^2}{2} - \frac{GM}{r_1} &= \frac{v_2^2}{2} - \frac{GM}{r_2} \\
\frac{1}{2}(v_1^2 - v_2^2) &= GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \\
\frac{v_1^2}{2}\left(1 - \frac{v_2^2}{v_1^2}\right) &= GM\frac{r_2 - r_1}{r_1 r_2} \\
\frac{v_1^2}{2}\left(1 - \frac{r_1^2}{r_2^2}\right) &= GM\frac{r_2 - r_1}{r_1 r_2} \\
\frac{v_1^2}{2}\frac{r_2^2 - r_1^2}{r_2^2} &= GM\frac{r_2 - r_1}{r_1 r_2} \\
\frac{v_1^2}{2}\frac{r_2 + r_1}{r_2} &= GM\frac{1}{r_1} \\
\frac{v_1^2}{2}\frac{2a}{r_2} &= GM\frac{1}{r_1} \\
\frac{v_1^2}{2} &= GM\frac{r_2}{2ar_1} \\
&= GM\frac{2a - r_1}{2ar_1} \\
&= GM\left(\frac{1}{r_1} - \frac{1}{2a}\right)
\end{aligned}$$

The discussion for v_2 is similar, or we can shortcut from the beginning of the last three lines above.

$$\begin{aligned}
\frac{v_2^2}{2} &= \frac{v_1^2}{2}\frac{r_1^2}{r_2^2} = GM\frac{r_1}{2ar_2} \\
&= GM\frac{2a - r_2}{2ar_2} \\
&= GM\left(\frac{1}{r_2} - \frac{1}{2a}\right)
\end{aligned}$$

In summary the two velocities are

$$v_1 = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)} \qquad v_2 = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)}$$

Newton generalized these to the *vis viva equation* for any velocity on the ellipse, distance r from the central body (Earth or Sun).

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

This is as it must be to conserve total energy: to the kinetic energy $GM(1/r - 1/(2a))$ we must add the potential energy $-GM/r$.

We can now see that the angular momenta at these extremes are the same.

$$v_1 r_1 = \sqrt{GM\left(2r_1 - \frac{r_1^2}{a}\right)} = \sqrt{GM\frac{b^2}{a}} \qquad v_2 r_2 = \sqrt{GM\left(2r_2 - \frac{r_2^2}{a}\right)} = \sqrt{GM\frac{b^2}{a}}$$

By conservation this is the angular momentum for all r. But we cannot write it vr because we must use the component of v perpendicular to r and this is vr itself only at the extremes.

Before we complete the argument to find the *changes* in velocities we need for the orbit changes, we should backtrack to momentum and energy.

17. Momentum and kinetic energy. Three conservation laws capture a lot of physics.

- Linear momentum $mv =$ constant.
- Angular momentum $mvr =$ constant.
- Energy = constant.

Because they are physics, they are based on experience. They can be explained in terms of other aspects of physical experience but only with a great deal of sophistication.

For us, we must just accept them.

But we can discuss them. First m and v are the mass and velocity of a moving body. If the body is "attached" (maybe by a rope, maybe by gravity) to some central point which it is orbiting, r is the distance of the body to the central point.

In our discussion of orbits we do not need conservation of linear momentum. But it is a fundamental property and the easiest to understand (although not that easy), so I'll discuss it first.

Let's compare what happens when a marble hits another marble with what happens when it hits a billiard ball. Let's suppose the second marble and the billiard ball are initially stationary.

In the first case the other marble will roll away quite fast.

In the second case, the billiard ball will also roll away, but much more slowly.

The difference is the difference in mass. Suppose the marbles are 1.5 grams each and the billiard ball is 150 grams.

Then it seems likely that the billiard ball rolls away 100 times slower than the second marble.

This is not quite true. We'll need to solve two equations to see what happens exactly. Conservation of momentum gives us the first equation.

Suppose m is the mass of each of the marbles, M the mass of the billiard ball, v_0 is the initial velocity of the first marble, v is the final velocity of the first marble, and u is the final velocity of the other object, be it the other marble in the first example or the billiard ball in the second.

Then conservation of linear momentum gives

in the first example	and, in the second example
$mv_0 = mv + mu$	$mv_0 = mv + Mu$

If v were zero (it's not) then u would be 100 times less in the billiard ball case than in the case of the second marble.

To resolve this we need the second equation and we get it from conservation of energy.

I did not give a formula for energy yet because there are many different kinds. But for the collisions we are discussing, only *kinetic* energy is involved.

$$\text{Kinetic energy} = \frac{mv^2}{2}$$

This also depends only on mass m and velocity v but in a different way.

The difference can be seen if the first marble hits a *wall* but we won't go into that.

For these collisions the only energy is kinetic. Conservation of energy here means conservation of kinetic energy. We have a second equation for each example.

In the first example

$$m\frac{v_0^2}{2} = m\frac{v^2}{2} + m\frac{u^2}{2}$$

and, in the second example

$$m\frac{v_0^2}{2} = m\frac{v^2}{2} + M\frac{u^2}{2}$$

If we divide everything by m the two equations for the second example give

$$\begin{aligned}
\left(v + \frac{M}{m}u\right)^2 &= v^2 + 2\frac{M}{m}uv + \left(\frac{M}{m}\right)^2 u^2 = v^2 + \frac{M}{m}u^2 \\
v &= \frac{1}{2}\left(1 - \frac{M}{m}\right) u \\
v_0 &= \frac{1}{2}\left(1 + \frac{M}{m}\right) u
\end{aligned}$$

The last two give u from v_0 and v from u.

For the first example, we just set $M = m$ and so

$$\begin{aligned}
v &= 0 \\
u &= v_0
\end{aligned}$$

which is to say that the first marble stops dead and the second marble rolls away with the original velocity of the first marble.

For the billiard ball $M/m = 100$ and the two results boil down to

$$\begin{aligned}
u &= \frac{2}{1 + M/m}v_0 = \frac{2}{101}v_0 \\
v &= \left(\frac{1 - M/m}{1 + M/m}\right) v_0 = -\frac{99}{101}v_0
\end{aligned}$$

and you should check that momentum and energy *are* conserved.

Note that the billiard ball starts moving not with 1/100 the speed of the initial marble but about twice this, and that the marble recoils with almost its full speed but backwards because of the minus sign.

You can also see what happens if the marble hits a wall: M/m becomes very large so u is effectively zero and the marble recoils with its full, if negative, original speed.

That was a long discussion of linear momentum and kinetic energy. But it gave you actual numbers to explore.

In orbital problems we are concerned with angular momentum. The mass of the Earth or of the Sun is enormous compared with the mass of a spacecraft, so the central body rather acts as though it were the wall of the linear momentum discussion. Linear momentum in this case however does not tell us much.

Angular momentum concerns motion around a central point, a distance r away. It is most easily seen in circular motion. A spinning top keeps spinning, until friction slows it down. A figure skater in a spin brings her arms in, reducing the radius, and so spins faster. Most experiments you can do to illustrate angular momentum will make you dizzy so these examples must suffice.

The combination of velocity and radius was given in the previous Note for the perigee angular momentum mv_1r_1 and the apogee angular momentum mv_2r_2.

In that Note I left off the m just because it is the same everywhere and so is just an additional symbol which we don't need. I was really defining the *specific* angular momentum, per unit mass.

I did the same thing for *specific* energy, both kinetic and potential.

18. Potential energy. In a gravitational setting the other form of energy is *potential* energy.

When you stand on top of a hill you have energy which is not kinetic—you are not moving. That energy is *potentially* kinetic energy. If the hill were smooth and you were a ball, you would roll down it acquiring speed. Your speed at the bottom would give you kinetic energy equal to the potential energy you had at the top.

In the uniform, non-varying, gravitational field close to the surface of the Earth, your potential energy at height h is mgh where m is your mass and g ("one gee") is the acceleration due to gravity at the Earth's surface, about 10 meters per second per second (9.8 is a slightly better approximation).

So you can immediately calculate how fast you would be going after falling off an 80 meter cliff.

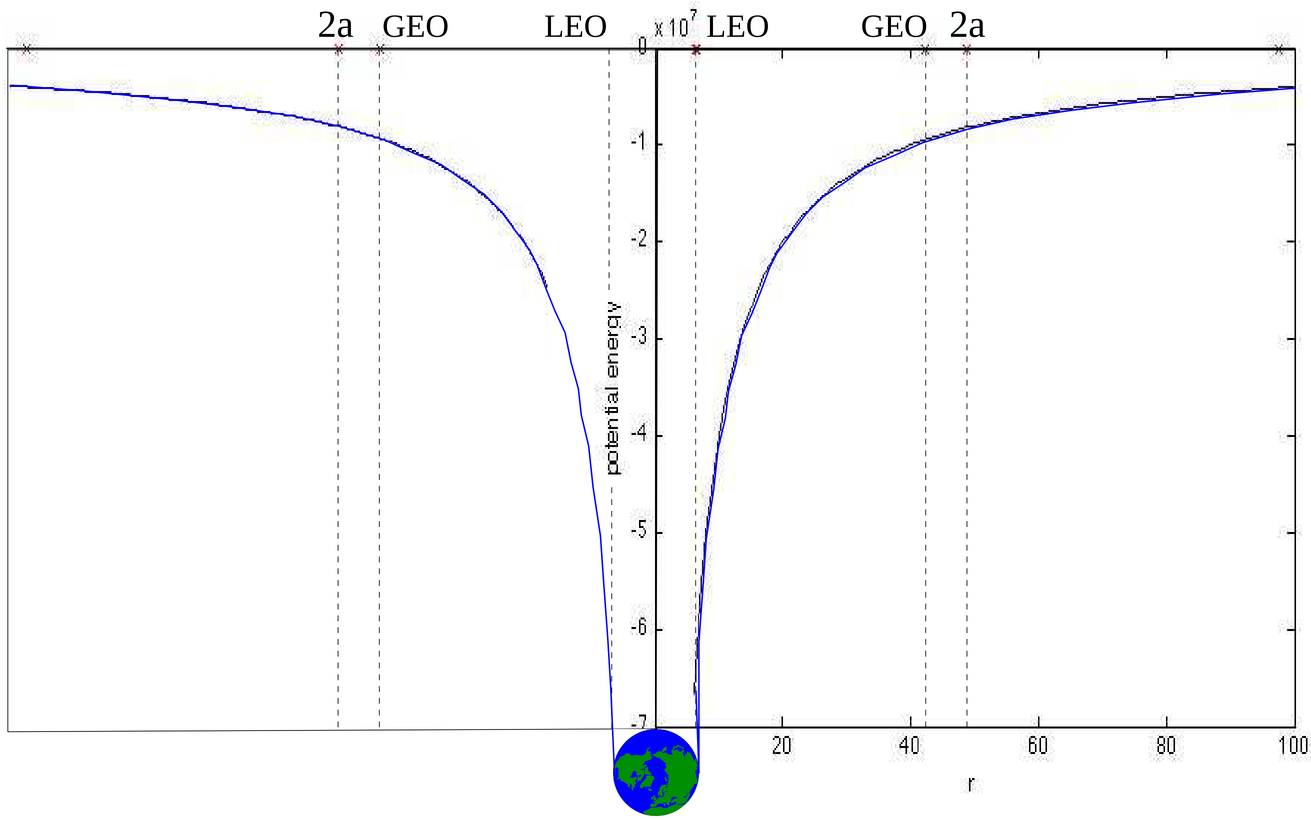
$$\begin{aligned} m\frac{v^2}{2} &= mgh \\ v &= \sqrt{2gh} = 40 \text{ meters/sec} \end{aligned}$$

Note that the m disappears because it occurs on both sides of the equation. So we can also here use *specific* energy.[3]

In interplanetary space the gravitational potential energy is more complicated and goes as $-GM/r$, as given in Note 16. Again, this is a *specific* energy and the formula would usually be multiplied by m.

We can think of this as a hill, too, which gets steeper and steeper the closer we get to the centre.

[3]It is a matter of some sophistication that m is the same on both sides of the equation. On the kinetic energy side m is *inertial* mass but on the potential energy side it is the *gravitational* mass. Einstein saw the distinction and then *postulated* that two are equivalent, leading him to his general relativity theory of gravity.



Note that the minus sign in $-GM/r$ means that the potential energy is always negative, climbing to zero only at $r = \infty$. This is OK because in the end it will only be *differences* in energy that matter. And it enables us to see the potential energy as a "well" into which you will fall unless you have an orbital speed whose centrifugal acceleration counters the downward acceleration.

That gravitational acceleration is GM/r^2, as some sophisticated math (calculus) can show directly from the potential energy.

19. Delta-V for orbit changes. Now we can use the vis viva equation from Note 16 to calculate the velocity changes needed to make the two orbit changes from the r_1 circular orbit to the elliptical transfer orbit, and from that to the r_2 circular orbit.

These velocity changes will be just the "delta-V" we will need to determine fuel requirements from the rocket equation of Note 1 in Part I.

We will be able to express all the delta-V in terms of the r_1 orbit velocity $v_1 = \sqrt{GM/r_1}$ and the ratio $\rho = r_2/r_1$ of the two orbital radii.

r_1 to transfer

$$
\begin{aligned}
\Delta v &= \sqrt{GM\left(\frac{2}{r_1}-\frac{1}{a}\right)}-\sqrt{\frac{GM}{r_1}} \\
&= \sqrt{\frac{GM}{r_1}}\left(\sqrt{2-\frac{2r_1}{r_1+r_2}}-1\right) \\
&= \sqrt{\frac{GM}{r_1}}\left(\sqrt{2\left(1-\frac{1}{1+r_2/r_1}\right)}-1\right) \\
&= v_1\left(\sqrt{2\left(1-\frac{1}{1+\rho}\right)}-1\right) \\
&= v_1\left(\sqrt{\frac{2\rho}{1+\rho}}-1\right)
\end{aligned}
$$

transfer to r_2

$$
\begin{aligned}
\Delta v &= \sqrt{\frac{GM}{r_2}}-\sqrt{GM\left(\frac{2}{r_2}-\frac{1}{a}\right)} \\
&= \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{r_1}{r_2}}-\sqrt{2\frac{r_1}{r_2}-\frac{2r_2}{r_1+r_2}}\right) \\
&= \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{r_1}{r_2}}-\sqrt{2\frac{r_1}{r_2}\left(\frac{r_1}{r_1+r_2}\right)}\right) \\
&= \sqrt{\frac{GM}{r_1}}\sqrt{\frac{r_1}{r_2}}\left(1-\sqrt{\frac{2}{1+r_2/r_1}}\right) \\
&= v_1\sqrt{\frac{1}{\rho}}\left(1-\sqrt{\frac{2}{1+\rho}}\right)
\end{aligned}
$$

For the LEO-to-GEO transfer, we must find v_1 given $r_1 = 6.8$ megameters, and r_2 given $v_2 = 2\pi r_2/24$ hr. Then we can apply the formulas.

For LEO

$$
\begin{aligned}
v_1 &= \sqrt{\frac{GM}{r_1}} \\
&= \sqrt{\frac{400_{10}12}{6.8_{10}6}} \\
&= 7.7 \text{ km/s}
\end{aligned}
$$

For GEO

$$
\begin{aligned}
\frac{2\pi r_2}{T} &= \sqrt{\frac{GM}{r_2}} \\
r_2^3 &= \left(\frac{T}{2\pi}\right)^2 GM \\
&= \left(\frac{24\times 3600}{2\pi}\right)^2 400_{10}12 \\
r_2 &= 42 \text{ megameters}
\end{aligned}
$$

Now we can find the two ΔV. We'll need $\rho = 42/6.8 = 6.2$.

