Excursions in Computing Science:
Week v Milli-micro-nano-..math Part II

T. H. Merrett*

McGill University, Montreal, Canada

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I. Prefatory Notes

1. **Cube root of 8.** Almost every calculator has a square-root button, $\sqrt{}$. But how would we calculate a *cube* root?
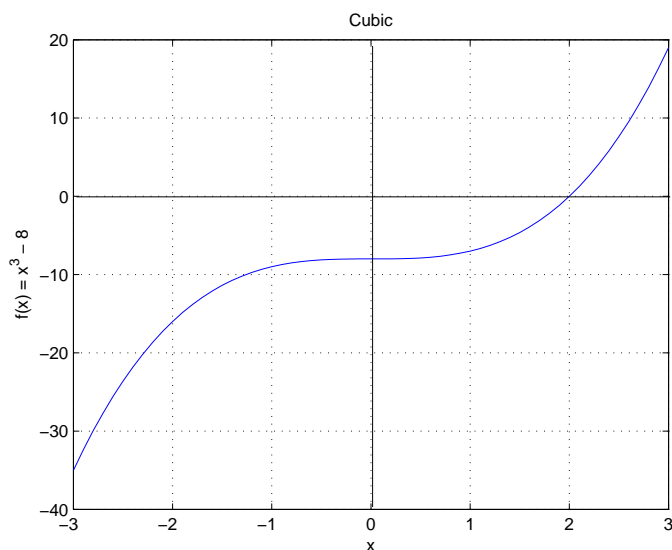
Let's start with one we know: $\sqrt[3]{8} = 2$.

We are going to find this by finding the "zeros" of a "function". For this case

$$0 = f(x) = x^3 - 8$$

will do it: if we can find the value of x which makes this $f(x)$ zero, we will know $\sqrt[3]{8}$.

Here is a plot of this function.

Note that the line crosses the horizontal (x) axis at $x = 2$, which we know to be the cube root of 8.

2. Slope. Now we introduce the idea of a *slope*. Loosely speaking, the slope of a function $f(x)$ at a specific value of x is the tangent to the function curve at that point.

To be precise, we define the slope to be a *number*: the height divided by the base of the triangle generated by that tangent. In the figure

$$\text{slope}_{x=x_1} f(x) \stackrel{\text{def}}{=} \frac{f(x_1)}{\text{base}}$$

Root of f(x)
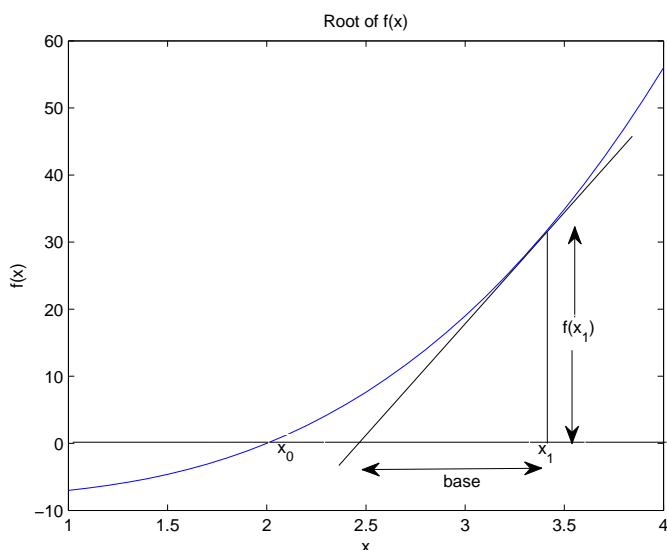
3. Approximations. Now we are going to start making approximations. These will be true enough when things get close enough together, so in the end we will have exact results.

(We'll use the symbol $\approx$ to mean "approximately equals". If we use it in a chain of equalities, the last expression in the chain will not exactly equal the first expression—even though there may be many $=$ signs and only one $\approx$ sign—but will only approximately equal it.)

In the above figure, we don't know what the *base* is, so we'll approximate it by $x_1 - x_0$.

We don't know what x_0 is, either, but we want to find it: it is the unknown place where $f(x_0) = 0$. So we'll use it to stand for what we don't yet know and do some algebra until we can eventually pin it down.

Now we can write the slope down approximately.

