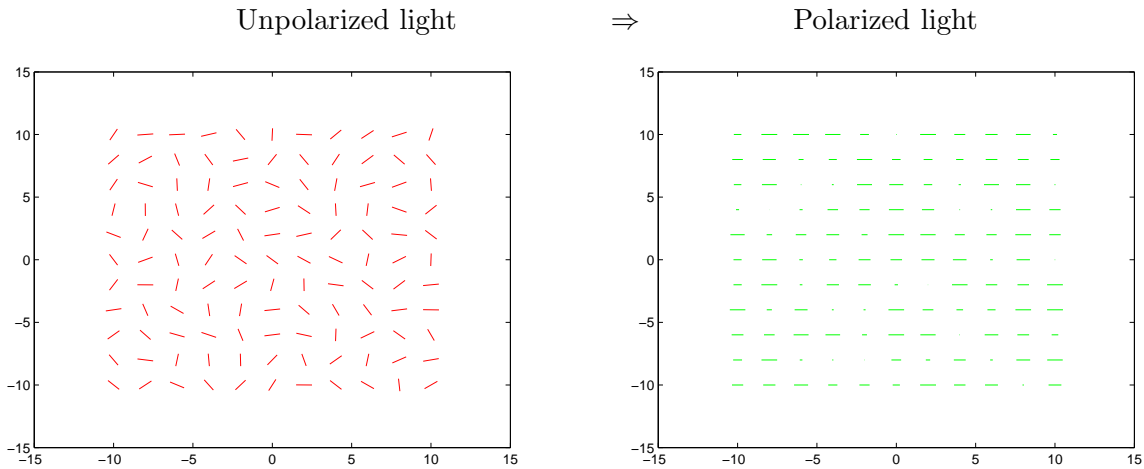# Excursions in Computing Science:
Week 2. Operators

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I. Prefatory Notes

1. A polarizing filter does something to light.

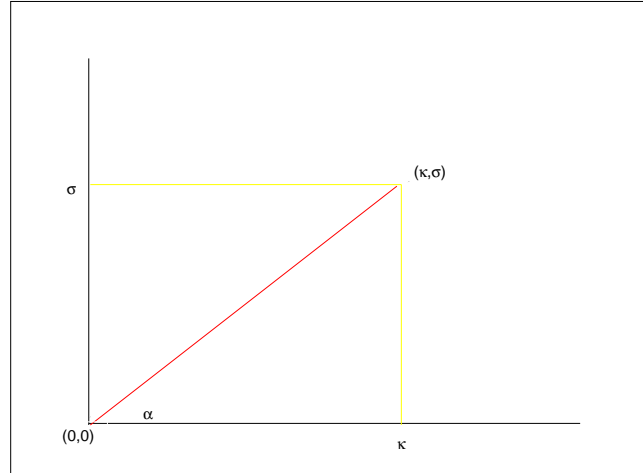We should think of filtering as an operation and the filter as an operator.

Unpolarized light $\quad\quad\Rightarrow\quad\quad$ Polarized light

2. We'll call a light beam a *vector*

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

(using κ, the Greek k, for $\cos(\alpha)$, and σ, the Greek s, for $\sin(\alpha)$).

3. A horizontal polarizing filter extracts the κ component.
(Let's work with amplitudes: we can always square them at the end to get intensities.)

For a vector ($\vec{u}$ for "unpolarized"),

$$\vec{u} = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

what operator finds the κ component?

Easy. Just set $\sigma \rightarrow 0$.

4. What if the polarizing filter is at angle θ?

Describe the filter as c, s: $c = \cos\theta, s = \sin\theta$.

So the result will be along the vector ($\vec{p}$ for "polarized")

$$\vec{p} = \begin{pmatrix} c \\ s \end{pmatrix}$$

Here's a way to figure out the matrix that maps ("projects") *every* point in 2-dimensional space onto a single line such as $\vec{p}$.