LEO to transfer

$$
\begin{aligned}
\Delta V_{Lt} &= v_1\left(\sqrt{\frac{2\rho}{1+\rho}}-1\right) \\
&= 7.7\times 0.31 \\
&= 2.4 \text{ km/sec}
\end{aligned}
$$

Transfer to GEO

$$
\begin{aligned}
\Delta V_{tG} &= v_1\sqrt{\frac{1}{\rho}}\left(1-\sqrt{\frac{2}{1+\rho}}\right) \\
&= 7.7\times 0.19 \\
&= 1.5 \text{ km/sec}
\end{aligned}
$$

So the total delta-V is 3.9 km/sec.

20. Surface to LEO. We need a different approach to launch from the surface of the Earth to low Earth orbit. The delta-V has two components which are perpendicular to each other, and which must therefore be combined using Pythagoras.

$$\Delta V = \sqrt{(\Delta V_{\text{horizontal}})^2 + (\Delta V_{\text{vertical}})^2}$$

The horizontal component is the difference between the orbital velocity v_1 and the velocity of rotation of the Earth's surface. We would take advantage of the Earth's rotation to give us a boost in the direction we will be orbiting in.

That rotational velocity is the Earth's circumference divided by 24 hours.

$$v_{\text{rotE}} = \frac{2\pi 6.4 \text{ megameters}}{24\times 3600} = 465 \text{ m/sec}$$

So the horizontal component is

$$\Delta V_{\text{horizontal}} = v_1 - v_{\text{rotE}} = 7.7 - 0.5 = 7.2 \text{ km/sec}$$

The vertical component does not come out to be a change in velocity but only the equivalent due to the change in potential energy.

$$\begin{aligned}
\frac{(\Delta V_{\text{vertical}})^2}{2} &= -\frac{GM}{r_1} - \left(-\frac{GM}{r_{\text{Earth}}}\right) \\
&= GM_{\text{Earth}}\left(\frac{1}{r_{\text{Earth}}} - \frac{1}{r_1}\right) \\
&= 400_{10}12\left(\frac{1}{6.4_{10}6} - \frac{1}{6.8_{10}6}\right) \\
&= 3.7_{10}6 \\
\Delta V_{\text{vertical}} &= 2.7 \text{ km/sec}
\end{aligned}$$

Finally we combine horizontal and vertical components.

$$\Delta V = \sqrt{(7.2_{10}3)^2 + 2 \times 3.7_{10}6} = 7.7 \text{ km/sec}$$

Since we can just add delta-Vs, the total to get from the surface of the Earth to geosynchronous orbit is

$$\Delta V = 7.7 + 2.4 + 1.5 = 12 \text{ km/sec}$$

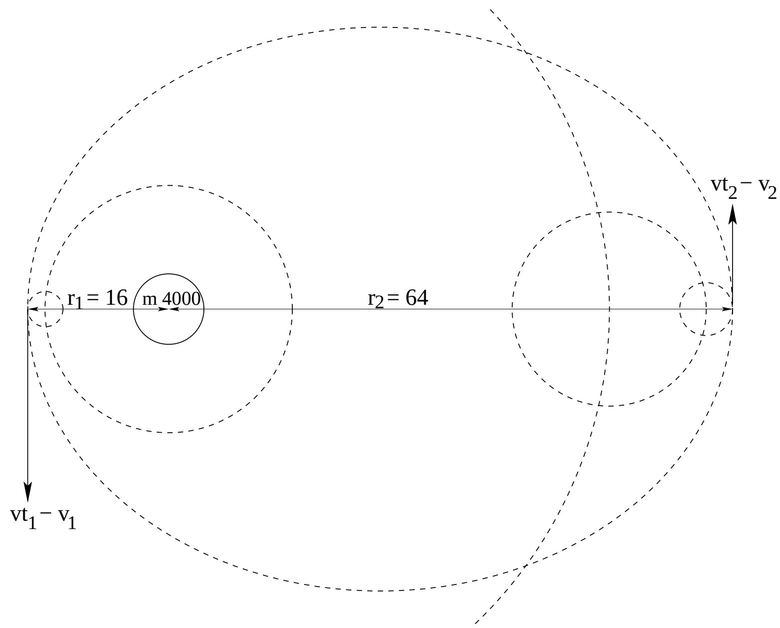
I must add that this is simplistic. First I have limited the calculations to only one or two significant figures.

More importantly, the launch from the Earth's surface is assumed to be from the Equator, or we would have had a smaller boost from the Earth's rotation. And we would have had to change the *plane* of the orbit, either LEO or GEO: a launch from Baikonur at 45.6^o north latitude or from Kennedy at 28.5^o would have resulted in an orbit in a plane at the same angle from the equatorial plane. Changing planes to an equatorial orbit costs delta-V, more or less depending on whether we do it at LEO or at GEO. The calculation involves rotation matrices.

21. Travelling the solar system. Sol, our sun, supports a complicated system of planets orbiting Sol, moons orbiting planets and, who knows, sub-moons orbiting moons, or at least artificial satellites. Humanity has already travelled to Luna, Earth's moon, and sent probes to most of the planets and several of their moons.

So let's make a simplified treatment of how to do this. We will suppose that all planet and moon orbits are circular, that our vehicles will start and end their journeys in circular orbit around the body of origin (e.g., the Earth) and around the destination body (e.g., Titan, a moon of Saturn).

We will calculate the ΔV required to change from circular orbit to elliptical transfer orbit, and from transfer orbit to circular orbit again at the other end. Here is a picture of a toy solar system in which we start in orbit around an orbit around the Sun, and end in orbit around a moon of another planet.



The “toy” aspect of this is that I have set the mass of the Sun, 4000 units (say exatonnes), Newton’s gravitational constant $G = 1$, and the transfer ellipse perihelion, 16 units (say gigameters), and aphelion, 64 units, so that you can do the math in your head or at most with pencil and paper but no calculator.

The velocities of the circular orbits are additive. We suppose that every orbit is followed counterclockwise (we are looking down on the solar system from somewhere above Earth’s north pole) and that our vehicle is at the extreme point furthest from the Sun when it changes to the transfer orbit, and again when it changes from the transfer orbit into circular orbit about its destination. (This is called a Hohmann transfer orbit and requires the least energy of all ways of travel between heavenly bodies.)

Let’s suppose that the sums of those circular velocities are $v_1 = 6$ units (say km/sec) at perihelion and $v_2 = 4$ units at aphelion. To find the ΔV at each end we need the differences between these velocities and the corresponding velocities for the elliptical orbit.

In fact, we need the *absolute* differences, the positive number that is the difference if the difference is positive, or minus the difference if the difference is negative. This is because we must burn fuel whether we are speeding up to catch the ellipse or, possibly, slowing down to leave it.

The radii of the circular orbits are also additive. They are given in the diagram as r_1 at perihelion and r_2 at aphelion.

We know how to find the velocities at the extremes of the elliptical orbit.

$$vt_1 = \sqrt{GM\left(\frac{2}{r_1} - \frac{1}{a}\right)} = \sqrt{GM\frac{r_2}{ar_1}} \qquad vt_2 = \sqrt{GM\left(\frac{2}{r_2} - \frac{1}{a}\right)} = \sqrt{GM\frac{r_1}{ar_2}}$$

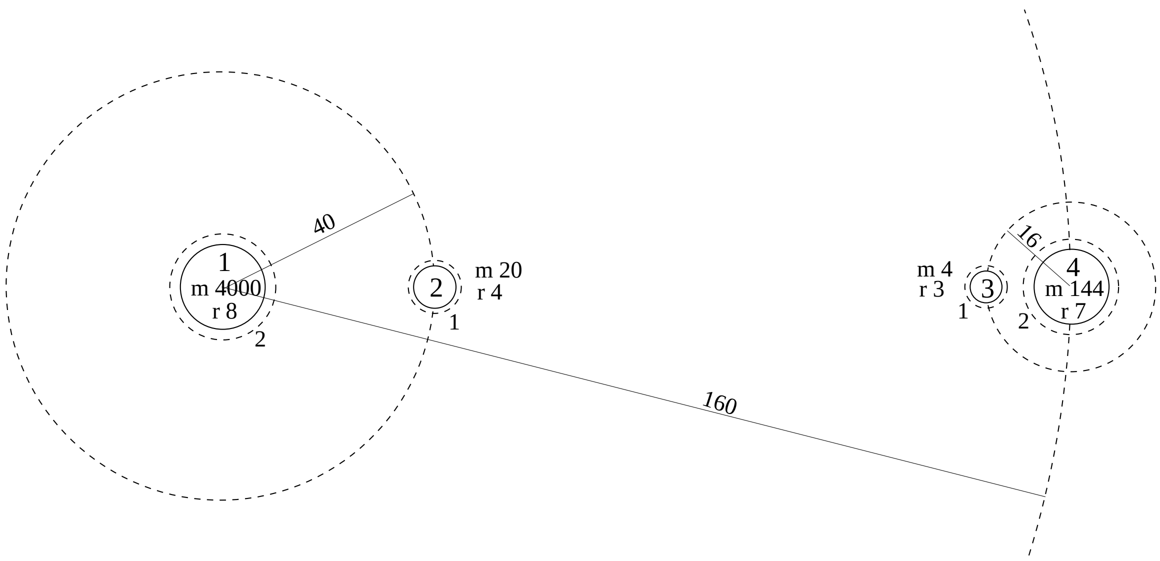
because $a = (r_1 + r_2)/2$. And you can work out for the toy system that

$$vt_1 = 20 \qquad vt_2 = 5$$

So $20 - 6 + 5 - 4$ gives a total ΔV of 15 for the voyage.

We now must do the work to find those accumulated circular velocities v_1 and v_2: we only *supposed*the 6 and the 4 above.

Toillustrate that I’m going to give you another toy solar system, a more complicated one, but different, in which the numbers also can be calculated with little effort.



There are four bodies. The Sun still has mass 4000. Orbiting it are two planets, body 2 of mass 2 units orbiting 40 units away, and body 4 of mass 144, 160 units away. Planet 4 has a moon, body 3, of mass 3, 16 units away from it.

We need the masses of the other bodies because they all have circular orbits *around* them, which will be our departure and destination orbits. The radii of these orbits are shown relative to the surfaces of their primary bodies, so we also show the radii of those bodies.

The velocity around each circular orbit, which we must find and add up, is given by

$$v = \sqrt{\frac{GM}{r}}$$

With $G = 1$ still, you can check that the added-up distances and velocities are

Body	Orbital distance	Orbital velocity
1	10	20
2	40+5	10+2
4	160+9	5+4
3	160+20	5+3+1

The transfer ellipses between the orbits around any two planets, or between the orbit around Sol and said planets, are ellipses with r_1 and r_2 given by the two "Orbital distances". Thus from Sol to the inner planet $r_1 = 10$ and $r_2 = 45$. These numbers won't give the simple calculations we saw in the first toy solar system but your calculator can find the square roots of $4000 \times 20/55/45$ and $4000 \times 90/55/10$.

The transfer from Sol to the outer planet, or between planets, is similar.

ΔV	1	2	3	4
1		11.8987	15.0004	14.8566
2			6.0929	5.9962
3				8.4197
4				

To find ΔV for transfers *within* a planetary system we must find distances and velocities relative to the planet, not to Sol.

Body	Orbital distance	Orbital velocity
4	9	4
3	20	3+1

Here are the transfers.

ΔV	3	4
3		2.5838
4		

Since the 3-to-4 entry in the larger table incorrectly supposes travel from moon 3 to its primary planet 4 via a transfer orbit around the Sun, we replace it by the entry in the smaller table (transfer within the system of planet 4), which is, of course, a smaller ΔV.

Here is the final chart of ΔV connecting all bodies in this toy solar system.

ΔV	1	2	3	4
1		11.8987	15.0004	14.8566
2			6.0929	5.9962
3				2.5838
4				

Now for the real solar system. I'll include Ceres as a representative of the asteroids, and the now demoted planet Pluto as a representative of the Kuiper belt. The Oort cloud where the comets mainly live is much further out.

The left four columns are the basic data, found from various Internet sources, mostly NASA (except see below for the third column, *park*). The *orbit* column gives the semi-major axis (a) for the orbit of the body around its primary, but all the calculations assume circular orbits for planets and moons.

	orbit(m)	radius(m)	park(m)	mass(kg)	v(m/s)	r(m)	parkV(m/s)	parkR(m)
Sol	0	$695_{10}6$	$10_{10}6$	$1989_{10}27$	0	0	$434_{10}3$	$705_{10}6$
Mercury	$58_{10}9$	$2.44_{10}6$	$100_{10}3$	$329_{10}21$	47875	$58_{10}9$	50813	$58_{10}9$
Venus	$108_{10}9$	$6.05_{10}6$	$400_{10}3$	$4.9_{10}24$	35023	$108_{10}9$	42118	$108_{10}9$
Luna	$385_{10}6$	$1.74_{10}6$	$100_{10}3$	$73_{10}21$	30804	$150_{10}9$	32437	$150_{10}9$
Earth	$150_{10}9$	$6.38_{10}6$	$250_{10}3$	$5.972_{10}24$	29786	$150_{10}9$	37540	$150_{10}9$
Deimos	$23_{10}6$	6200	1000	$1.5_{10}15$	23201	$228_{10}9$	23205	$278_{10}9$
Phobos	$9_{10}6$	11100	1000	$11_{10}15$	23985	$228_{10}9$	23993	$278_{10}9$
Mars	$228_{10}9$	$3.40_{10}6$	$200_{10}3$	$639_{10}21$	21853	$228_{10}9$	25296	$278_{10}9$
Ceres	$414_{10}9$	$0.47_{10}6$	$100_{10}3$	$938_{10}18$	17900	$414_{10}9$	18231	$414_{10}9$
Callisto	$1.9_{10}9$	$2.41_{10}6$	$100_{10}3$	$108_{10}21$	21259	$780_{10}9$	22950	$780_{10}9$
Ganymede	$1.1_{10}9$	$2.63_{10}6$	$100_{10}3$	$148_{10}21$	23936	$780_{10}9$	25837	$779_{10}9$
Europa	$0.7_{10}9$	$1.56_{10}6$	$100_{10}3$	$48_{10}21$	26798	$780_{10}9$	28186	$779_{10}9$
Io	$0.4_{10}9$	$1.82_{10}6$	$100_{10}3$	$89_{10}21$	30386	$780_{10}9$	32147	$779_{10}9$
Jupiter	$778_{10}9$	$71.4_{10}6$	$2_{10}6$	$1.9_{10}27$	13058	$780_{10}9$	54597	$778_{10}9$
Titan	$1.2_{10}9$	$2.57_{10}6$	$1_{10}6$	$132_{10}21$	16596	$1.43_{10}12$	18167	$1.43_{10}12$
Saturn	$1.43_{10}12$	$60_{10}6$	$2_{10}6$	$568_{10}24$	9648	$1.43_{10}12$	40218	$1.43_{10}12$
Uranus	$2.87_{10}12$	$25.6_{10}6$	$1_{10}6$	$868_{10}24$	6800	$2.87_{10}12$	53508	$2.87_{10}12$
Neptun	$4.50_{10}12$	$24.8_{10}6$	$1_{10}6$	$102_{10}24$	5432	$4.50_{10}12$	21718	$4.50_{10}12$
Pluto	$5.91_{10}12$	$1.15_{10}6$	$100_{10}3$	$13_{10}21$	4741	$5.91_{10}12$	5575	$5.9_{10}12$

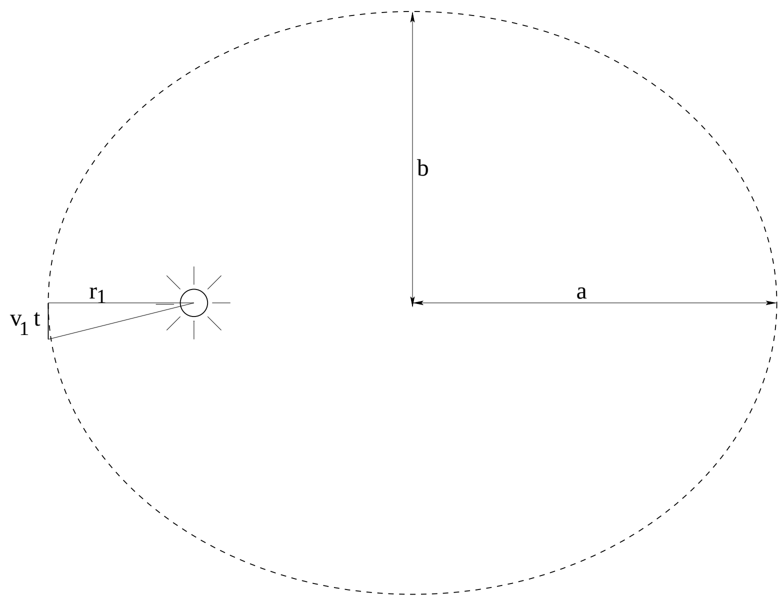
The four columns on the right are values I calculated for combined velocities and radii: v is the total velocity of the body itself around the Sun when, if it is a moon, it is at its furthest from the Sun; r is that furthest distance; *parkV* is the maximum velocity for a *parking* orbit around the body, the distance from the surface of which is given in the third column, *park*; and *parkR* is the sum of that orbit's radius with r.