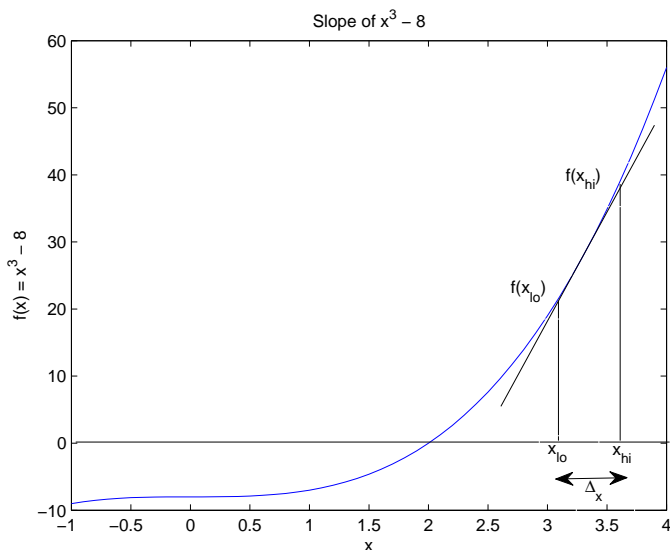$$\text{slope}_{x=x_1} f(x) = \frac{f(x_1)}{\text{base}} \approx \frac{f(x_1)}{x_1 - x_0}$$

So we can rearrange this to write down x_0, our quarry, approximately.

$$x_0 \approx x_1 - \frac{f(x_1)}{\text{slope}_{x=x_1} f(x)}$$

Since we know $f(x) = x^3 - 8$ we can calculate $f(x_1)$ for any given x_1.

We must now figure out how to calculate $\text{slope}_{x=x_1} f(x)$.

4. Slope of cubic. Here is $f(x) = x^3 - 8$ again, with all the pieces needed to find its slope at x_{lo}.

Slope of $x^3 - 8$

The definition of slope tells us, if x_{lo} and x_{hi} are close enough together,

$$\text{slope}_{x=x_{\text{lo}}} f(x) = \frac{f(x_{\text{hi}}) - f(x_{\text{lo}})}{x_{\text{hi}} - x_{\text{lo}}} = \frac{f(x_{\text{lo}} + \Delta x) - f(x_{\text{lo}})}{\Delta x}$$

So let's try it for $f(x) = x^3 - 8$.

$$
\begin{aligned}
\text{slope}_{x=x_{\text{lo}}} x^3 - 8 &= \frac{(x_{\text{lo}} + \Delta x)^3 - (x_{\text{lo}})^3}{\Delta x} \\
&= \frac{3(x_{\text{lo}})^2 \Delta x + 3x_{\text{lo}}(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\
&\approx \frac{3(x_{\text{lo}})^2 \Delta x}{\Delta x} \\
&= 3(x_{\text{lo}})^2
\end{aligned}
$$

That is, slope $x^3 - a = 3x^2$ for any a.

The $\approx$ in the above chain holds more and more exactly the smaller Δx gets, i.e., the closer x_{hi} and x_{lo} get together. But this is what we intend to happen in calling the result the "slope *at*" the specified value. What we really do is take the *limit* as the two points get closer and closer together. This justifies using the exact $=$ in saying above "slope $x^3 - a = 3x^2$".

5. The root. So now we can go back to Note 3 where we had

$$x_0 \approx x_1 - \frac{f(x_1)}{\text{slope}_{x=x_1} f(x)}$$

and replace it for $f(x) = x^3 - 8$ by

$$x_0 \approx x_1 - \frac{(x_1)^3 - 8}{3(x_1)^2}$$

Let's try it. We'll need a value for x_1. We'll just guess. We know for this case that the final cube root will be 2, but it is a special case we chose to make it easy to check the answer. Usually we

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can only guess the value of the final cube root, maybe from plotting the function. So we'll pretend we don't know and try $x_1 = 3$ as a guess.

$$x_2 = x_1 - \frac{(x_1)^3 - 8}{3(x_1)^2} = 1.9444$$

This is pretty close to 2, but let's try the result as the *next guess* (call it x_2).

$$x_3 = x_2 - \frac{(x_2)^3 - 8}{3(x_2)^2} = 2.0302$$

Better. Try again.

$$x_4 = x_3 - \frac{(x_3)^3 - 8}{3(x_3)^2} = 1.9856$$

Keep trying.

$$x_5 = x_4 - \frac{(x_4)^3 - 8}{3(x_4)^2} = 2.0074$$

$$x_6 = x_5 - \frac{(x_5)^3 - 8}{3(x_5)^2} = 1.9964$$

$$x_7 = x_6 - \frac{(x_6)^3 - 8}{3(x_6)^2} = 2.0018$$

$$x_8 = x_7 - \frac{(x_7)^3 - 8}{3(x_7)^2} = 1.9991$$

$$x_9 = x_8 - \frac{(x_8)^3 - 8}{3(x_8)^2} = 2.0005$$

$$x_{10} = x_9 - \frac{(x_9)^3 - 8}{3(x_9)^2} = 1.9998$$

$$\vdots$$

How many more steps before the result is 2.0000? 2.000000000000000?

So we see $x_n \to x_0$ in the limit as n gets arbitrarily large.