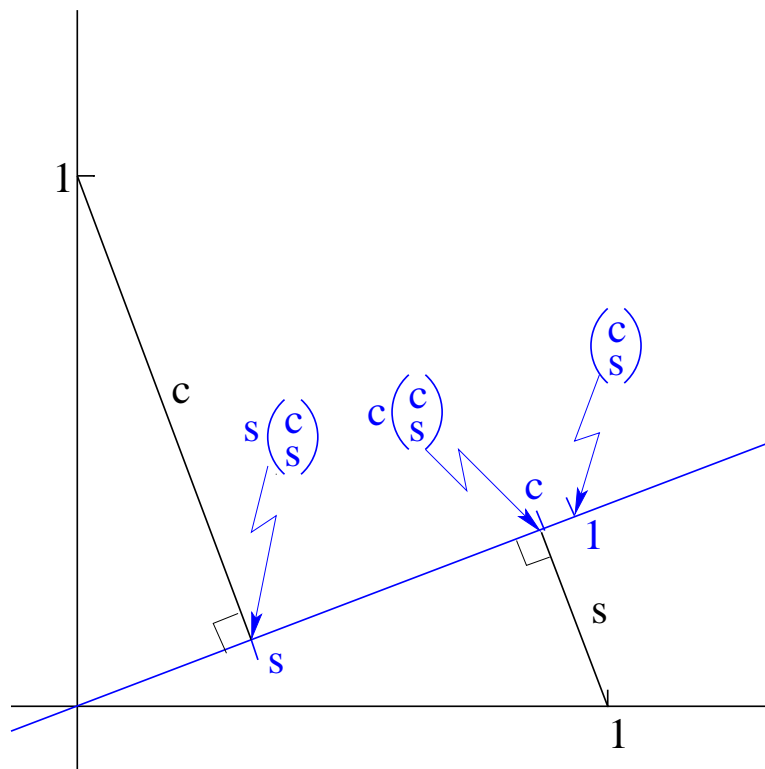A vector such as $\vec{u}$ can be thought of as the sum of two components.

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} = \kappa \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So it suffices to find out how to map the two points

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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into the line having direction

$$\begin{pmatrix} c \\ s \end{pmatrix}$$

From this we see that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

maps to

$$c\begin{pmatrix} c \\ s \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

maps to

$$s\begin{pmatrix} c \\ s \end{pmatrix}$$

Now we ask: which matrix will do these mappings? It should be clear that to do the first, the matrix must have the first column

$$\begin{pmatrix} c^2 \\ cs \end{pmatrix}$$

And to do the second, it must have the second column

$$\begin{pmatrix} cs \\ s^2 \end{pmatrix}$$

So we get the *projection* matrix from 2-dimensional space to the 1-dimensional line

$$\begin{pmatrix} c \\ s \end{pmatrix}$$

namely

$$\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

5. Now some new math. Two kinds of "vector multiplication":

i) "inner product" (or "dot product"), $\vec{p}^T\vec{u}$, (1 by n) × (n by 1):

$$\begin{pmatrix} c \\ s \end{pmatrix} \underline{(c,s) \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}} = \begin{pmatrix} c \\ s \end{pmatrix} (c\kappa + s\sigma) \qquad \text{(1 by 2)} \times \text{(2 by 1)} \rightarrow \text{(1 by 1)}$$

ii) "outer product", $\vec{p}\vec{p}^T$, (n by 1) × (1 by n):

$$\underline{\begin{pmatrix} c \\ s \end{pmatrix} (c,s)} \begin{pmatrix} \kappa \\ \sigma \end{pmatrix} = \begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix} \begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \qquad \text{(2 by 1)} \times \text{(1 by 2)} \rightarrow \text{(2 by 2)}$$

iii) There is also a "scalar product", (m by n) × "1 by 1" or "1 by 1" × (m by n):

$$\begin{pmatrix} c \\ s \end{pmatrix} (c\kappa + s\sigma) = \begin{pmatrix} c^2\kappa + cs\sigma \\ sc\kappa + s^2\sigma \end{pmatrix} \qquad \text{(2 by 1)} \times \text{(1 by 1)} \rightarrow \text{(2 by 1)}$$

$$= (c\kappa + s\sigma) \begin{pmatrix} c \\ s \end{pmatrix} \qquad \text{(1 by 1)} \times \text{(2 by 1)} \rightarrow \text{(2 by 1)}$$

iv) And the general "matrix product", (k by m) × (m by n):