The distance in column *park* is an arbitrarily chosen orbital distance from the surface of the body. The radius of that orbit is the sum of *park* and the radius of the body. We will consider a vehicle moving from one parking orbit to another, starting and ending at the extreme points of those two orbits from the Sun. That saves us worrying about launching from the surface to the parking orbit, with all the complications of off-equator launch sites and orbit plane changes.

Connecting these parking orbits are elliptical transfer orbits. The period of an elliptical orbit is conveniently the period of a circular orbit with a substituted for r: a circle is an ellipse with $a = b = r$, so an ellipse is a squashed circle of area πab instead of πr^2; Kepler says orbits sweep out equal areas in equal times, and this is a consequence of the constancy of angular momentum, always, say,

$$v_1 r_1 = r_1 \sqrt{GM \frac{r_2}{a r_1}} = \sqrt{GM \frac{r_1 r_2}{a}} = \sqrt{GM \frac{b^2}{a}}$$

The area swept out in a short time t at the perihelion is the triangle in the figure, whose base is r_1 and whose height is $v_1 t$.



So $v_1 r_1 t$ is twice the area swept out in time t.

Kepler thus tells us that in time T which is the period for the whole orbit, $v_1 r_1 T$ is twice the area of the whole ellipse.

$$2\pi ab = v_1 r_1 T = \sqrt{GM \frac{b^2}{a}} T$$

giving the period

$$T = \frac{2\pi ab}{b\sqrt{GM/a}} = 2\pi a^{3/2} \sqrt{GM}$$

The time required for a voyage between, say, Earth and Mars, is half the period of the elliptical transfer orbit, or $\pi a^{3/2} \sqrt{GM}$. Here are all the transfer times between planets, omitting the moons, to the nearest numbers of days.

	Trip time (Earth days)										
	Sol	Mercury	Venus	Earth	Mars	Ceres	Jupiter	Saturn	Uranus	Neptune	Pluto
Sol	0	16	40	65	122	298	768	1902	5428	0648	16019
Mercury	16	0	76	105	171	362	854	2017	5591	10851	16252
Venus	40	76	0	146	217	421	932	2121	5735	11031	16458
Earth	65	105	146	0	259	472	998	2207	5855	11180	16628
Mars	122	171	217	259	0	574	1127	2374	6085	11464	16952
Ceres	298	362	421	472	574	0	1454	2786	6642	12149	17730
Jupiter	768	854	932	998	1127	1454	0	3653	7777	13525	19285
Saturn	1902	2017	2121	2207	2374	2786	3653	0	9936	16090	22154
Uranus	5428	5591	5735	5855	6085	6642	7777	9936	0	22317	29011
Neptune	10648	10851	11031	11180	11464	12149	13525	16090	22317	0	37447
Pluto	16019	16252	16458	16628	16952	17730	19285	22154	29011	37447	0

Note that returning takes as long as going: the table is "symmetrical".

If we extended this table to include the moons, they would be hardly distinguishable from their primary planets. To find transfer times *within* any particular planetary system we must redo the calculation, stopping short of the planet's orbit around Sol.

	Luna	Earth
Luna	0	5
Earth	5	0

	Titan	Saturn
Titan	0	3
Saturn	3	0

	Callisto	Ganymede	Europa	Io	Jupiter
Callisto	0	6	5	4	3
Ganymede	6	0	3	2	1
Europa	5	3	0	1	1
Io	4	2	1	0	0
Jupiter	3	1	1	0	0

(Note that Deimos and Phobos are so close to each other and to Mars that transfer time in the Mars system is less than half a day.)

These transfer times are rather long. Try converting some of the longer ones to Earth years. (I gave them in Earth days.) Don't forget that the Hohmann transfer orbits start on the other side of the primary body and as far away as possible from where they meet the destination orbit. They are thus worst-case scenarios—the maximum possible times.

The energies required, on the other hand, are the best cases, the minimum possible. We can get a feel for them by using the transfer ellipses plus the velocity data already recorded for the bodies to calculate ΔV.

On the next page are the ΔV to the nearest number of kilometers per second for all transfers including both planets and moons. I've had to insert little patches which I've calculated separately for the four intraplanetary systems (Luna-Earth, Mars, Jupiter and Saturn).

Notice again that coming and going have the same costs. Accelerating from orbit a to orbit b and decelerating from b to a require the same ΔV.

A third interplanetary measure, which is useful for simple calculations, is the difference between potential energies of each pair of planets in the solar gravitational well. This is calculated as in Note 18 for Earth, except $GM_{\text{Sol}} = (400/3) \times 10^{18}$ (Note 13).

	Delta-V (km per sec)																		
	Sol	Mer	Ven	Lun	Ear	Dei	Pho	Mar	Cer	Cal	Gan	Eur	Io	Jup	Tit	Sat	Ura	Nep	Plu
Sol	0	219	216	208	213	202	203	204	196	202	205	207	211	233	197	214	233	201	185
Mer	219	0	17	17	22	20	21	22	22	33	35	38	42	64	31	47	68	37	21
Ven	216	17	0	10	15	7	8	10	9	21	24	26	30	52	20	36	58	27	12
Lun	208	17	10	0	6	4	5	6	9	22	25	27	31	53	22	38	60	29	14
Ear	213	22	15	6	0	9	10	11	7	17	19	22	26	48	17	33	55	24	9
Dei	202	20	7	4	9	0	1	2	5	19	22	24	28	50	19	36	58	28	12
Pho	203	21	8	5	10	1	0	1	4	18	21	23	27	50	19	35	57	27	11
Mar	204	22	10	6	11	2	1	0	3	17	20	22	26	48	17	33	56	26	10
Cer	196	22	9	9	7	5	4	3	0	14	17	20	23	46	16	32	56	25	10
Cal	202	33	21	22	17	19	18	17	14	0	3	5	8	24	18	34	56	25	9
Gan	205	35	24	25	19	22	21	20	17	3	0	3	6	24	21	37	59	28	12
Eur	207	38	26	27	22	24	23	22	20	5	3	0	3	23	23	40	61	30	14
Io	211	42	30	31	26	28	27	26	23	8	6	3	0	22	27	44	65	34	18
Jup	233	64	52	53	48	50	50	48	46	24	24	23	22	0	50	66	87	56	41
Tit	197	31	20	22	17	19	19	17	16	18	21	23	27	50	0	15	55	24	9
Sat	214	47	36	38	33	36	35	33	32	34	37	40	44	66	15	0	71	40	25
Ura	233	68	58	60	55	58	57	56	56	56	59	61	65	87	55	71	0	63	47
Nep	201	37	27	29	24	28	27	26	25	25	28	30	34	56	24	40	63	0	17
Plu	185	21	12	14	9	12	11	10	10	9	12	14	18	41	9	25	47	17	0

Here is the table, in gigajoules per kilogram of the spacecraft mass. (A joule is about the kinetic energy of a very fast small bird. A gigajoule is about the energy of a large passenger airplane at cruising speed.)

Mercury	Venus	Earth	Mars	Ceres	Jupiter	Saturn	Uranus	Neptune	Pluto
0	1.0653	1.4046	1.7095	1.9714	2.1213	2.1987	2.2455	2.2623	2.2693
0	0	0.3393	0.6442	0.9061	1.0560	1.1334	1.1802	1.1970	1.2040
0	0	0	0.3049	0.5668	0.7167	0.7941	0.8409	0.8576	0.8647
0	0	0	0	0.2619	0.4118	0.4892	0.5360	0.5527	0.5598
0	0	0	0	0	0.1499	0.2273	0.2741	0.2909	0.2979
0	0	0	0	0	0	0.0774	0.1243	0.1410	0.1480
0	0	0	0	0	0	0	0.0468	0.0636	0.0706
0	0	0	0	0	0	0	0	0.0167	0.0238
0	0	0	0	0	0	0	0	0	0.0070
0	0	0	0	0	0	0	0	0	0

(The zeros in the table, apart from the diagonal, should be the negatives of the numbers shown symmetrically across the diagonal from them: "falling" down the potential energy well gains energy. Not surprisingly, the numbers add up to each other.)

So a 1-tonne spacecraft flying from the orbit of the Earth to the orbit of Mars would need to gain some $1000 \times 0.3049_{10}9 = 305$GJ of energy,

22. Launch windows. Unfortunately, all we've done so far is travel from orbit to orbit. That does not do us any good if we have reached, say, Mars' orbit, but Mars is on the other side of the Sun.

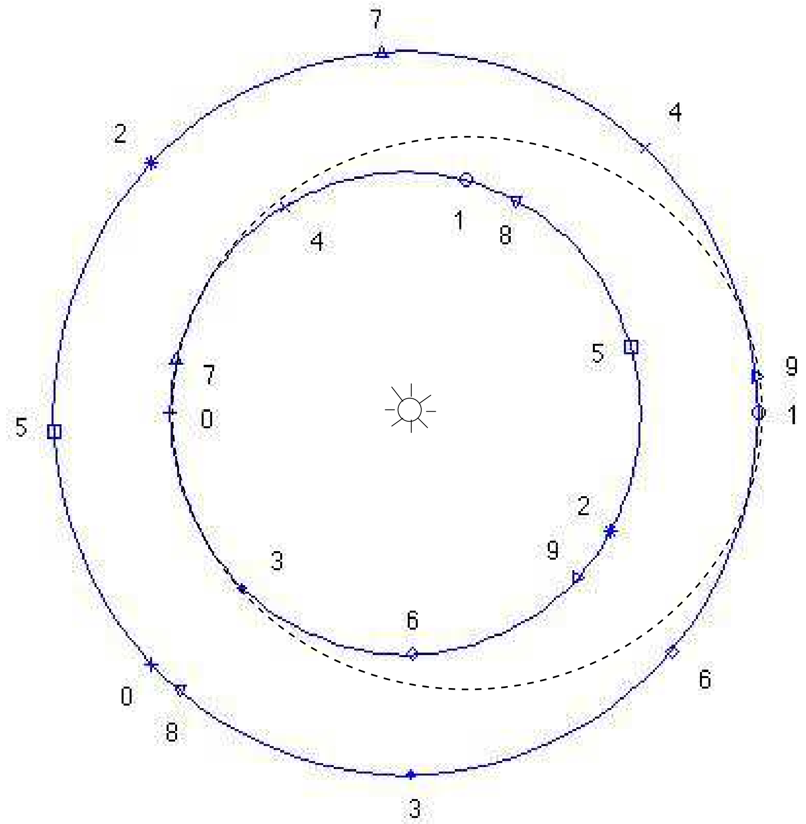
So we must adhere to a *launch window*. If we launch our spaceship from Earth orbit, aiming for the point on Mars orbit on the other side of the Sun, we must time it so that Mars gets there when we do.

Since Mars takes 687 (Earth) days to complete an orbit around the Sun, and since it will take us 259 days on the elliptical transfer orbit to get from now to rendez-vous, "now" must be the moment when Mars has

$$\frac{259}{687}360 = 135$$

degrees to go before rendez-vous.

That moment in time gives Earth and Mars the positions marked "0" on the diagram.



Position “1” is the rendez-vous, 259 days later. We can see, too, that in that time, Earth has come around to the same side of the Sun as Mars, passed the closest approach of the two planets, and is heading away from Mars again.

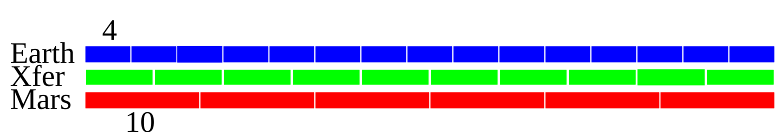
The other positions marked on the diagram are separated from each other by 259 days, the duration between perihelion and aphelion of the transfer orbit, which I’m using as a clock.

We can see that this particular transfer orbit cannot be reused in anything like the near future after its first use.

In fact, it will take 501,510 “ticks” of the “clock”—129,891,090 Earth days—before the original configuration lines up again: the *least common multiple* of the three intervals,

$$\mathrm{lcm}(365, 518, 687) = 129,891,090 = 501,510 \times 259 = 355,866 \times 365 = 189,070 \times 687$$

To see this, let’s simplify the numbers. Suppose the three intervals were 4, 6 and 10, respectively.



Here, I’ve stretched out the three orbits into a line.

A way to find the least common multiple is to break each number into its prime factors and then gather just enough of these to make a number which each of the three divides exactly.

$4 = 2 \times 2$ $6 = 2 \times 3$ $10 = 2 \times 5$

$$2 \times 2 \times 3 \times 5 = 60 = \mathrm{lcm}(4, 6, 10)$$

We can do this[4] for the big numbers too.

$$518 = 2 \times 7 \times 37 \qquad 365 = 5 \times 73 \qquad 587 = 3 \times 229$$

$$2 \times 3 \times 5 \times 7 \times 37 \times 73 \times 229 = 129,891,090$$

Of course, in astronomy, the numbers won't usually be exact integers. Indeed, the Earth's orbit is closer to 365.25 days (hence a "leap year" every four years), and even that is not exact (hence leap year is cancelled every century).

Can we imagine a "convenient" solar system in which the periods have simple least common multiples? Such a solar system would permit *cyclers*—large, safe structures travelling permanently between planets on transfer orbits, which could be met by ferries at both ends and would provide safe, if slow, voyages between planets, spinning for artificial gravity and with plenty of shielding against cosmic rays and interplanetary dust and debris.

It is not easy. The requirement is that three periods, p_1, p_2 and p_t, have an lcm which is not too many years, say, lcm/p_2, of the outer planet's period, p_2. The transfer orbit's semi-major axis, r_t, gives its period $p_t = r_t^{(3/2)}$ and is given by

$$r_t = \frac{r_1 + r_2}{2} = \frac{p_1^{(2/3)} + p_2^{(2/3)}}{2}$$

(Well, there is some constant c in there, $p_t = cr_t^{(3/2)}$, but we can work entirely with ratios in which case c cancels out everywhere.)

