

Appendix: Bernoulli Numbers

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In section 1.4 we summed numbers from 1 to n , getting the triangular number

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

and in section 4.4 we summed the squares of the numbers from 1 to n , getting

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 &= 2\Phi_n - \Delta_n \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \end{aligned}$$

These can be expanded in powers of n , and we can also give the sum of the zeroth powers of 1 to n , which is just the sum of n ones.

$$\begin{aligned} 1^0 + 2^0 + 3^0 + \cdots + n^0 &= n \\ 1^1 + 2^1 + 3^1 + \cdots + n^1 &= \frac{1}{2}(n^2 + n) \\ 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n) \end{aligned}$$

We also, in section 5.2, discovered Pascal's triangle. Let's call the elements of row k

$$\binom{k}{r}$$

(pronounced “ k choose r ”) with r ranging from 0 to k :

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$$\begin{array}{ccccccc}
 & & & & & & \binom{0}{r} \\
 & & & & & & \binom{1}{r} \\
 & & & & & & \binom{2}{r} \\
 & & & & & & \binom{3}{r} \\
 & & & & & & \binom{4}{r} \\
 & & & & & & \binom{5}{r} \\
 & & & & & & \binom{6}{r} \\
 & & & & & & r
 \end{array}$$

Now, in our equations for the sums of powers, if we multiply and divide each term by $\binom{k+1}{r}$, with k being the exponent in the sum and r ranging (downwards) over the exponents of n , an interesting thing happens.

$$\begin{aligned} 1^0 + 2^0 + 3^0 + \cdots + n^0 &= 1 \times \binom{1}{0} n \\ 1^1 + 2^1 + 3^1 + \cdots + n^1 &= \frac{1}{2} \left(1 \times \binom{2}{0} n^2 + \frac{1}{2} \binom{2}{1} n \right) \\ 1^2 + 2^2 + 3^2 + \cdots + n^2 &= \frac{1}{3} \left(1 \times \binom{3}{0} n^3 + \frac{1}{2} \binom{3}{1} n^2 + \frac{1}{6} \binom{3}{2} n \right) \end{aligned}$$

(I did the division by putting the numerical value of $\binom{k+1}{r}$ in the denominator, then dividing.)

What we see is that all the multiplier fractions are the same. These are the *Bernoulli numbers* and we call them B_r (with the exception of B_1^+ which we'll look at in a moment).

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{1}{k+1} (B_0 \binom{k+1}{0} n^{k+1} + B_1^+ \binom{k+1}{1} n^k + B_2 \binom{k+1}{2} n^{k-1} + \cdots)$$

This is the basis for “Faulhaber’s fabulous formula” for sums of powers.

We have $B_0 = 1$, $B_1^+ = 1/2$, $B_2 = 1/6$. How can we calculate the other B s in order to find sums of higher powers?

Well, it happens that

$$\binom{k+1}{0} B_0 + \binom{k+1}{1} B_1 + \cdots + \binom{k+1}{k} B_k = 0$$

for any $k \geq 1$, giving the sequence of equations

$$\begin{array}{rcl} B_0 + 2B_1 & & = 0 \\ B_0 + 3B_1 + 3B_2 & & = 0 \\ B_0 + 4B_1 + 6B_2 + 4B_3 & & = 0 \\ B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 & & = 0 \\ B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5 & & = 0 \\ B_0 + 7B_1 + 21B_2 + 35B_3 + 35B_4 + 21B_5 + 7B_6 & & = 0 \\ & & \vdots \end{array}$$

and these, starting with $B_0 = 1$, give

B_0	B_1	B_2	B_3	B_4	B_5	B_6	$\dots$
1	$-1/2$	$1/6$	0	$-1/30$	0	$1/42$	$\dots$

Note that B_1 differs in sign from B_1^+ above: this is the only difference. All the other B_k s with odd k are zero.

This discussion leaves open a number of “why?” questions. If you become a number theorist, you’ll answer them. Ramanujan was a number theorist. His first paper was on Bernoulli numbers.

Let’s conclude with a very brief excursion into number theory.

Warning: number theory is rarely useful but sometimes addictive.

The von-Staudt-Clausen theorem says that the denominator of any non-zero B_k ($k > 0$) is the product of all primes p whose predecessors $p - 1$ divide k .

(If you didn’t notice this, you may not get addicted.)

Let’s see. We introduced prime numbers in section 8.2. For the B_k we’ve seen, we need primes 2, 3, 5 and 7.

k	1	2	4	6
divisible by	$2 - 1$	$2 - 1, 3 - 1$	$2 - 1, 3 - 1, 5 - 1$	$2 - 1, 3 - 1, 7 - 1$
B_k denominator	2	$6 = 2 \times 3$	$30 = 2 \times 3 \times 5$	$42 = 2 \times 3 \times 7$

and so on (that’s the theorem).

It follows that every prime appears with powers of either 0 or 1 in every Bernoulli denominator, and appears with power 1 in some Bernoulli denominator. Which is a little boggling.