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix} \begin{pmatrix} \kappa \\ \sigma \end{pmatrix} = \begin{pmatrix} c^2\kappa + cs\sigma \\ sc\kappa + s^2\sigma \end{pmatrix} \qquad \text{(2 by 2)} \times \text{(2 by 1)} \rightarrow \text{(2 by 1)}$$

In MATLAB:

```
alpha = pi/6;            p'                      p * pu
kappa = cos(alpha);
sigma = sin(alpha);      ans =                   ans =
u = [kappa;sigma]
                             0.8090    0.5878        0.8046
u =                                                  0.5846
                         pu = p' * u
    0.8660                                       pu * p
    0.5000               pu =
theta = pi/5;                                    ans =
c = cos(theta);              0.9945
s = sin(theta);                                      0.8046
p = [c;s]                                            0.5846
                         pp = p * p'
p =                                              pp * u
                         pp =
    0.8090                                       ans =
    0.5878                   0.6545    0.4755
                             0.4755    0.3455        0.8046
                                                     0.5846
```
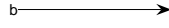
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6. So the polarizing filter is the operator

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}$$

Let's try this for the bottom, middle, and top filters, **b**, **m**, **t**.
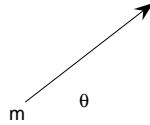**b**: $\theta = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

m: $\theta = $ any angle

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}$$

t: $\theta = \pi/2$
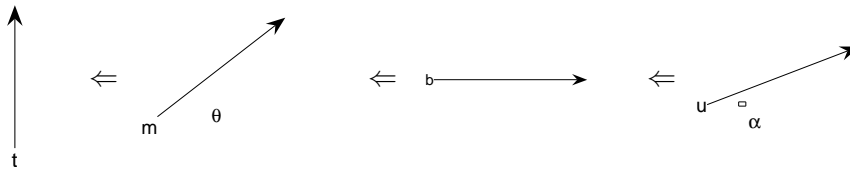
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

7. We filter

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

by **b** then **m** then **t**:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^2 & 0 \\ sc & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ sc & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ sc\kappa \end{pmatrix}$$

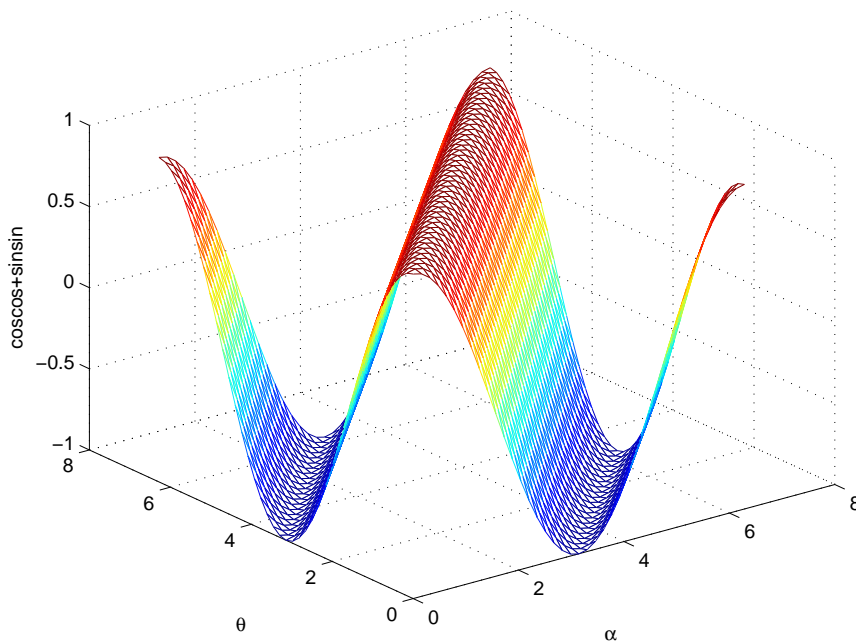where sc is $\sin\theta\cos\theta$ as before.