Here are some numbers, insisting only that the periods be integers so that they have least common multiples and their configurations will repeat. (In some of these cases, the "radii"—by which we can refer to semi-major axes too—are also integers.)

Periods					Radii				
p_1	p_2	p_t	lcm	lcm/p_1	a_1	a_2	a_t	a_2/a_1	Planets
362911	912673	614125	585395^3	$548.241_{10}9$	5041	9409	7225	1.9	Me:Ve
117649	357911	226981	212219^3	$81.239_{10}9$	2401	5041	3721	2.1	Ve:Ma
11	560	220	6160	560	4.9461	67.94	36.443	13.7	Me:Ju
1	343	125	42875	42875	1	49	25	49.0	Me:Ur

I've selected these examples from an extensive (but of course not complete) set of numbers I've experimented with because the ratios a_2/a_1 approximately match the orbital ratios of actual solar system planets.

The results are not promising. The Mercury-Venus system repeats every 548 billion Mercury years (132 billion Earth years). The Venus-Mars system repeats every 81 billion Venus years (49 billion Earth years). Mercury to Uranus, the final example in which both the periods and the radii are whole numbers, does rather better, but the repeat still takes 10 thousand Earth years, and the orbital ratio should be 49.5 not 49. The best result of all, Mercury to Jupiter, still needs over a century (135 Earth years) to repeat the cycle.

And Earth itself figures nowhere in the results, except for a billion-year cycle with Mercury, almost.

So the cycler is hardly a commercial prospect. It is not the ocean liner of interplanetary travel.

It may be possible to alter a cycler orbit each time around, say at perihelion, so that it will meet the target planets in a reasonable cycle, but this takes us into the realm of gravity assists, the "three-body" problem, and complicated rocketry.

23. Conic sections. The parameter a in the vis-viva equation of Note 16 can become quite large. A comet falling sunward from the Oort cloud at 10,000 to 100,000 AU follows a very elongated

[4]Note that this result is very sensitive to changes. Increasing 10 by 10% to 11 more than doubles the lcm from 60 to 132, and increasing it again by 9% to 12 changes it right back down to 12.

ellipse. A body from outside the solar system altogether would have $a = \infty$. Indeed this value of a defines what we would mean by "outside".

The velocity required to get us outside the solar system, e.g., for interstellar flight, the *escape velocity*, is given by the vis-viva equation with this value of a.

$$v_{\text{escape}} = \sqrt{\frac{2GM}{r}}$$

where r is the radius of the orbit we're "escaping" from.

Since $\sqrt{GM/r}$ is the velocity in a circular ($a = r$) orbit of that radius, it follows that

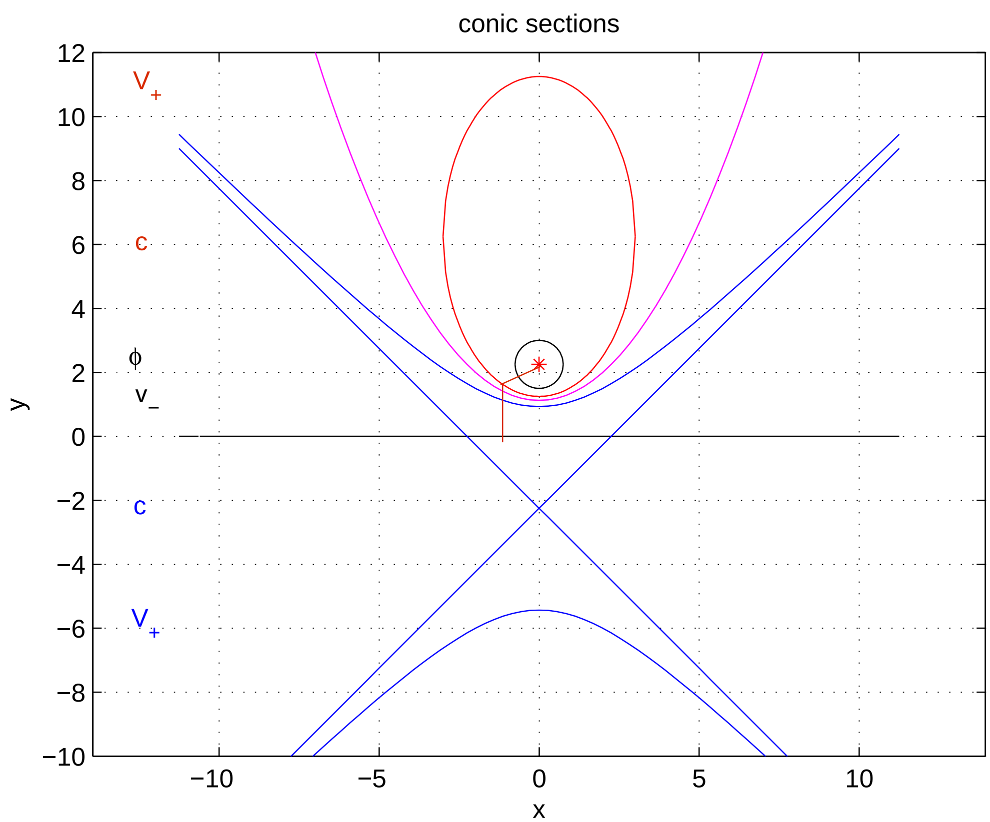
$$v_{\text{escape}} = \sqrt{2}\, v_{\text{circular}}$$

The escape orbit is no longer an ellipse. It must be open-ended or else the escaping body would loop back. It is a *parabola*.

The circle and the ellipse belong to a wider family of curves, called the *conic sections*, which include the parabola and also the *hyperbolas*. (When a ship, starting at escape velocity for some orbit, is far enough away from the Sun its velocity is zero. If we want a velocity higher than that, so we can get somewhere having left the solar system, we must follow a hyperbola.)

To work with conic sections we must know about *eccentricity*. This is a parameter, ϵ, which varies from 0 for a circle, through the ellipses to 1 for a parabola and then on through the hyperbolas.

Here are circle, parabola, and an example each of ellipse and hyperbola.



What they have in common is that the distance from the focus (the red asterisk) to any point on the curve equals the eccentricity times the distance from that point to the *directrix* (the horizontal black line at $y = 0$).

If the focus is at point $(0, \phi)$ and (x, y) is the point on the curve, then this gives, for all curves except the circle

$$\begin{aligned}
\sqrt{x^2 + (y - \phi)^2} &= \epsilon y \\
x^2 + (y - \phi)^2 &= \epsilon^2 y^2 \\
x^2 + y^2 - 2\phi y + \phi^2 &= \epsilon^2 y^2 \\
x^2 + (1 - \epsilon^2) y^2 - 2\phi y + \phi^2 &= 0
\end{aligned}$$

and I'm going to extract a perfect square from everything involving y, so I'll invent q and p temporarily

$$\begin{aligned}
x^2 + (1 - \epsilon^2)(y - q)^2 &= p^2 \\
x^2 + (1 - \epsilon^2) y^2 - (1 - \epsilon^2) 2yq + (1 - \epsilon^2) q^2 - p^2 &= 0
\end{aligned}$$

(The first of the above two lines will be a conic section centred at $(0, q)$ and with "radius" p: you can compare it to the equation for the ellipse in Note 9.)

In order to make the last line in each of the above two sets of equations equal to one another, we must have

$$q = \frac{\phi}{1 - \epsilon^2}$$

and

$$(1 - \epsilon^2) q^2 - p^2 = \phi^2$$

from which

$$p^2 = (1 - \epsilon^2) q^2 - \phi^2 = \phi^2 \left(\frac{1}{1 - \epsilon^2} - 1 \right) = \phi^2 \frac{\epsilon^2}{1 - \epsilon^2}$$

Thus we have finally for the equation for the conic section

$$x^2 + (1 - \epsilon^2) \left(y - \frac{\phi}{1 - \epsilon^2} \right)^2 = \frac{\phi^2 \epsilon^2}{1 - \epsilon^2}$$

Or, if we want to plot y against x

$$y = \frac{\phi}{1 - \epsilon^2} \pm \sqrt{\frac{\phi^2 \epsilon^2}{(1 - \epsilon^2)^2} - \frac{x^2}{1 - \epsilon^2}}$$

where we must take care to keep the x term from getting so big as to make the argument of the square root negative. (These equations don't work for the circle ($\epsilon = 0$) or the parabola ($\epsilon = 1$). I drew a circle with radius $\phi/3$ above. We'll come back to the parabola.)

We can investigate some particular values of y when $x = 0$.

$$y = \frac{\phi}{1 - \epsilon^2} \pm \frac{\phi \epsilon}{1 - \epsilon^2} = \frac{\phi}{1 \mp \epsilon}$$

These two numbers are the *vertices*, the points at which the curve crosses the y-axis. They are shown as v_+ and v_- respectively. Note that both ellipses and hyperbolas cross the y-axis in two places. For the hyperbola v_+ is lower than v_-.

The average of the vertices is the *centre* of the conic section.

$$c = \frac{v_+ + v_-}{2}$$

In making the plots I chose $\phi = 9/4$ because this turns out to give the ellipse the same proportions as in Note 9. Here are the numbers for the values of ϵ shown.

	ϵ	v_+	v_-	c
ellipse	0.8	11.25	1.25	6.25
hyperbola	$\sqrt{2}$	-5.43	0.93	-2.25

The distance from c to ϕ can be defined as f. For the ellipse this is 4, which is the f used in Note 9. The distance from c to $v_\pm$ can be defined as a. For the ellipse this is 5, which is the a used in Note 9. The eccentricity of the ellipse is also $\epsilon = f/a$:

$$\begin{aligned} c &= \frac{v_+ + v_-}{2} = \frac{\phi}{2}\left(\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon}\right) = \frac{\phi}{1-\epsilon^2} \\ a &= c - v_- = \frac{v_+ - v_-}{2} = \frac{\phi}{2}\left(\frac{1}{1-\epsilon} - \frac{1}{1+\epsilon}\right) = \frac{\phi\epsilon}{1-\epsilon^2} \\ f &= c - \phi = \phi\left(\frac{1}{1-\epsilon^2} - 1\right) = \frac{\phi\epsilon^2}{1-\epsilon^2} \end{aligned}$$

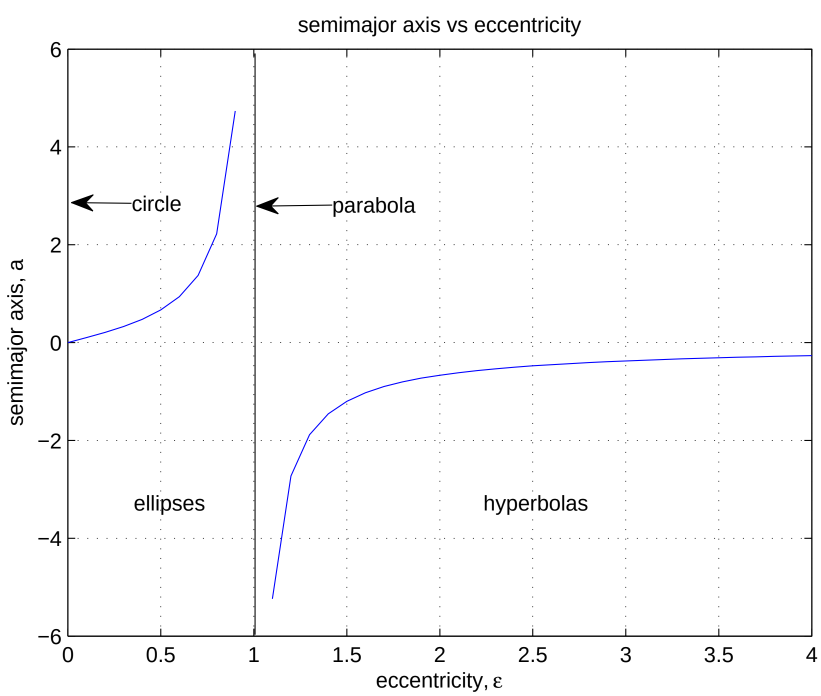
and these equations hold also for the hyperbola. (Note that the equation $\epsilon = f/a$ does allow $\epsilon = 0$ whether a is zero or not, and gives a circle of radius a.)

The centre of the hyperbola, also negative, is the crossing point of two *asymptotes*. These are the straight lines that the hyperbola approaches as x gets very large, both positively and negatively. They would give the final trajectory of a ship moving in excess of escape velocity.

The special case of the parabola, with $\epsilon = 1$, must be treated differently right from the start.

$$\begin{aligned} x^2 + (\phi - y)^2 &= y^2 \\ x^2 - 2y\phi + \phi^2 &= 0 \\ y &= \frac{x^2 + \phi^2}{2\phi} = \frac{x^2}{2\phi} + \frac{\phi}{2} \end{aligned}$$

Finally, we can see what happens to a as ϵ passes through 1.



We see an ellipse getting more and more elongated as a gets bigger, until it is a parabola at $a = \infty$ when $\epsilon = 1$. As ϵ grows beyond 1, a switches suddenly to $-\infty$ and continues to get bigger, approaching zero again.

The corresponding hyperbola, as ϵ increases from 1, starts very narrow, a hairpin turn around the Sun, with the asymptotes almost parallel and the centre where they cross a very long way from the vertex. Then the angle between the asymptotes increases, becoming a right angle at $\epsilon = \sqrt{2}$, and the centre approaches the vertex as we see in the diagram. Eventually the angle between the asymptotes is 180^o and the hyperbola is a straight line—the "eccentricity" is zero again.

The vis viva equation (Note 16) holds for hyperbolas as well, only the $1/a$ second term under the square root is negative and so subtracting it increases the velocity. This increase is small when ϵ is just above 1, i.e., the orbit is just beyond being a parabola, and gets arbitrarily large as the hyperbolic orbit approaches a straight line.

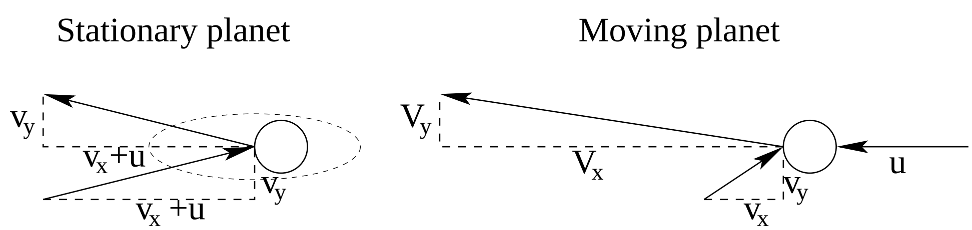
24. Gravity assist. Suppose we have a two-stage rocket giving a delta-V of 9 km/s (Note 11 of Part I) and we want to put a probe in orbit around Saturn's moon Titan. That requires a delta-V of 17 km/s (Note 21).

We can find an extra delta-V of up to 26 km/s by "bouncing off" Jupiter. This is twice Jupiter's orbital speed (Note 14).

We accomplish this by treating Jupiter as a moving wall and the spacecraft as a billiard ball (Note 17).

Of course it's not really like this. But if we start far enough from Jupiter we can imagine the spacecraft moving in a straight line which is the asymptote of a hyperbola with Jupiter at its focus. Then when we look again, the spacecraft is on the outward asymptote, again in a straight line. Ignoring all the gravitational details and the curved part of the hyperbola, this is just as if the probe had indeed bounced off Jupiter.

We need two points of view. First, a stationary Jupiter. I've shown the probe bouncing off, but I've ringed the region we're ignoring, which would show the hyperbola and Jupiter at its focus.



Here the horizontal velocity of the spacecraft, $v_x + u$, is just reversed, and its vertical velocity, v_y, is unchanged.