6. Square roots. This process works for square roots, too. In fact, it is the basis for the $\sqrt{}$ button on your calculator, and for the `sqrt` function in any programming language.

We just need to work out that $\text{slope}(x^2 - a) = 2x$. Now try finding the square root of 4.

$$x_0 \approx x_1 - \frac{(x_1)^2 - 4}{2(x_1)}$$

7. Antislopes. Now we know about slopes we might wonder about going the other way. Just as division undoes multiplication, *antislopes* undo slopes[1]. Here are some examples.

$\text{slope}_x f$	$f(x)$	$\text{antislope}_x f$
0	1	$x + C$
1	x	$x^2/2 + C$
$2x$	x^2	$x^3/3 + C$
$3x^2$	x^3	$x^4/4 + C$
$4x^3$	x^4	$x^5/5 + C$
$\vdots$	$\vdots$	$\vdots$

---

[1] We saw that division goes with finding slopes and we'll see that multiplication goes with finding antislopes, so it would be more appropriate to reverse this phrase and say "Just as multiplication undoes division, antislopes undo slopes."

where C in each case is some "constant", that is, some number or expression which does not depend on x.

Check these examples by finding the slopes of the antislope$_x$ f column. Then see if you can figure out the antislopes of the slopes$_x$ f column.

8. Areas. In addition to slopes, the other major use of arbitrarily shrinking interval, Δx, is to find areas. Let's find the area of an equilateral triangle.
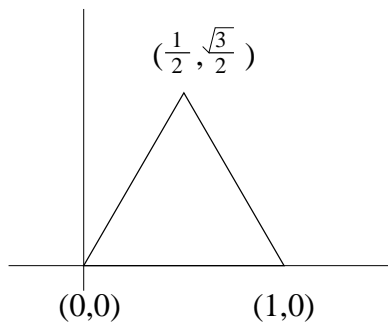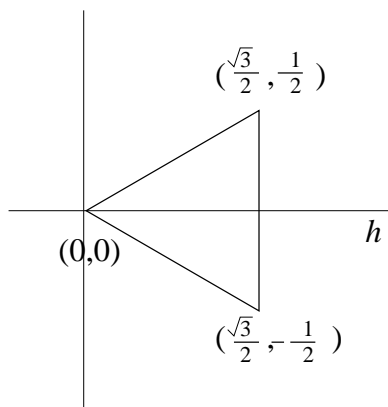

You probably know that the area of a triangle is $bh/2$ where b is the *base* and h is the *height*. For the triangle of the figure, $b = 1$ and $h = \sqrt{3}/4$ so
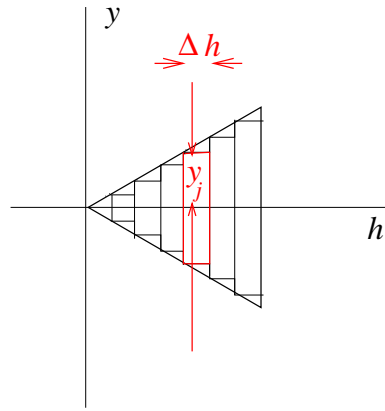
$$\text{area} \;=\; \frac{\sqrt{3}}{4}$$

Let's see if we can work out *why* this is the answer.

We'll redraw the triangle rotated so that its "height" is horizontal.

Now slice it into vertical strips of width Δh, and make them rectangles just touching the sides of the triangle at two of the four corners of each rectangle.

The area of the triangle will be

$$\text{area} \approx \sum_{j=0}^{(\sqrt{3}/2)/\Delta h - 1} 2y_j \Delta h$$

where we must figure out what each y_j is.

(Do you see why the number of steps of width Δh is $(\sqrt{3}/2)/\Delta h$?)

I used the approximation sign, $\approx$, because of the missing bits of area between the steps and the sides of the triangle. The important thing about this approximation is that it gets better and better the smaller Δh is: this is where the milli-micro-nano aspect of this Week comes in.

Let's think of y as a *function* of h

$$y(h) = sh$$

where s is the slope of the upper side of the triangle.