8. We keep seeing c^2, cs, s^2, i.e., $\cos^2(), \cos()\sin(), \sin^2()$, and even $c\kappa + s\sigma$.

What do these combinations of trig. functions mean?

Let's try MATLAB

```
alpha = 0:pi/20:2*pi;
theta = 0:pi/20:2*pi;
mesh(theta',alpha,cos(theta')*cos(alpha) + sin(theta')*sin(alpha))
xlabel('\alpha'),ylabel('\theta'),zlabel('coscos+sinsin')
```


Look carefully! This goes through two cycles as α and θ each go through one.

We can get at this by looking at another operator, *rotate*.

Rotate: $(x, y) \rightarrow (x', y')$

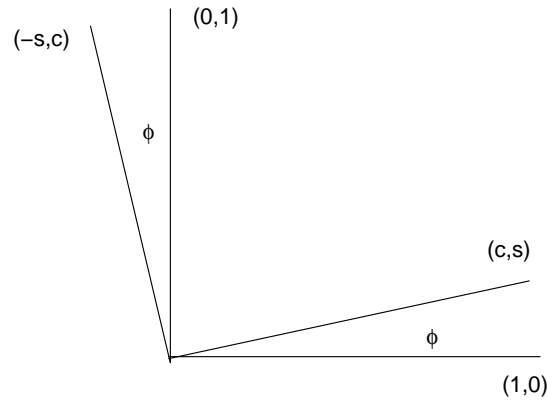So let's consider rotation to be an operator, say

$$R = \begin{pmatrix} p & u \\ q & v \end{pmatrix}_\phi$$

where ϕ is the angle of rotation.

Rotating the x-axis, $R\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, should give the line oriented at angle ϕ:

$$R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ s \end{pmatrix}_\phi$$

This means $p = c$ and $q = s$. (Why?)

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Similarly, rotating the y-axis, $R \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, should give the line oriented at angle ϕ *to the y axis*:
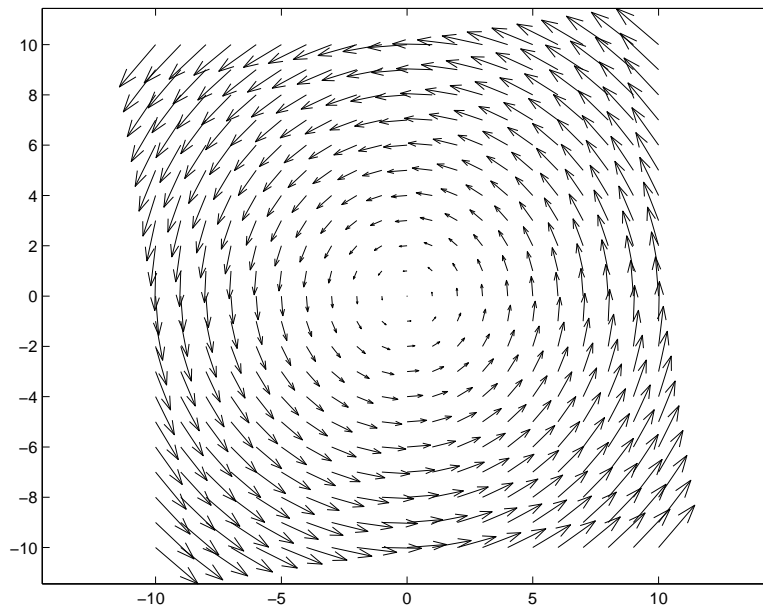
$$R \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -s \\ c \end{pmatrix}_\phi$$

This means $u = -s$ and $v = c$. (Why?)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{i.e.,}$$

$$\begin{aligned} x' &= x \cos \phi - y \sin \phi \\ y' &= x \sin \phi + y \cos \phi \end{aligned}$$

9. What is the operator for *two* rotations?

ϕ then ψ is $\phi + \psi$

$$\begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix}_\psi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \rightarrow \begin{pmatrix} C & -S \\ S & C \end{pmatrix}_{\phi+\psi}$$