The second point of view is Jupiter moving at velocity u. The encounter is brief enough that we can suppose u is a straight line rather than a circular (or elliptical) orbit. This is the point of view of the Sun and let's say it is our point of view from Earth's orbit, and the point of view of the probe.

Now the probe's incoming velocity is (v_x, v_y) and its outgoing velocity will be $(V_x, V_y) = (-(v_x + 2u), v_y)$.

The magnitude of the outgoing velocity is

$$\sqrt{(v_x + 2u)^2 + v_y^2} = \sqrt{v^2 + 4v_x u + u^2}$$

where $v^2 = v_x^2 + v_y^2$ is the magnitude of the incoming velocity.

So the "collision"—flyby, really—can add up to $2u$ to the velocity of the spacecraft, which is double the orbital velocity of the planet.

If we point everything in the right direction, this can get us to Titan with enough fuel to insert into orbit around the moon.

To get from Earth orbit to Jupiter we need the vis viva equation of Note 16 (the Δv table at the end of Note 21 includes the expensive insertion into orbit around Jupiter, which we don't need this time). It gives

$$v_{\text{Earth orbit}}\left(\sqrt{\frac{\rho}{1+\rho}}-1\right) = 30\left(\sqrt{\frac{10}{6}}-1\right) = 8.7 \text{ km/s}$$

And to go from Jupiter's orbit to Saturn's needs only $13(\sqrt{4/3}-1) = 2$ km/s. So we have plenty of leeway to get to Titan, and well beyond, which is what Voyagers 1 and 2 did in the 1980s.

The *Oberth maneuver* is a powered variant of the flyby. It takes advantage of the fact that kinetic energy goes as the square of the velocity (Note 17) by doing a burn at the closest point to the planet and thus at the highest speed.

If we can get Δv extra speed from the burn and if the velocity at which the burn is started is q times this, $v = q\Delta v$, then the resulting additional (specific) kinetic energy is half of

$$(v+\Delta v)^2 - v^2 = ((q+1)^2 - q^2)\Delta v^2 = (2q+1)\Delta v^2$$

This is half Δv^2 itself if $v = 0$ and grows at more than v as v gets bigger.

q	0	1	2	3	4
$(2q+1)/2$	0.5	1.5	2.5	3.5	4.5
$(q+1)^2/2$	0.5	2	4.5	8	12.5

Note that the calculations assume the burn is instantaneous. So the effect is best for a rocket with a lot of thrust.

25. Resonances. In Note 22 we wished for planets in commensurate orbits so that we could fly permanent "cyclers" among them. In fact, commensurate orbits are unusually common in the Solar system, at least among the moons of planets. They are called "resonances".

Here are data on the four "Galilean" moons of Jupiter (Galileo's discovery of them in 1609–10 was the first glimpse of imperfection in the heavenly sphere—bodies not orbiting Earth).

	Io	Europa	Ganymede	Callisto
a Mm	424	678	1072	1896
e	0.007	0.001	0.009	0.004
n deg/day	201.8	99.8	50.2	21.3
P days	1.78	3.61	7.17	16.87
v Km/s	17.3	13.7	10.9	8.2

Note the 4:2:1 ratio of n, the angular speed, of the inner three moons, and the corresponding 1:2:4 ratio of P, their orbital periods.

The data on the semimajor axes, a in megameters, and on the eccentricities e, is from NASA's Planetary Fact Sheets. I calculated

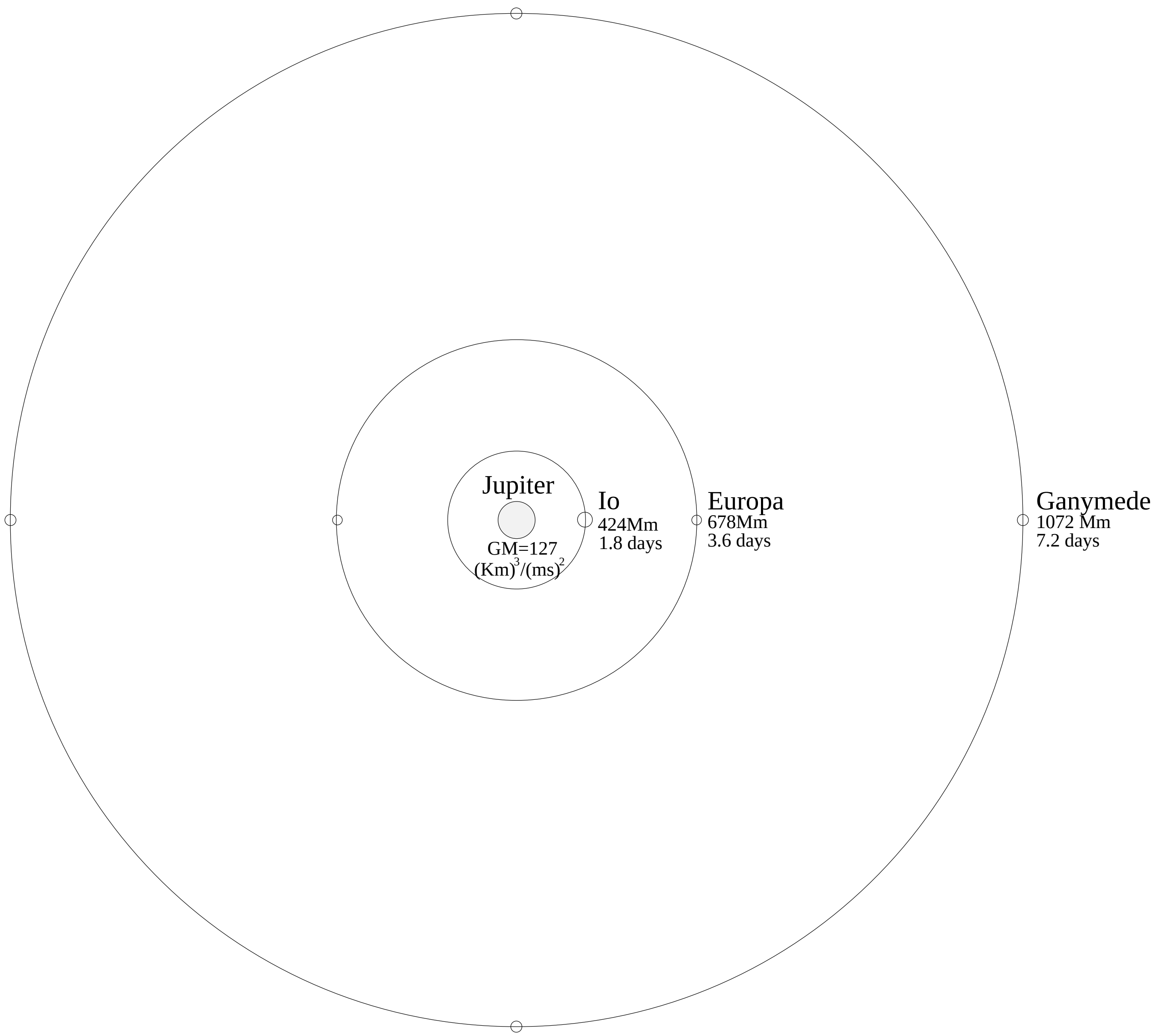
$$n = \sqrt{\frac{GM}{a^3}} \times 24 \times 3600$$

where GM for Jupiter is $6.674_{10}{-}11 \times 1.898_{10}27 = 12.667_{10}16$, and later multiplied it by $180/\pi$ to convert from radians to degrees. And I used

$$P = \frac{2\pi}{n}$$
$$v = \frac{an}{24 \times 3600 \times 1000}$$

for the period and orbital velocity, respectively (the eccentricities are so small that a serves fine as a radius: n is, strictly, the *mean* angular velocity).

Here is a picture, more or less to scale, of the inner three moons. It shows two possible positions of Europa and four of Ganymede for one position of Io. The three lined-up positions is a double *conjunction*. Callisto is omitted from the picture, but demonstrates that commensurability is not necessary or even usual.



The data I've shown do not give exactly integer ratios. Here are some more precise data.

	Io	Europa	Ganymede
n deg/day	203.4890	101.3747	50.3176

We can notice the following.

$$n_{\mathrm{Io}} - 2n_{\mathrm{Europa}} = 203.4890 - 202.7494 = 0.7396$$

$$\begin{aligned} n_{\text{Europa}} - 2n_{\text{Ganymede}} &= 101.3747 - 100.6352 = 0.7395 \\ n_{\text{Io}} - 3n_{\text{Europa}} + 2n_{\text{Ganymede}} &\approx 0 \end{aligned}$$

with the ≈ 0 holding to at least seven significant figures.

How do these simple integer relationships arise? Let's look at conjunctions of two moons, the outer, larger one in a circular orbit, and the inner, smaller one in an elliptical orbit. We'll suppose the inner moon is so small as to have no gravitational influence on the outer moon. But the outer moon does attract the inner, as well, of course, as both of them being attracted to the primary body they are orbiting.

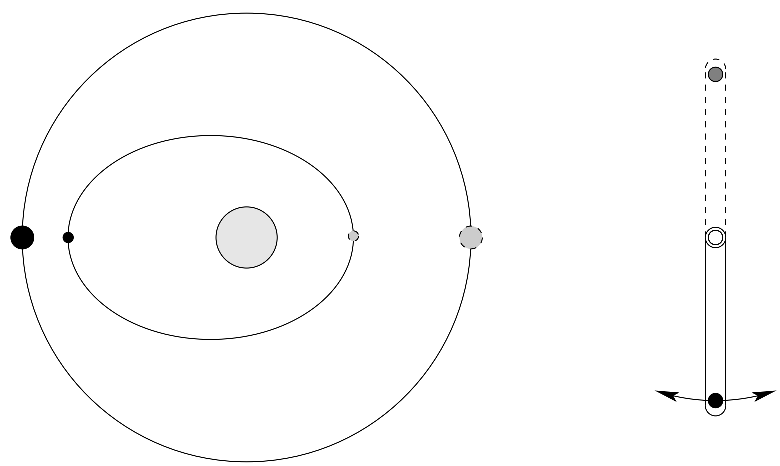
I've shown two positions for each moon, leading us to four possible configurations.

I've also shown a rigid pendulum in two positions, which we can use for a discussion of equilibrium and stability.

When the pendulum is hanging straight down from its pivot (which is in the centre of the diagram) it is in *equilibrium*: all the forces on it balance. If it is displaced slightly left or right, gravity pulls it back to its equilibrium position: the equilibrium is *stable*.

The motion it follows is an *oscillation*: it swings back and forth, until friction with the air damps down that motion and the pendulum winds up in stable equilibrium again.

When the pendulum is "hanging" straight up—the position shown by the dashed lines—it is also in equilibrium, but *unstable* equilibrium: any disturbance will cause it to fall down towards its stable position. But in the absence of any disturbance—slight motions of the air for instance—it will stay pointing straight up.



We can imagine similar behaviour in the two orbiting moons. Both are moving, of course, the inner one faster than the outer, so we must think of *averages* of the forces involved, not just simple forces.

If the conjunction of the two moons is shown by their two black positions (or their two grey positions) then it is plausible that they have a stable equilibrium: the inner moon will be retarded by the outer as it passes, but this is made up for by it being accelerated when it is catching up again one orbit later.

(If the inner is orbiting twice as fast as the outer—a 2:1 resonance—then when the outer has reached its grey position the inner is back at its black position. They are again in equilibrium.)

If the two are in resonance, this conjunction will happen repeatedly. The influences will go on averaging out and *if the orbits do not change* the outer will not affect the inner.

Of course, the orbits do change over time under these influences, but let's stick to (relatively) short timescales.

Now suppose the closest approach of the two moons does not happen when the inner is at apoapsis (furthest away from the primary: the black position) but is just a little bit off. This is like displacing the pendulum from stable equilibrium: the conjunction will oscillate around the two black positions, sometimes happening earlier, sometimes later than the diagram shows.

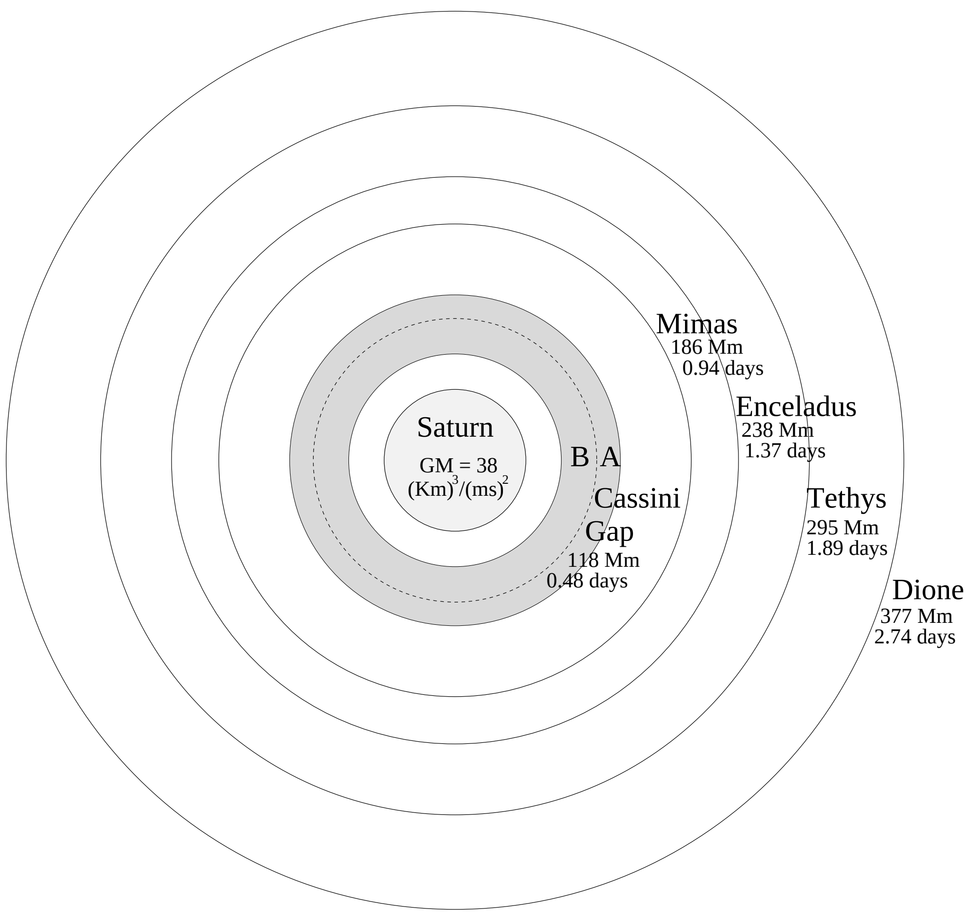
Or, if the inner *orbit* is inclined a little so apoapsis and the inner black position is, say, higher than the line from outer to primary, then the *orbit* may "precess" and that precession may oscillate around the equilibrium configuration shown in the diagram. This will have some consequences for clearing gaps in rings.

The above discussion is not very satisfactory. It is a gloss over the notorious "three-body problem": the gravitational attraction among three bodies has no analytical solution such as the conic sections of two-body gravitation. So we must resort to computer simulation of the forces among the bodies. And even computer simulation fails in many cases because the behaviour is *chaotic*, meaning that extraordinarily small changes in the input configurations lead to changes in the results so large as to make them effectively unpredictable.

Much effective work has been done despite these obstacles. The Excursion for this Note gives a couple of pointers.

Gaps in rings bring us to Saturn.