We can see

$$s = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

and we can check

$$y(0) = 0 \qquad y(\frac{\sqrt{3}}{2}) = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2} = \frac{1}{2}$$

So

$$h_j = j\Delta h$$

between

$$j = 0 \quad \text{and} \quad j = \frac{\sqrt{3}/2}{\Delta h}$$

and we have

$$y_j = y(h_j) = \frac{1}{\sqrt{3}}h_j = \frac{j\Delta h}{\sqrt{3}}$$

So

$$\begin{aligned}
\text{area} \;&\approx\; \sum_{j=0}^{(\sqrt{3}/2)/\Delta h - 1} 2y(h_j)\Delta h \\
&=\; \sum_{j=0}^{(\sqrt{3}/2)/\Delta h - 1} \frac{2\Delta h}{\sqrt{3}}j\Delta h
\end{aligned}$$

$$= \frac{2}{\sqrt{3}}(\Delta h)^2 \sum_{j=0}^{(\sqrt{3}/2)/\Delta h - 1} j$$

$$= \frac{2}{\sqrt{3}}(\Delta h)^2 \frac{(\sqrt{3}/2)/\Delta h)((\sqrt{3}/2)/\Delta h - 1)}{2}$$

$$= \frac{1}{\sqrt{3}}(\Delta h)^2 \frac{\sqrt{3}/2}{\Delta h}(\frac{\sqrt{3}/2}{\Delta h} - 1)$$

$$\approx \frac{1}{\sqrt{3}}(\Delta h)^2 (\frac{\sqrt{3}/2}{\Delta h})^2$$

$$= \frac{1}{\sqrt{3}}\frac{3}{4}$$

$$= \frac{\sqrt{3}}{4}$$

which is the same result as the $bh/2$ calculation at the beginning of this Note.

Notice that I made *two* approximations, which happened, this time, to cancel each other out. The second approximation was that $\frac{\sqrt{3}/2}{\Delta h}$ is big enough that $\frac{\sqrt{3}/2}{\Delta h} - 1$ essentially equals $\frac{\sqrt{3}/2}{\Delta h}$. This approximation also gets better and better as Δh gets arbitrarily small: more milli-micro-nano-.. math.

Note also that the sum, $\sum$, disappears in the fourth line above and is replaced by $n(n-1)/2$ (letting n be $\frac{\sqrt{3}/2}{\Delta h}$): the sum is just a triangular number (Week i Note 1). (Please do not confuse the Δ in Δh (this Note) with the Δ in Δ_n etc. (Week i Notes). Here Δ modifies the variable following it to mean "a little bit of". There the Δ means "triangle" and has a *subscript n* to say how big the triangle is.)

To summarize this procedure a little differently, we summed the base, b, as a function of the height, h
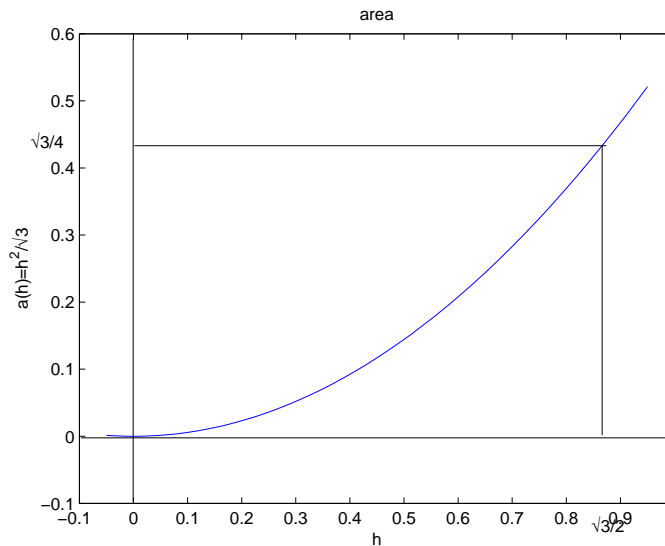
$$b(h) = \frac{2}{\sqrt{3}}h$$

over a discrete set of values, h_j, separated by steps of size Δh from 0 to $\sqrt{3}/2$.

base
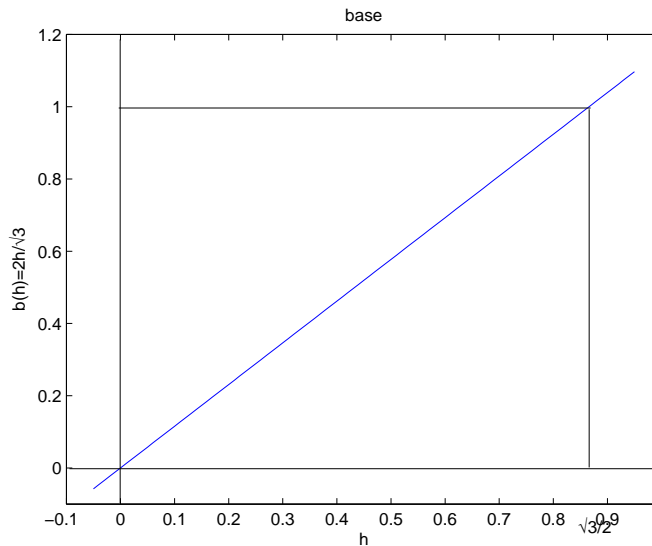
We can use $b = 2h/\sqrt{3}$ to generalize the area from the height $h = \sqrt{3}/2$ triangle to a triangle with *any* height h:

$$\text{area} \;=\; \frac{1}{2}bh = \frac{1}{\sqrt{3}}h^2$$

9. Volumes. We can use the same technique to find the volume of a regular tetrahedron. This has all six edges of the same length, say 1, and height $\sqrt{2/3}$ in the third dimension. (Each of the four faces is an equilateral triangle of sides 1 and "height" on the face—we won't call this the height any more—$\sqrt{3}/2$ as in the previous Note.)