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$$\begin{aligned}
C &= c'c - s's \\
(\cos(\phi + \psi)) &= \cos\psi\cos\phi - \sin\psi\sin\phi) \\
S &= s'c + c's \\
(\sin(\phi + \psi)) &= \sin\psi\cos\phi + \cos\psi\sin\phi)
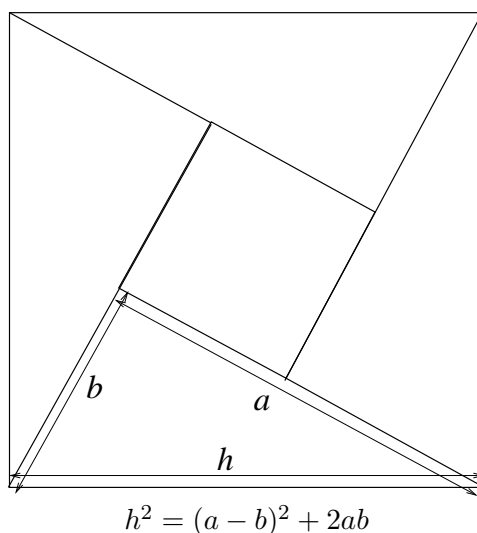\end{aligned}$$

Try $\psi = \phi$

$$\begin{aligned}
\cos(2\phi) &= \cos^2\phi - \sin^2\phi \\
\sin(2\phi) &= 2\sin\phi\cos\phi
\end{aligned}$$

Try $\psi = -\phi$

$$\begin{aligned}
0 = \sin(0) &= -\sin\phi\cos\phi + \cos\phi\sin\phi \\
1 = \cos(0) &= \cos^2\phi + \sin^2\phi
\end{aligned}$$

NB $\cos()$ is even, $\sin()$ is odd.

Pythagoras!

$$h^2 = (a - b)^2 + 2ab$$

Again, try $\psi = -\phi$

$$\begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix}_\psi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi = \begin{pmatrix} c' & s' \\ -s' & c' \end{pmatrix}_\psi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Identity operator.

10. Note that $R(\psi)R(\phi) = R(\psi + \phi)$ for rotations $R()$
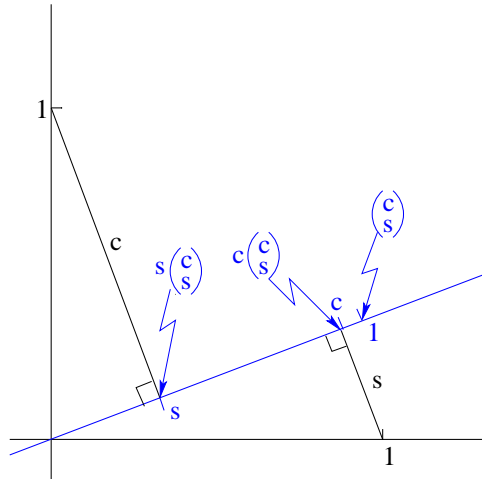
What other kind of function, *multiplied* by itself, gives itself on the *sum* of its arguments?

11. **Summary**

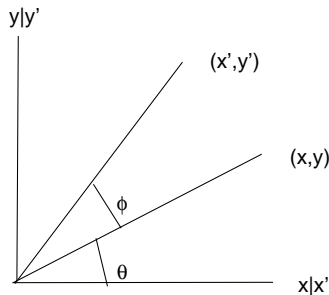(These notes show the trees. Try to see the forest!)

- Vector and matrix products: inner, outer, scalar.

- Projection operator, $\vec{u}$ on $\vec{p}$

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}_\theta \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

- Polarizing filter is a projection.

- Rotation operator by ϕ

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}_\phi \begin{pmatrix} x \\ y \end{pmatrix}$$

- Identity operator.

- Multiplying cos and sin by each other adds angles.

II. The Excursions
You've seen lots of ideas. Now *do* something with them!

1. Show that $\cos^2(\theta) = (1 + \cos(2\theta))/2$ and that $\sin^2(\theta) = (1 - \cos(2\theta))/2$. "Squaring is doubling": discuss!