	Cassini	Mimas	Enceladus	Tethys	Dione	Rhea	Titan	Hyperion	Iapetus
a Mm	118	186	238	295	377	527	1221	1481	3561
e		0.0201	0.0044	0.0000	0.0022		0.0289	0.1042	
n deg/day	756.7	381.4	262.5	190.6	131.5	79,7	22.6	16.9	4.5
P days	0.476	0.94	1.37	1.89	2.74	4.52	15.95	21.29	79.38
v Km/s	18.0	14.3	12.6	11.3	10.0	8.5	5.6	5.1	3.3



(My data source is NASA again but I took the eccentricities e that I've listed from Peale's review cited in the Excursions. And I did the same calculations, with $GM = 6.674_{10}{-}11 \times 0.568_{10}27$, as for Jupiter's moons, even though the eccentricities are larger for Saturn.)

Note that the first entry, "`Cassini`", is not a moon but the gap between Saturn's two brightest rings, the A ring and the B ring. As well as stabilities, resonances cause instabilities and can clear gaps in, say, the rings of Saturn or the asteroid belt.

Before we get to the rings, note that Mimas and Tethys, Enceladus and Dione, and Titan and Hyperion further out, each have 2:1 resonances. The inner four moons are shown in the diagram.

Finally, any object in the Cassini gap between rings A and B would have a 2:1 resonance with Mimas, the innermost moon. (Saturn's rings have many other gaps, mostly in simple resonances with Mimas or other moons.)

The mechanism of how gaps are cleared is complex, involving chaotic behaviour which permits apparently stable orbits to last a long time then suddenly fall out. A sketch might involve precession of the apoapsis and consequent collision with nearby bodies.

Such processes are particularly significant in the asteroid belt between Mars and Jupiter, with Jupiter acting as the disturbing body. Some more data:

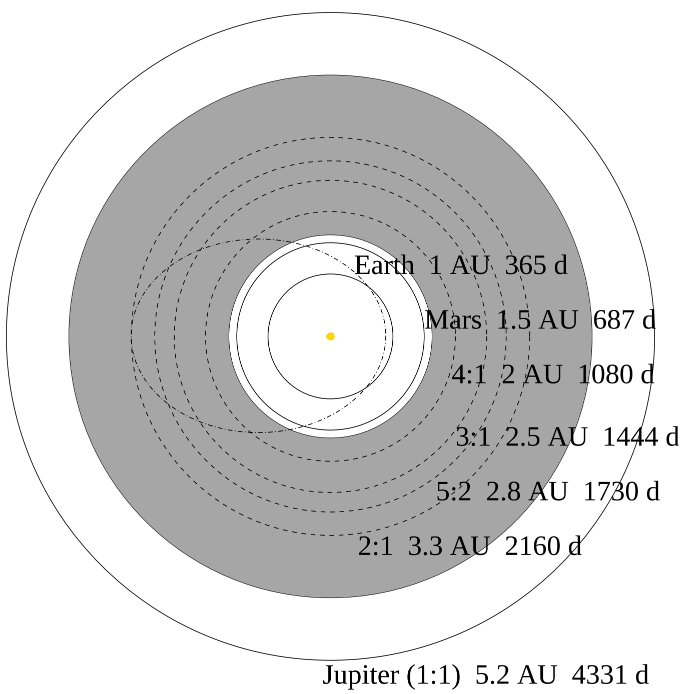
	Earth	Mars	4:1	3:1	5:2	2:1	Jupiter
a Gm	149.6	227.8	308.2	374	421.9	489.2	777.9
n deg/day	0.986	0.524	0.333	0.249	0.208	0.167	0.083
P days	365	687	1080	1444	1730	2160	4331
v Km/s	29.8	24.1	20.8	18.8	17.7	16.5	13.1

An orbit deflected from 3 AU might reach Mars if its eccentricity (Notes 23 and 9)

$$e = \frac{f}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4r_1 r_2}{(r_1 + r_2)^2}}$$

were

$$\sqrt{1 - \frac{4 \times 3 \times 1.5}{(3 + 1.5)^2}} = \frac{1}{3}$$



and Earth if its eccentricity were

$$\sqrt{1 - \frac{4 \times 3 \times 1}{(3+1)^2}} = \frac{1}{2}$$

From the 2:1 resonance at 3.3 AU these eccentricities become 0.375 and 0.535 respectively. I've suggested the latter orbit in the diagram.

Mars has played a role in removing asteroids from these gaps. They get pulled out by resonance and wind up colliding with Mars.

Earth is a less likely target, but there is a danger. Because of the chaotic aspects, asteroids heading our way cannot be predicted until they are on the way. So we must keep a watch.

The delta-V required to change from a 3 AU orbit (asteroid Ceres) to the Earth-crossing orbit is (Note 15)

$$\begin{aligned}\sqrt{\frac{GM}{r_1}} - \sqrt{GM\left(\frac{2}{r_1} - \frac{2}{r_1+r_2}\right)} &= \sqrt{\frac{GM}{r_1}}\left(1 - \sqrt{\frac{2r_2}{r_1+r_2}}\right) \\ &= \sqrt{\frac{400_{10}18}{3 \times 3 \times 0.15_{10}12}}\left(1 - \sqrt{\frac{2}{4}}\right) = \frac{20_{10}3}{3}\frac{1}{\sqrt{0.15}}\left(1 - \frac{1}{\sqrt{2}}\right) \\ &= 5.04 \text{ Km/sec.}\end{aligned}$$

For the 2:1 resonance at 3.3 AU this number becomes 5.22 Km/sec. If we spot the rogue asteroid just as it gets this kick, and find a way to reduce the kick to, say, 5 Km/sec, we can remove the risk.

Chaotic dynamics will also oblige us to be circumspect if we ever later on start mining the asteroids. Ships and even robots should stay clear of the gaps.

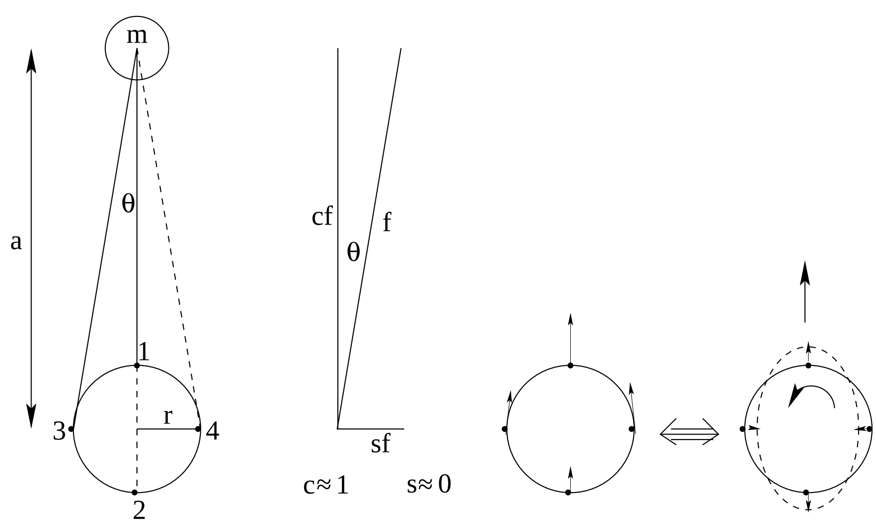
How all these resonances came about, and the issues of stability and instability, are ongoing problems, some of which have been cleverly solved. One important consideration is tides.

26. Tides. We mentioned friction in passing in the previous Note, discussing the drag of the air on a pendulum, slowing it down and eventually stopping it.

There is negligible such friction in the vacuum of space. However, one significant source of friction in the motion of the planets and moons is the tides.

We will also see that tides give a way to change the radius of an orbit, in fact to make it bigger.

Tides are caused by the force of gravity being stronger or weaker at different points in space.



The force from the Moon (on top of the diagram) at point 1 on the Earth's surface is

$$f_1 = \frac{Gm}{(a-r)^2}$$

while the force at point 2 is

$$f_2 = \frac{Gm}{(a+r)^2}$$

(I've used small fs here to indicate that I'm referring to *specific* forces, that is forces per unit mass.)

These differ by

$$f_1 - f_2 = Gm\left(\frac{1}{(a-r)^2} - \frac{1}{(a+r)^2}\right) = Gm\frac{4ar}{(a^2-r^2)^2} \approx Gm\frac{4r}{a^3}$$

So there is a net up-and-down stretching force trying to separate points 1 and 2.

The $\approx$ means approximately equal, and is true since the distance to the moon, a, is much bigger than the radius r of the Earth, so $(a^2 - r^2)^2$ is very nearly a^4.

Points 3 and 4, on the other hand, get squeezed together. The forces

$$f_3 = \frac{Gm}{a^2 + r^2} = f_4$$

have vertical and horizontal *components* which I've shown as cf and sf respectively.

$$\vec{f_3} = f_3(s, -c) \qquad \text{and} \qquad \vec{f_4} = f_3(-s, -c)$$

Since a is so much bigger than r the angle θ is very small and cf almost equals f while sf almost equals zero.

But it's not exactly zero and the difference

$$\vec{f_4} - \vec{f_3} = -2sf_3(1, 0)$$

contributes to the squeeze.

These forces act particularly on the Earth's oceans, which squeeze and bulge, as shown by the dashed ellipse.

You can see why we have two high tides and two low tides a day. As the Earth rotates under the Moon, the bulges appear to circle around the planet from east to west.

You can also appreciate that the tides experience friction as they try to circle the planet, dashing against the land masses but mainly against the ocean floor in shallow parts of the seas.

This friction will tend to slow down the rotation of the Earth. Indeed, there is geological and paleontological evidence that, 620 million years ago, the day was 21.9 hours and has now slowed down to 24 hours.

The effect of this is to force the Moon further away from the Earth. Why? Because angular momentum is conserved.

Briefly, as the Earth slows down it loses angular momentum. To be conserved, that angular momentum must go somewhere, and it goes to the orbiting Moon.

It is easy to work out the angular momentum of the Moon in its orbit (we'll neglect the angular momentum of the *Moon*'s rotation and assume incorrectly that it doesn't change). It is

$$J_{\text{Moon}} = mva$$

where m is the mass of the Moon, a is the radius of its orbit (actually, semi-major axis, but I'm making another incorrect assumption that the orbit is circular), and v is its orbital speed

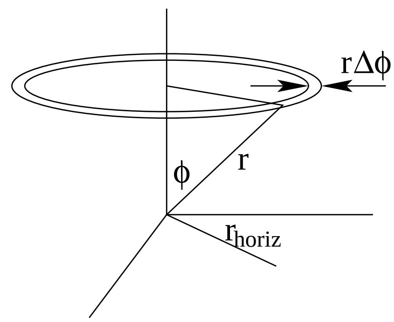
$$v = a\omega$$

That ω is the Moon's *angular velocity*, in radians per second, which, if a is measured in meters, gives a speed v in meters per second.

To find the angular momentum of the rotating Earth requires a digression.

We must find the angular momentum of a rotating sphere. This is the sum of the angular momenta of its parts.

We can start with a simple ring around the axis of rotation. Every part of the ring has the same speed, $r_{\text{horiz}}\omega$, where r_{horiz} is the radius of the ring, which I've distinguished from the radius r of the spherical shell of which the ring is a part, and from the radius R of the whole sphere, of which the shell is a part.



So the angular momentum of this ring is its mass times that speed times r_{horiz} again.

To discuss the mass of a part of a sphere it's best to work with the *density* ρ of the sphere. This is mass per unit volume.

$$\rho = \frac{M}{V} = \frac{M}{4\pi R^3/3}$$

where the whole sphere masses M Kg and has volume V given by the formula in its radius R meters.

So the mass of the ring is ρ times its volume, which is $2\pi r_{\text{horiz}}$ times its width $r\Delta\theta$, shown, times the thickness of the spherical shell it is part of, Δr.

I've cheated in drawing the picture: the ring is not flat as shown, but must be thought of as being a part of a hollow spherical shell of radius r. The width, which I've called $r\Delta\theta$, and the thickness, which I've called Δr, are supposed to be so small that this distortion does not make any difference.

So the angular momentum of this ring part of a spherical shell inside the sphere is

$$(\rho(2\pi r_{\text{horiz}})(r\Delta\theta)(\Delta r))(r_{\text{horiz}}\omega)(r_{\text{horiz}})$$

What we must now do is add up all the different contributions of the little $\Delta\theta$s, for all angles θ, and of the little Δrs for all the radii r inside the sphere. That is, θ goes from 0 to $180^o = \pi$ radians, and r from 0 to R. This can be done by lots of arithmetic or, better, by *calculus*.

I'll just give the result for the whole sphere of radius R and mass M rotating at ω radians per second. It is

$$J = 2\frac{1}{5}\frac{4\pi}{3}R^5\rho\omega = \frac{2}{5}MR^2\omega$$

where I've gone back to $M = \rho V = \rho 4\pi R^3/3$.

That's the angular momentum. We're also going to need the kinetic energy of the rotation of the Earth, in order to find out how much energy has been lost to friction.

It turns out to be the same calculation. Instead of adding up mvr_{horiz} for each ring we add up $mv^2/2$. So the answer is very similar.

$$K = \frac{1}{5}\frac{4\pi}{3}R^5\rho\omega^2 = \frac{1}{2}\frac{2}{5}MR^2\omega^2$$

We see that there is a common element in both results. It is called the *moment of inertia* for the sphere

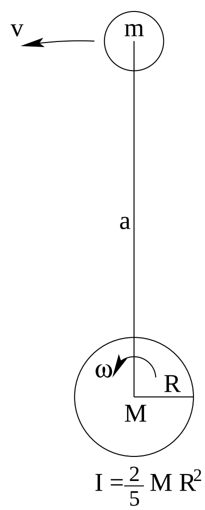
$$I = \frac{2}{5}MR^2$$

and it makes rotational momentum and kinetic energy look almost the same as their ordinary counterparts.

$$\begin{aligned} J &= I\omega \\ K &= \frac{1}{2}I\omega^2 \end{aligned}$$

That ends the digression. Now we can see what happens to the Moon as it slows down the Earth's rotation.

We'll be interested in the angular momentum $I\omega$ and the rotational kinetic energy $I\omega^2/2$ of the Earth, and the angular momentum mva of the Moon.



For Earth (pretending it is a sphere with radius intermediate between polar and equatorial)

$$I = \frac{2}{5}MR^2 = \frac{2}{5}5.97_{10}24 \times (6.37_{10}6)^2 = 96.9_{10}36$$

So at 1 revolution per day which is

$$\omega = \frac{2\pi}{24 * 3600} = 7.27_{10}\text{–}5 \text{ radians/sec}$$

we have

$$J_E = I\omega = 7.05_{10}33 \text{ joule–sec}$$

But $620_{10}6$ years ago this was higher: 24/21.9 of this

$$J'_E = I\omega' = 7.72_{10}33 \text{ joule–sec}$$

for a difference of $0.67_{10}33$ joule-sec.

This difference cannot be lost, by conservation of angular momentum, so it must go to changing the angular momentum of the Moon.

Today the Moon's angular momentum is

$$J_M = mva = ma\sqrt{\frac{GM}{a}} = m\sqrt{GMa} = 73.46_{10}21\sqrt{400_{10}12 \times 384.4_{10}6} = 28.8_{10}33 \text{ joule–sec}$$

where I've used the result for v from Note 11 and for GM_{Earth} from Note 12.

620 million years ago it was lower:

$$J'_M = J_M + J_E - J'_E = 28.8_{10}33 - 0.67_{10}33 = 28.13_{10}33$$

So

$$a' = \left(\frac{J'_M}{m}\right)^2 \frac{1}{GM} = \left(\frac{28.13_{10}33}{73.46_{10}21}\right)^2 \frac{1}{400_{10}12} = 366.6 \text{ megameters}$$

This is closer than the present-day orbit of 384 megameters by 17.8 megameters or 29 millimeters per year for the last 620 million years.