We'll turn the tetrahedron sideways, so its 3-D height is shown horizontally, and slice it into triangular slabs of thickness Δh and sizes from 0 at the apex to sides=1 at the base.
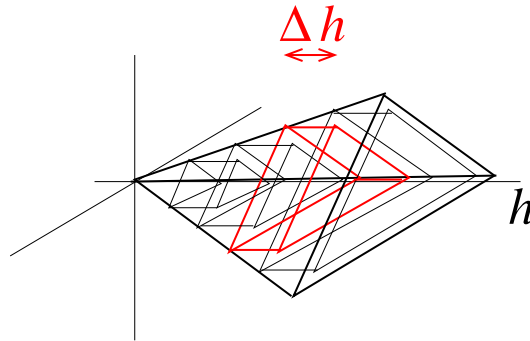
$$area_j = \tfrac{1}{\sqrt{3}}\, h_j^2$$

$$\Delta vol_j = \tfrac{1}{\sqrt{3}}\, h_j^2 \Delta h$$

So the sum we need this time is

$$\text{vol} \;\approx\; \sum_{j=0}^{\sqrt{2/3}/\Delta h - 1} \frac{1}{\sqrt{3}}h_j^2 \Delta h$$

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with $h_j = j\Delta h$ as before, and with the same reasons for noting that this is an approximation (which will improve as Δh gets small).

$$
\begin{aligned}
\text{vol} \;\approx\;& \sum_{j=0}^{\sqrt{2/3}/\Delta h - 1} \frac{1}{\sqrt{3}} j^2 (\Delta h)^3 \\
=\;& \frac{(\Delta h)^3}{\sqrt{3}} \sum_{j=0}^{\sqrt{2/3}/\Delta h - 1} j^2 \\
=\;& \frac{(\Delta h)^3}{\sqrt{3}} \frac{1}{3} \frac{\sqrt{2/3}}{\Delta h} \left(\frac{\sqrt{2/3}}{\Delta h} - \frac{1}{2}\right)\left(\frac{\sqrt{2/3}}{\Delta h} - 1\right) \\
\approx\;& \frac{1}{3\sqrt{3}} \left(\frac{\sqrt{2/3}}{\Delta h}\right)^3 \\
=\;& \frac{2\sqrt{2}}{27}
\end{aligned}
$$

In the second line I summed j^2 from 0 to $n-1 = \sqrt{\frac{2}{3}}\frac{1}{\Delta h} - 1$ which we can work out from Week i Note 1 as $\frac{1}{3}n(n-\frac{1}{2})(n-1)$. I approximated this by $n^3/3$, once again an approximation which improves as Δh gets small, and an approximation which happens to counteract the first approximation above exactly, giving the exact right answer.

10. **Antislopes and areas.** In Notes 8 and 9 we found the areas under the following two functions.
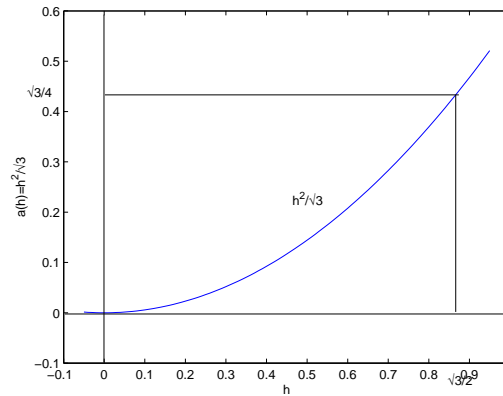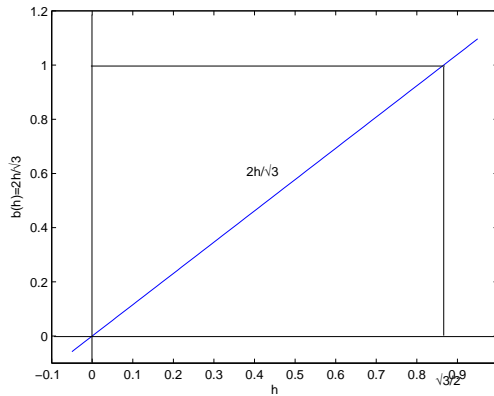
Let's just find the antislopes of these functions (Note.7).

$$
\text{antislope}_h \; \frac{2}{\sqrt{3}}h \;=\; \frac{2}{\sqrt{3}} \,\text{antislope}_h\; h = \frac{1}{\sqrt{3}}h^2 + C_1
$$

$$
\text{antislope}_h \; \frac{1}{\sqrt{3}}h^2 \;=\; \frac{1}{\sqrt{3}} \,\text{antislope}_h\; h^2 = \frac{1}{3\sqrt{3}}h^3 + C_2
$$

where the constants C_1 and C_2 can be any numbers not depending on h.