2. Use Pythagoras and the formula for adding angles to derive the following values for cos() (c) and sin() (s). What is another relationship between the 15-degree and 75-degree functions? How does the table extend beyond 90 degrees to 360 degrees?

deg	0	15	30	45	60	75	90
c	1	$(\sqrt{2+\sqrt{3}})/2$	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	$(\sqrt{2-\sqrt{3}})/2$	0
s	0	$(\sqrt{2-\sqrt{3}})/2$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	$(\sqrt{2+\sqrt{3}})/2$	1

3. What does MATLAB give for

```
u = 0:2
v = (0:2)'
```

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```
uv
vu
```

4. A "normalized" vector has length 1, and so (in two dimensions) can be written as $(\cos(\theta),\sin(\theta))$, where ($\theta$ is its angle with the x-axis. Show that the scalar product of two such vectors

$$(c, s)\begin{pmatrix} C \\ S \end{pmatrix} = (C, S)\begin{pmatrix} c \\ s \end{pmatrix} = \cos(\theta_1 - \theta_2)$$

where $c = \cos(\theta_1), s = \sin(\theta_1), C = \cos(\theta_2), S = \sin(\theta_2)$ and $\theta_1 - \theta_2$ is the angle between the two vectors.

Show that the vector can also be written as $(\cos(\theta),\cos(\theta'))$ where (θ is its angle with the x-axis, as before, and θ' is its angle with the y-axis.

Using the scalar product, above, show that $\cos\theta_1 \cos\theta_2 + \cos\theta_1' \cos\theta_2' = \cos(\theta_1 - \theta_2)$ where $\theta_1' = \pi/2 - \theta_1$ and $\theta_2' = \pi/2 - \theta_2$.

Show that a three-dimensional normalized vector has components $(\cos(\alpha),\cos(\beta),\cos(\gamma))$, where α is its angle with the x-axis, β its angle with the y-axis, and γ its angle with the z-axis. What does this say about $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$? What expression do you think gives the cosine of the angle between two 3-D unit vectors?

5. In MATLAB write the operator that projects onto orientation $\pi/4$, and try it on half a dozen vectors at different angles. (At least one of these angles should be a multiple of one of the others, say twice it.) Describe your experiments in terms of a polarizing filter and light beams. With more projections, model the 2- and 3-filter polarizers discussed in class.

6. Using the diagram
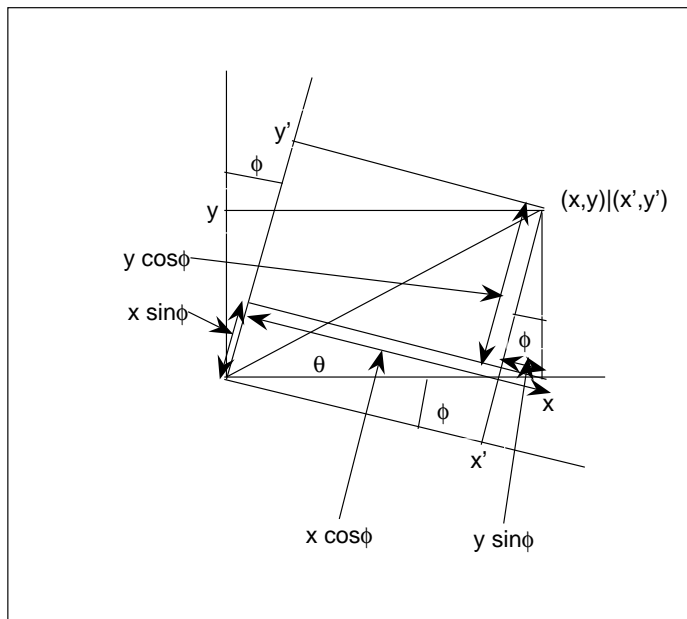
derive the rotation operator

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \begin{pmatrix} x \\ y \end{pmatrix}$$

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The diagram views the vector at angle θ staying still and the *axes* rotating through the negative angle, $-\phi$. Since the rotation operator just changes the coordinates describing the vector, it is the same thing if the vector rotates one way or if the axes rotate the other way.

7. In MATLAB write the operator that rotates by angle $\pi/4$, and try it on half a dozen vectors at different angles. (At least one of these angles should be a multiple of one of the others, say twice it.) Write other rotations by other angles, such as $-\pi/4, \pi/2$, and 0, and try them on your vectors.

8. Use MATLAB to generate the 25 coordinate pairs

$$(x, y) = (-2, -2), (-2, -1), .., (0, 0), .., (2, 2)$$

to rotate all through $\pi/20$

$$(u, v) = R(x, y)$$

and to display the rotation as arrows from each (x, y) to the corresponding (u, v). (Try `help quiver`)

9. Do the same for a projection, say to orientation $\pi/5$.