The Moon is moving away from Earth at 29 mm/year.

How much energy has been lost to friction?

The Earth's present-day rotational energy is

$$K_E = \frac{1}{2}I\omega^2 = \frac{1}{2}96.9_{10}36(7.27_{10} - 5)^2 = 2.561_{10}29 \text{ joules}$$

It was more 620 million years ago,

$$K'_E = \left(\frac{24}{21.9}\right)^2 K_E = 3.075_{10}29 \text{ joules}$$

a loss, over that time of $0.514_{10}29$ joules, or $2.65_{10}12$ joules/sec.

This is 2.65 terawatts, or pretty close to total U.S. power consumption. How much of the Moon's tidal energy, which otherwise just produces heat, could we convert to useful work?

The heating caused by tides affects other planetary moons. Although Jupiter is five times further from the Sun than we are, Jupiter's second moon, Europa, is a water world, albeit under 100 kilometers of ice. And Jupiter's innermost moon, Io, is volcanic, with its rocks being water ice and its lava being water.

Tides work not only from Moon to Earth but also the other way around. The Moon does not have oceans to slosh around but the tidal forces stressed the Moon anyway. This has slowed its rotation until its day is a month long: we now always see the same side of the Moon. At this point, the tidal stresses have stopped[5] because the Moon is not rotating relative to Earth's gravity.

Just as tides enlarge the orbit of Earth's Moon, they can enlarge the orbits of other satellites, or even planets. Changing the orbit changes its period and can bring two or more satellites into

[5]That is, tidal stresses due to Earth. The Sun still has a tidal effect on both Earth and Moon.

commensurate orbits as seen in the previous Note.

27. The Lagrange points. The resonances in Note 25 involved orbital periods in simple whole-number relationships. But we did not discuss the simplest of all. Are there 1:1 resonances?

Yes. For any planet-primary system, or moon-planet system, there are five of them. They are called the *Lagrange points*.

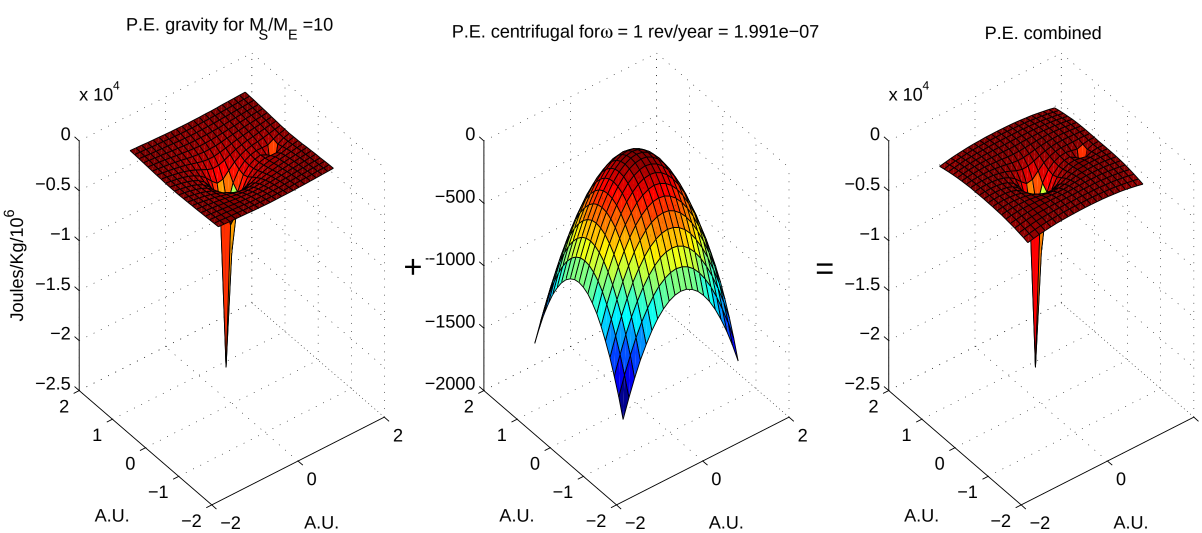
They result from the combination of gravity with rotational forces.

In Note 18 we found that potential energy is a handy way of visualizing forces. And we already there discussed gravitational potential energy as a kind of funnel, getting steeper as it narrows in on the gravitational source.

Centrifugal force can also be thought of in terms of potential energy. If you are sitting on a spinning turntable you tend to slide towards the outer rim. If you are in the middle there is no force on you. But as you get closer to the edge the outwards force increases.

From the viewpoint of potential energy you can imagine you are sitting on an upside-down bowl. It is horizontal in the middle and so you experience no force. But it gets steeper and steeper further from the centre, and you slide harder and harder.

Here are pictures of the two potential energies and what they look like combined as a sum. Note that the gravitational potential energy of the Sun is accompanied by a dimple which is the gravitational potential energy of the Earth.



We see that the combined potential energy has a maximum somewhere around the orbit of the Earth, beyond which the centrifugal forces bend it down, overcoming the gravitational attraction.

We can see this better with a contour plot of the same data. First we have the two gravitational potential energies alone.

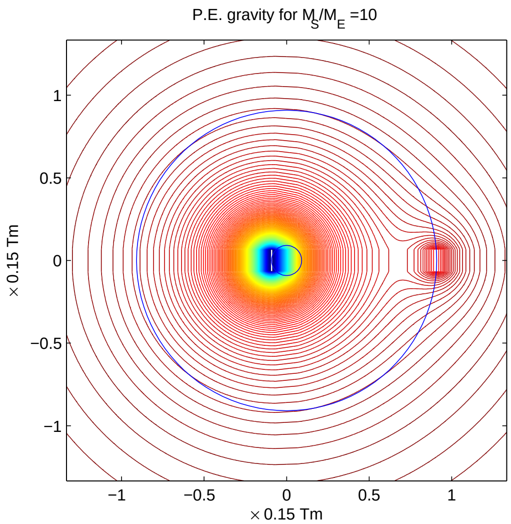
$$PE_{\text{grav}} = -\frac{GM_{\text{Sun}}}{d_{\text{Sun}}} - \frac{GM_{\text{Earth}}}{d_{\text{Earth}}}$$

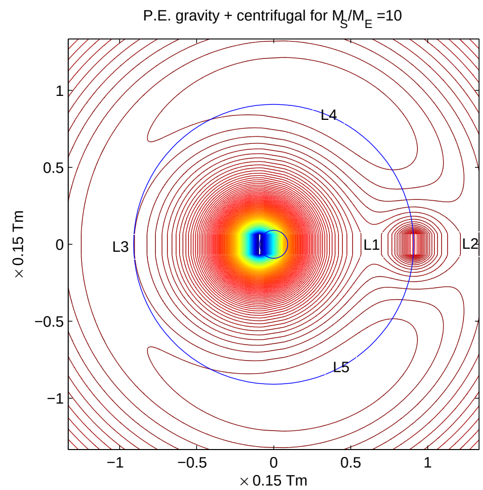
where

$$\begin{aligned} d_{\text{Sun}} &= \sqrt{(x + r_S)^2 + y^2} \\ d_{\text{Earth}} &= \sqrt{(x - r_E)^2 + y^2} \end{aligned}$$

are the respective distances from an arbitrary point (x, y) to the Sun and the Earth. The Sun is located at $(-r_S, 0)$ and Earth is at $(r_E, 0)$ with

$$r_S = \frac{M_{\text{Earth}}}{M_{\text{Sun}} + M_{\text{Earth}}} \text{ A.U.}$$
$$r_E = \frac{M_{\text{Sun}}}{M_{\text{Sun}} + M_{\text{Earth}}} \text{ A.U.}$$





Then we add the centrifugal force caused by a rotation of 1 revolution per year or

$$\omega = \frac{2\pi}{365.25 \times 24 \times 3600} \text{ radians/sec.}$$

which has a (pseudo) potential energy

$$PE_{\text{centrif}} = -\frac{\omega^2(x^2 + y^2)}{2}$$

an inverted paraboloid.

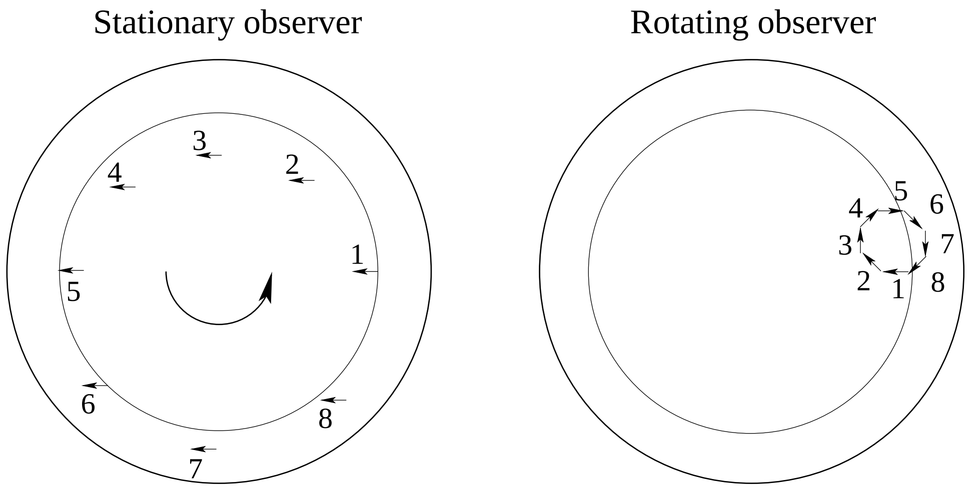
The L1 and L2 Lagrange points are at “saddle points” on the potential energy surface. If you check the contours closely you’ll see that L1 is on a ridge, dropping left and right into the gravity funnels for Sun and Earth, but going uphill above and below to the higher contours around the L5 and L4 points.

The L4 and L5 Lagrange points are at the tops of the two hills which are in fact the highest points of the combined potentials. And the L3 point is another saddle point in the flattest region of the whole potential energy surface.

All five Lagrange points are equilibrium points: the surface is horizontal where they are so there is no net force. However, the equilibria do not look stable: if something disturbs a body sitting on the horizontal surface, it will move away from equilibrium, either in any direction from the top of a hill or in one of two opposite directions from a saddle point.

This instability is not just apparent for L1, L2 and L3. A satellite parked in any of them will need thrusters to keep itself there. Calculations show that such corrections must be made, statistically, every 23 days for L1 and L2, and every 150 years for L3. Even so, they are good places to park because they don’t need *much* energy to stay there. So NASA put a solar observatory at L1, where it can see the Sun 24/7, and a satellite studying the cosmic microwave background at L2, where it can avoid the Sun 24/7.

For L4 and L5, the instability is overcome by another rotational pseudoforce, the *Coriolis* force. This is an apparent force perceived by a rotating observer watching a body move in what to a stationary observer is a constant direction while also being carried along with the rotation.



On the left are the successive legs of a body moving horizontally with velocity $\vec{v}$ while also sharing the rotation $\vec{\omega}$ of a turntable.

On the right is what this motion looks like to an observer on the turntable at, say, the starting point of the body's horizontal motion. Just compare the angles of positions 2, 3, $\cdots$ 8, on the right, to the inner circle marking the position of the observer.

We see that the motion appears to the rotating observer to be forced rightwards by a mysterious acceleration.

This acceleration is usually described mathematically as

$$\vec{v} \times \vec{\omega}$$

(Actually, the acceleration is twice this quantity, but a bane for theoretical physicists is tracking down elusive factors of 2, and I won't go into the explanation.)

Here $\vec{v}$ and $\vec{\omega}$ are the quantities v and ω *directed* in three dimensions, and $\times$ is a special kind of multiplication in which, among other things, reversing the order changes the sign ($\vec{\omega} \times \vec{v} = -\vec{v} \times \vec{\omega}$), has a direction perpendicular to both, and, if $\vec{v}$ and $\vec{\omega}$ themselves are perpendicular to each other, just has the value $v\omega$.

A "right-hand rule" is important in this new kind of three-dimensional math. First, the direction of $\vec{\omega}$ is given by the direction of the thumb of your right hand if you curl the fingers in the direction of the rotation. So in the picture, $\vec{\omega}$ points straight up out of the page.

Second, the direction of $\vec{v} \times \vec{\omega}$ is given by the direction of the thumb of your right hand if you curl the fingers so as to push $\vec{v}$ towards $\vec{\omega}$.

Check these out in the diagram and confirm that the acceleration apparent to the rotating observer is rightwards.

(Directed quantities such as these are called "vectors". Or $\vec{v}$ is: $\vec{\omega}$ is a "pseudovector" because it reverses direction on reflection.)

When all the numbers are put in, bodies at the L4 and L5 Lagrange points orbit those points stably because of this Coriolis acceleration.

Coriolis forces are significant on Earth, which is also rotating. The turntable we've discussed gives the behaviour of, say, winds near the North Pole[6].

If we consider a south wind (that is, blowing towards the north) as $\vec{v}$, at lower latitudes, it is no longer perpendicular to $\vec{\omega}$. This does not change the rightwards direction of the acceleration, but reduces its magnitude by some factor less than 1 which depends on the latitude: 1 at the North Pole but 0 at the Equator.

When we cross the Equator, the magnitude changes sign, reversing the direction of the Coriolis acceleration to leftwards. See if you can see why.

If we consider a west wind (blowing eastwards) at the North Pole, it curves rightward, too. As we go to lower latitudes the west wind remains perpendicular to the rotation, so neither magnitude nor direction of the acceleration change, but the direction becomes increasingly upwards as we slide sideways, as it were, down the globe of the Earth, until at the Equator it is entirely upwards.

These Coriolis accelerations are important for extreme weather. Consider a low-pressure region with winds therefore pointed into it from all directions. They all bend rightwards (in the northern hemisphere), giving a clockwise motion which can accelerate into a hurricane (cyclone) or even tornado. Such extremes occur if the air is warm enough to supply the energy needed. The low-pressure region is created in the first place by warm air rising over a hotspot.

In the southern hemisphere all these nasty things happen in the opposite direction.

To conclude our discussion of the L4 and L5 Lagrange points, the numbers require for stability that, approximately, $M_1 \geq 25M_2$, where M_1 is the primary and M_2 the satellite. This is certainly true for $M_{\text{Sun}} \approx (1_{10}6/3)M_{\text{Earth}}$. Where it is true, the Lagrange points are called "Trojans" because the asteroids discovered in Jupiter's L4 and L5 points were named after Greek and Trojan heros (Agamemnon, Achilles, Hector, etc.) in Homer (the Greeks being at one of the points and the Trojans at the other). A Sun-Earth Trojan asteroid was found in 2010.

Part III The Space Adventure.

28. Economics.
29. Microgravity.
30. Radiation.
31. Space debris.
32. Space elevator.
33. Ecology.
34. Population.
35. Genetics.
36. History.
37. Self-reproducing probes.
38. "Where Are They?"

Part IV Spaceship Earth.

39. Speeds.
40. Extinctions.
41. Herd science.
42. Climate.

Appendix. Trigonometry and calculus.

43. Trigonometry.
44. Integral calculus.
45. Differential calculus.

[6] The North Pole belongs to Canada: postal code H0 H0 H0.