Let's evaluate the first at $h = \sqrt{3}/2$ and at $h = 0$ and subtract.

$$
\text{antislope}_h \; \frac{2}{\sqrt{3}}h \Big|_0^{\sqrt{3}/2} = \frac{1}{\sqrt{3}}h^2 \Big|_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4} - 0 = \frac{\sqrt{3}}{4}
$$

and the second at $h = \sqrt{2/3}$ and at $h = 0$ and subtract.

$$
\text{antislope}_h \; \frac{1}{\sqrt{3}}h^2 \Big|_0^{\sqrt{2/3}} = \frac{1}{3\sqrt{3}}h^3 \Big|_0^{\sqrt{2/3}} = \frac{2\sqrt{2}}{27} - 0 = \frac{2\sqrt{2}}{27}
$$

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What is the connection with the triangular area in Note 8 and the tetrahedral volume in Note 9?

It turns out that antislopes give *areas* when suitably evaluated.

11. The Fundamental Theorem of Calculus. Areas between two points under a function are linked to the antislope of the function, i.e., to the function whose slope is that function. Let's see why.

Here is a function $f(x)$ and a small section of $f(x)\Delta x$ of the area under $f(x)$.
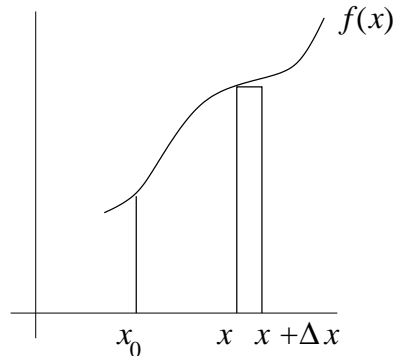
We can call the area $A(x_0, x)$, supposing that we are in the process of finding the area under $f(x)$ starting at x_0, and that we have so far reached x. If x_0 is fixed, we are interested only in the area as a function of x and we can write, simply, $A(x)$. Then

$$\Delta A(x) = f(x)\Delta x$$

for the small increment, $\Delta A(x)$, of the area.

This is very familiar: just rearrange

$$f(x) = \frac{\Delta A(x)}{\Delta x}$$

and we have that $f(x)$ is the slope of $A(x)$.

So $A(x)$ is the antislope of $f(x)$.

And if we want to find the area from say x_0 to say x_1, we just evaluate

$$A(x_1) - A(x_0)$$

We can write this

$$A(x) \mid_{x_0}^{x_1}$$

or

$$\text{antislope}_x \ f(x) \mid_{x_0}^{x_1}$$

Since these "definite integrals" are the same antislope evaluated at two different places and subtracted, the arbitrary constant that we introduced with antislopes in Note 7 disappears and we can forget about it.

12. Summary

(These notes show the trees. Try to see the forest!)

The reason I called these Notes "Milli-micro-nano-..maths" is because of the approximations. To make slopes exact we must apply the definition to ever smaller intervals.

The method I have presented to find cube and square roots is "Newton's method". It is an application of differential calculus, also invented by Isaac Newton, and independently by Gottfried

Leibniz whose notation is now used, although I have not used it. Newton's method works for finding roots (zeros) of almost any function and it converges very quickly.

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. **Throwing balls: parabola and square roots.** (This Excursion is meant to help if you found this topic tough going right from the start. It is about functions and their roots ("zeros").)

    Eric is throwing a ball for Iris to catch. Here is how the ball goes.
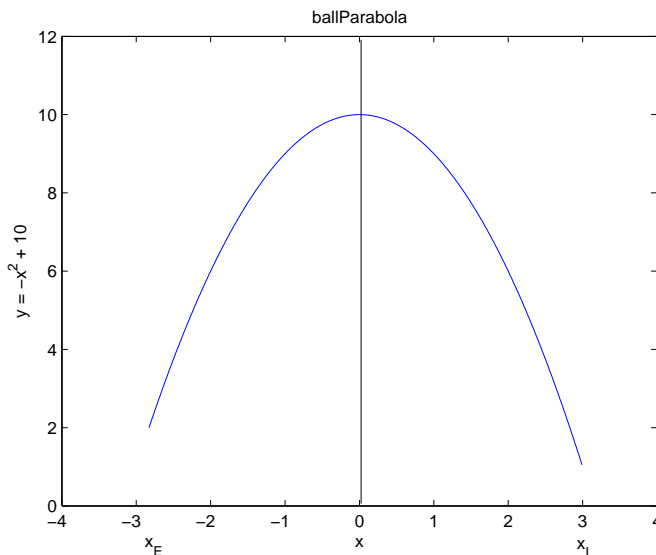
The ball leaves Eric's hand 2 meters from the ground, rises to a maximum height of 10 meters, then drops to 1 meter where Iris catches it. Gravity causes the ball to follow a *parabola* in between Eric and Iris. A parabola is an example of a *function*, which gives a unique value for y (which is what I've called the height in the graph) for each possible value of x (which is what I call the horizontal position of the ball).