10. What is the inverse of a projection?

11. How does the expression $\cos^2() + \sin^2() = 1$ relate to Pythagoras?

12. Why does the square-within-a-square diagram in lecture 4 prove Pythagoras' theorem, $a^2 + b^2 = h^2$, for h the hypotenuse of a right-triangle with other sides a and b?

13. (Eugene Lehman.) How can you find, by making a single measurement with a meter-stick, how much gravel you need to fill a circular ring, 10 cm deep, around a circular garden?
a) Given that the area of a circle of radius r is πr^2, what is the formula for the area of this ring ("annulus") of inner radius r_1 and outer radius r_2?
b) What is the single measurement you can make to find this annular area?

14. A matrix is an array of numbers laid out in two "dimensions". The positions of its elements might be given as the pairs of integers (0,0), (0,1), (1,0), (0,2), (1,1), (2,0), .. Computer memory is usually organized along only one "dimension": a sequence of locations numbered 0, 1, 2, ... Give formulas which map the array pairs into memory locations: a) running through all of row 0, then all of row 1, and so on; b) running through all of column 0, then all of column 1, and so on. You will need to use w, the width of the array or h, its height.

15. Matrix multiplication is made easy for us by MATLAB: `A*B`. Under the hood, a computer program to multiply the l-by-m matrix `A` by the m-by-n matrix `B` must evaluate the sum of products

$$\Sigma_{j=1..m} A_{ij} B_{jk}$$

for each $i = 1..l$ and $k = 1..n$, where A_{ij} is the element of `A` in row i and column j, and B_{jk} is the element of `B` in row j and column k. Do this in MATLAB using the for statement. How many element multiplications will be done in this whole matrix product?

16. a) An operator, L, is *linear* if it obeys two rules (called "axioms"):
　　　Axiom 1 $L(x + y) = L(x) + L(y)$ and
　　　Axiom 2 $L(ax) = aL(x)$.
Show that multiplication by matrix, M, is linear, if x and y, above, are matrices or vectors, and a, above, is a scalar (an ordinary number).
b) Translation by an amount (t_x, t_y) has the result of adding (t_x, t_y) to every vector being translated. Show that this is not linear. Which axiom is violated? What operation would

violate the other axiom but not this one?

c) Translation can be expressed by matrix multiplication by adding a third dimension, z, mapping the original two-dimensional space to the plane $z = 1$, and doing a shear operation which changes the z axis but not the x or y axes. Figure out how to do this. (Shear operations are discussed next week.)

17. a) Show that bisecting an angle involves extracting square roots: work backwards from doubling an angle. When should the positive root be used and when the negative?

b) If you know how to bisect an angle using ruler and compass, use (a) to make a procedure for finding the square root of any number between 0 and 1 using ruler and compass.

c) If you know how to n-sect (bisect, trisect, ..) a straight line segment using ruler and compass, extend your procedure in (b) to find the square root of any positive number.

d) Use Pythagoras to find the square root of any positive integer by ruler and compass.

e) Show that trisecting an angle involves extracting cube roots as well as square roots.

18. Look up Arthur Cayley, 1821-1895. How did he come up with matrices?

19. Any part of the Preliminary Notes that needs working through.