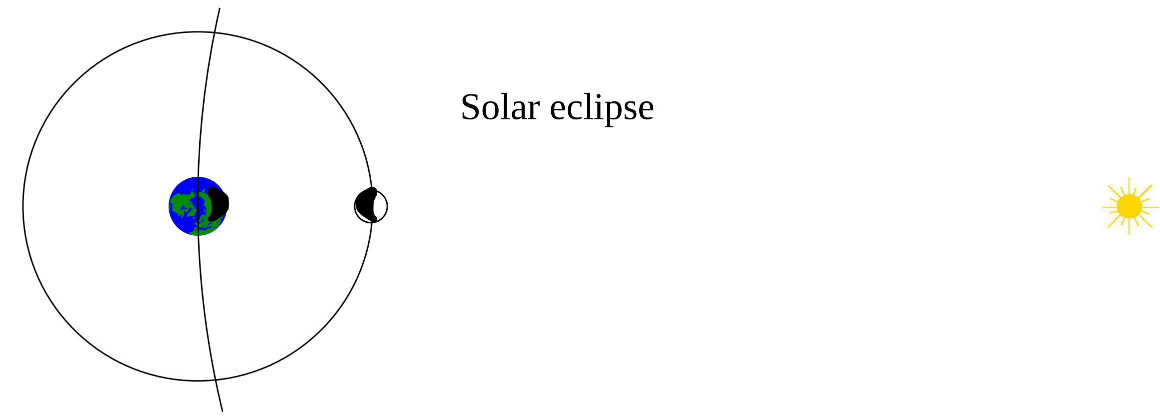
II. The Excursions

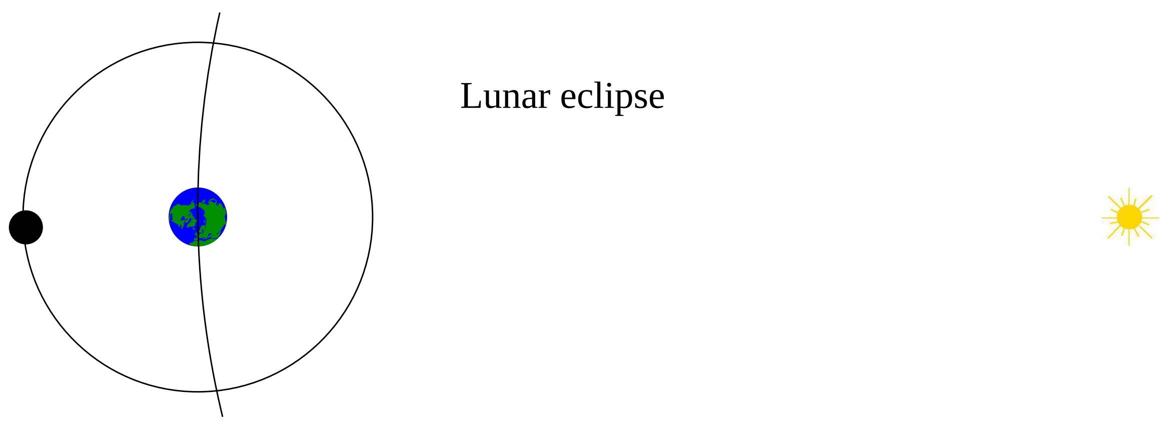
You've seen lots of ideas. Now *do* something with them!

1. Calculating solar eclipses

   With a little bit of math and astronomy we can say a little bit about eclipses.

   First, if the orbits of the Moon around the Earth and of the Earth around the Sun were in the same plane, we'd have a solar eclipse (the Moon blocks the Sun from the Earth) every New Moon. And we'd have a lunar eclipse (the Earth blocks the Sun from the Moon) every Full Moon.





   How often do New Moons (and hence Full Moons) happen? Relative to the fixed stars, the Moon orbits the Earth every 27.3 days and the Earth orbits the Sun every 365.25 days (365 plus a Leap Day every four years, almost).

   So we can describe the angles of the orbits as

$$\omega_{\text{Moon}} t = \frac{360}{27.3} t$$
$$\omega_{\text{Earth}} t = \frac{360}{365.25} t$$

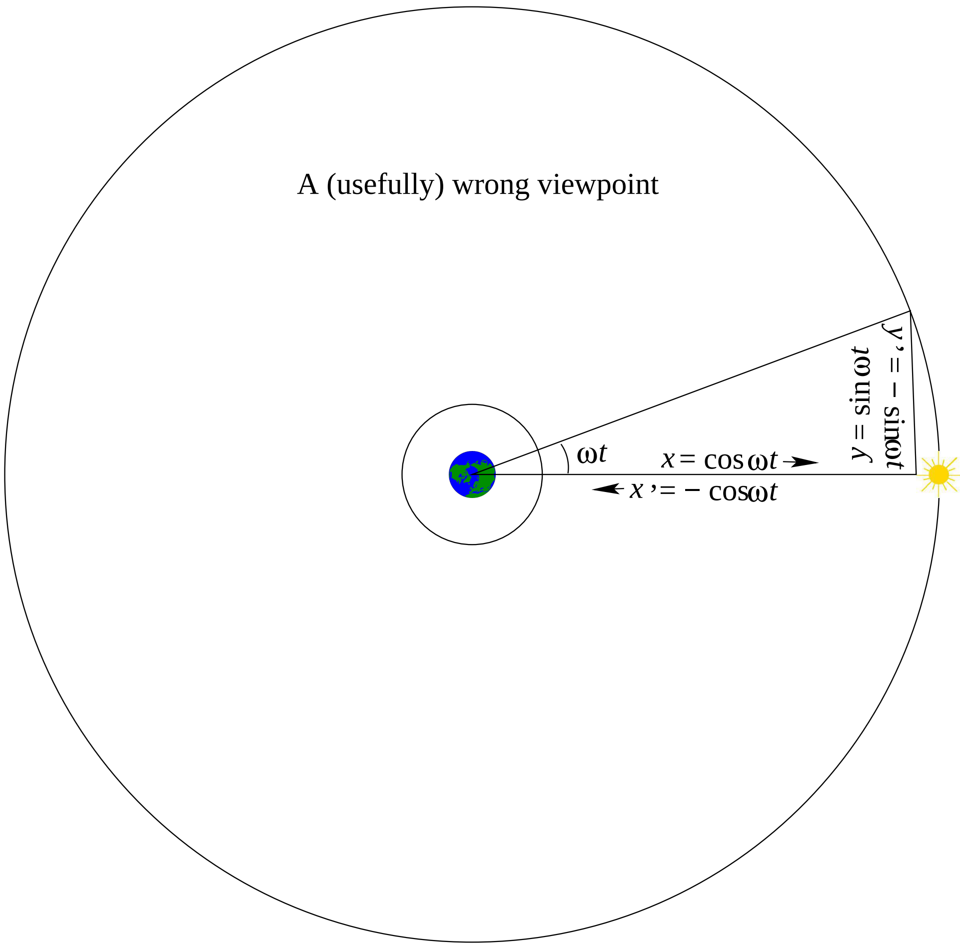
where t is the time in days since when the angles were 0 degrees, and the ωs are the *angular velocities* of the bodies in their orbits.

If we approximate both orbits as circles, we have

$$\begin{aligned} x &= \cos \omega t \\ y &= \sin \omega t \end{aligned}$$

in each case.

Now we know the Earth goes around the Sun, but mathematically it makes no difference if we say the Sun goes around the Earth, just as the Moon actually does. This will make it easier to calculate.



We have a New Moon every time both angles, $\omega_{\text{Moon}}t$ and $\omega_{\text{Earth}}t$—which I guess is now $\omega_{\text{Sun}}t$—are the same

$$\omega_{\text{Moon}}t + 360m = \omega_{\text{Sun}}t + 360s$$

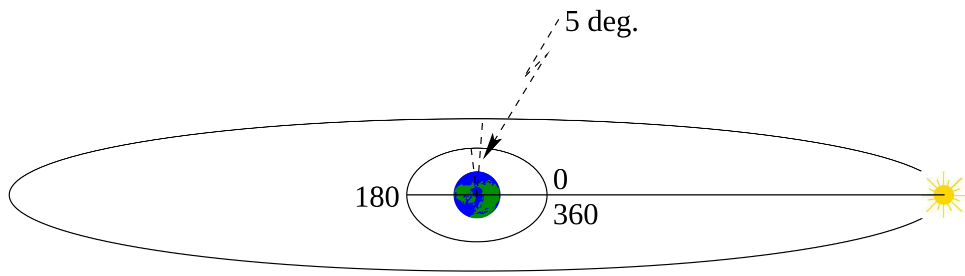
where m and s are just integers, and multiplying them by 360 degrees just says that we don't care how many complete orbits have happened before the alignment—so we can lump them together and simply say

$$\begin{aligned} \omega_{\text{Moon}}t &= \omega_{\text{Sun}}t + 360n \\ t &= 360n/(\omega_{\text{Moon}} - \omega_{\text{Sun}}) \\ &= 29.5053n \end{aligned}$$

where n is also just an integer ($s - m$ actually, but we'll forget about s and m and just use n from now) and the 29.5 days is the *sidereal* period of the Moon, relative to its alignment with the Sun, which is also "moving".

But we don't have solar eclipses every New Moon. That is primarily because the Moon's orbit is not *coplanar* with the Earth's orbit (or, from our usefully wrong perspective, the Sun's orbit).

Non−coplanar orbits



I've tried to show this in the drawing, with the Sun's orbit tilted away from us in perspective, and the Moon's orbit also tilted away from us but not so far.

The difference between the orbital planes is only about 5 degrees. But how much doesn't matter in the simple approach we are taking—which is to think of each body as a simple point. Once the moon is out of the solar plane, it cannot possibly form a straight line with Earth and Sun and so we cannot get an eclipse, either solar or lunar.

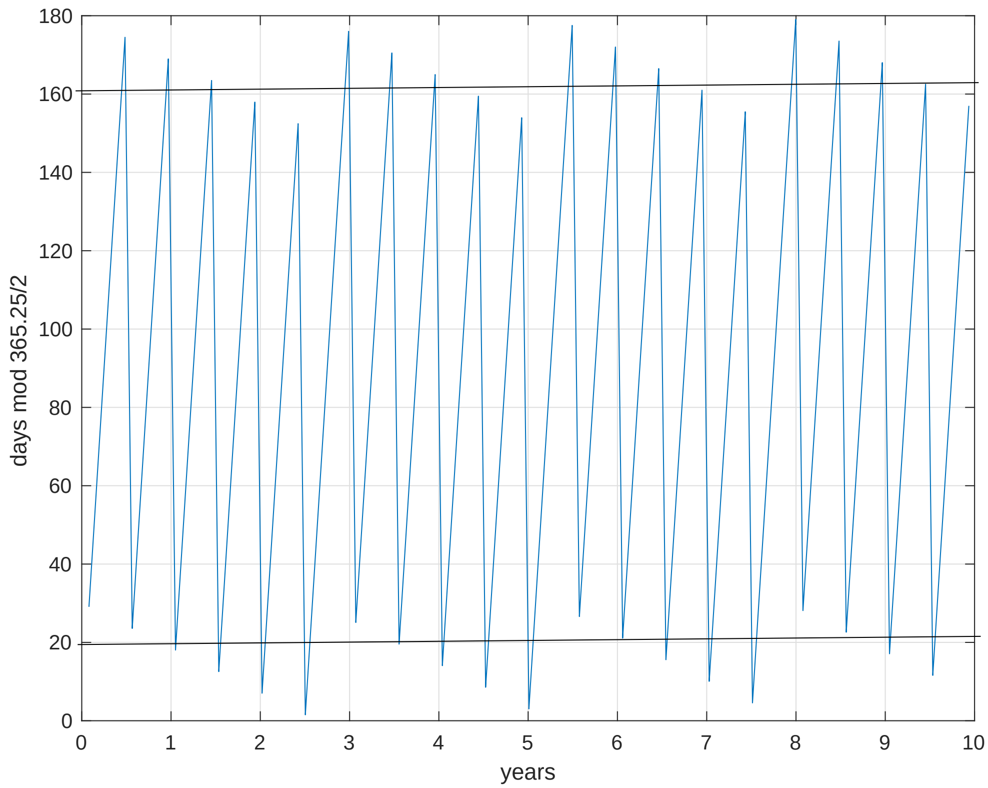
So the only way we can get, say, a solar eclipse, is when both Moon and Sun are at 0 degrees in their orbits, or both at 180 degrees, because that's when they and Earth are all in the same plane.

Here, I've plotted the remainders after dividing the angle the Sun traverses from New Moon 0 to new Moon n,

$$t(n) \times 360/365.25$$

by 180 degrees, where the number of days between those New Moons

$$t(n) \quad = \quad n \times \mathit{sidereal} = \frac{360n}{\omega_{\text{Moon}} - \omega_{\text{Sun}}}$$



In fact, there are 2 to 5 solar eclipses every year—much less often than the 29.5-day interval between New Moons, but surprisingly often. I've drawn two lines on the graph to show how this might happen, 20 degrees above 0 and 20 degrees below 180.

We can call these lines "tolerances". Clearly it is extremely unlikely that the angles will ever be exactly 0 or exactly 180. But the bodies are not simple points, either, and so we have some wiggle room. And the smallness of the 5-degree difference between planes is a consideration too. In our simple approach, the 20-degree tolerances give that wiggle room.

We haven't seen a solar eclipse in Montreal for years. Why is it claimed that there are at least a couple per year? Because not every eclipse shadows Montreal or eastern North America. Because the Moon and Earth wobble in their orbits.

In fact, the orbits are not circles but ellipses (but we could still pretend that the Sun goes around the Earth once a year). In fact they are not even ellipses because Newton's calculation that they are assumed that the Moon is influenced only by Earth's gravity. But it is also influenced by the Sun, and that "three-body problem" is no longer solvable in simple form but needs the detailed calculations that only computers can perform.

So to predict the next eclipse in Montreal will take more sophistication than we can muster now.

But we've done pretty well, and illustrated en route some important scientific strategies: make simple assumptions which you can calculate with, approximate, and when you can get no further check the facts and see if what you did was plausible. Then learn more math and science for the next attempt.

2. The orbital calculations of Notes 19 and 20 are based on [BW20] who use more precise numbers but consider a lower LEO. They give a detailed treatment of orbital plane changes.
3. The satellite and ring data for Jupiter and Saturn in Note 25 are from [Wil19]. The discussion of stabilities and instabilities in that Note is greatly amplified by Peale's review [Pea76] and brought up to date by [TW03].
   You can look at Jupiter's Galilean moons, and Saturn's rings, through a small telescope.
4. Look up values for the numbers calculated in Note 26, on tides. What you'll find won't agree exactly, but our very simplified calculations get pretty close.
5. The full mathematics of the Lagrange points of Note 27 are given in [Cor98].
6. Any part of the Preliminary Notes that needs working through.

References

[BW20] Marc Bulcher and Stephen Whitmore. MAE 5540 propulsion systems. URL = http://maenas.eng.usu.edu/MAE_5540_Web/propulsion_systems/section2/section2.4.pdf, last accessed 16 Oct. 2020, 2020.

[Cor98] Neil J. Cornish. The lagrange points. URL https://map.gsfc.nasa.gov/ ContentMedia/lagrange.pdf Last accessed Jan 28, 2021 Linked from https://solarsystem.nasa.gov/resources/754/what-is-a-lagrange-point/ June 23, 2020, by the same author., 1998.

[Pea76] S K Peale. Orbital resonances in the solar system. *Ann. Rev. Astro. Astrophys.*, 14(A76-46826 24-90):215–46, Sept. 1976.

[TW03] Jihad Touma and Jack Wisdom. Orbital dynamics. URL http://web.mit.edu/ 12.004/TheLastHandout/PastHandouts/Chap03.Orbital.Dynamics.pdf Notes for course 12.004 Introduction to Planetary Science, Last accessed 21/1/15., 2003.

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