The relationship between x and y is given by the *equation*

$$y = -x^2 + 10$$

That is the simplest equation to describe the particular path I've plotted. To make it this simple, I chose the axes so that the maximum height is attained when $x = 0$. I also chose the horizontal scale so that the ball is moving from somewhere around $x = -3$ to somewhere around $x = 3$. We'll see where exactly these endpoints are. In fact, that is the purpose of this Excursion.

Let's make a table showing some of the values for y that the rule $y = -x^2 + 10$ calculates for different values of x. The table spells this out in the second column. (We'll come to the meanings of the third and fourth columns shortly.)

x	$-x^2 + 10$	$= y$	$y - 1$	$y - 2$
-3	$-9 + 10$	$= 1$	0	-1
-2	$-4 + 10$	$= 6$		4
-1	$-1 + 10$	$= 9$		7
0	$0 + 10$	$=10$		8
1	$1 + 10$	$= 9$		7
2	$4 + 10$	$= 6$		4
3	$9 + 10$	$= 1$	0	-1

Check these calculations. What do you notice about the value of y for positive and negative values of x?

Now, where must Iris be to catch the ball 1 meter off the ground? The table says the ball is 1 meter off the ground at $x = -3$ and at $x = 3$. Since I've shown Iris on the x-*positive* side, we conclude that Iris is at $x_I = 3$.

The two possible positions, from which we've chosen the positive one, also correspond to $y - 1 = 0$, which I've marked with 0 in the $y - 1$ column.

Finding out where Eric must stand to throw the ball is trickier. We don't see any zeros in the $y - 2$ column. But we do see some negative numbers and some positive numbers in that column. $y - 2 = 0$, or $y = 2$, must occur:
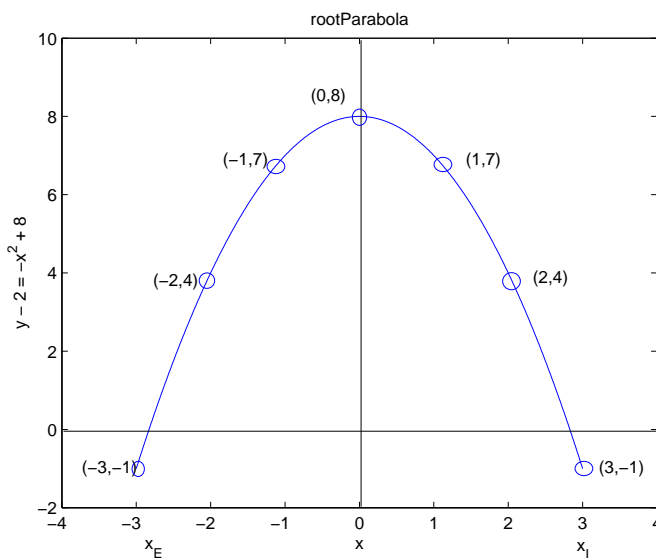
a) between $x = -3$, where $y - 2 = -1$, and $x = -2$, where $y - 2 = 4$—and it plainly must be somewhat closer to $x = -3$ than to $x = -2$; or

b) between $x = 2$, where $y - 2 = 4$, and $x = 3$, where $y - 2 = -1$—and it plainly must be somewhat closer to $x = 3$ than to $x = 2$.

Let's do some algebra.

$$\begin{aligned} 0 &= y - 2 \\ &= -x^2 + 10 - 2 \\ &= -x^2 + 8 \end{aligned}$$

and here is the graph of this $y - 2 = -x^2 + 8$. (I could have given $y - 2$ a new name, say y' or z, but why burden ourselves with additional symbols?)

We see it is another parabola, this time one which *crosses the x-axis* at the "zeros" of $y - 2$, i.e., where $y - 2 = 0$, or $y = 2$, which is just the height of the ball as it leaves Eric's hand. These zeros are the *roots* of the parabola.

(I've also shown the pairs (x, y) for the seven values of x that are in the table. Clearly these are only *some* of the possible pairs. How many could there be?)

To find where Eric is, we need the roots of this second parabola.

We must do some more algebra. Continuing from before

$$\begin{aligned} 0 &= -x^2 + 8 \\ x^2 &= 8 \\ x &= \sqrt{8} \\ &= 2.828\cdots \end{aligned}$$

Thus Eric is at $x_{\mathrm{E}} = -\sqrt{8} = -2.828\cdots$.

In the last line, I used the "square root" button on my calculator to find the number. Check this and try it on your own calculator. But why have I used the minus sign? Does it satisfy the requirement (b) above? What about requirement (a)?

Why does a square root almost always have two possible values? How do they relate to one another? What square root has only one value?

2. Write a program, say `cubeRT(a)`, which finds the cube root of a correctly to some number of decimal places which you can fix inside the program or else control by a second parameter, say `decpl`, for any number, a.

3. a) Show that slope $x^n = nx^{n-1}$ for any positive integer n. Hint: use the binomial coefficients of Week ii Note 6 in a way similar to the way they are used to expand $(1 + x)^n$ in Week ii Note 7.
   b) Show that slope $x^n = nx^{n-1}$ for any rational number n.

4. **Pythagoras.** In the next three excursions we are going to show that the height of an equilateral triangle of base b is indeed $(\sqrt{3}/2)b$ (i.e., $\sqrt{3}/2$ when $b = 1$ as in the example of Note 8). For this, we need to know *Pythagoras' theorem* that for a right-angled triangle of sides a, b and c (where c is the long side, the *hypotenuse*)

$$c^2 = a^2 + b^2$$

Show this using the following diagram
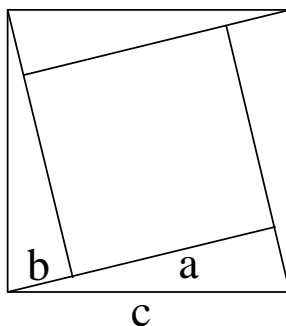
5. Using Pythagoras and the following diagram, show that the height of an equilateral triangle of base $b = 1$ is $\sqrt{3}/2$.
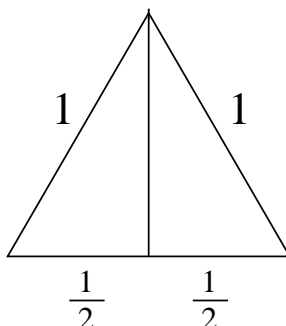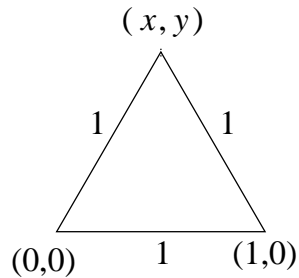

Why is the height $(\sqrt{3}/2)b$ for any base b?

6. Another way to find the height of an equilateral triangle is to start with its base, say of length 1

$$\overline{\underset{(0,0)\qquad 1\qquad (1,0)}{\qquad\qquad}}$$

and find the point (x, y) which is a distance 1 from both (0,0) and (1,0).

(x, y)

1 ⟋ ⟍ 1

(0,0)   1   (1,0)

a) Show that Pythagoras says the distance between two points (x, y) and (x', y') is

$$(x - x')^2 + (y - y')^2$$

b) Hence show that the two equations

$$\begin{aligned} x^2 + y^2 &= 1 \\ (x - 1)^2 + y^2 &= 1 \end{aligned}$$

must be solved.

c) Finally, use this to show that $x = 1/2$ and $y = \sqrt{3}/2$

7. The sum of j^2, $\sum_{j=0}^{n} j^2$ where $n = \sqrt{\frac{2}{3}}\frac{1}{\Delta h}$ in Note 9, could be approximated as $\frac{1}{2}\sum j(j+1)$ for large n (small Δh). This is the sum of triangular numbers in Week i Note 1, and produces the nth tetrahedral number. Show that it gives the same answer as the volume calculation in Note 9.

8. Pythagoras can show that the height of unit-sided tetrahedron is $\sqrt{2/3}$ as said in Note 9. Here is the geometrical approach.
The fourth point, or apex, of the tetrahedron will lie directly above the center of the equilateral triangle that is its base (why?)


$\dfrac{1}{2}$ $\dfrac{1}{2}$
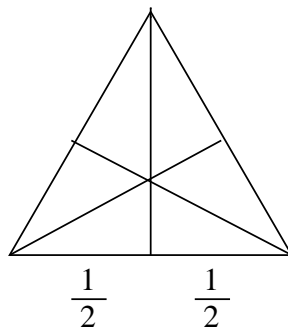
a) This can be found by bisecting the angle at each corner and noting where the bisectors meet. By symmetry, all three bisectors meet at the same point, and each is perpendicular to the edge of the triangle opposite to the angle it bisects.

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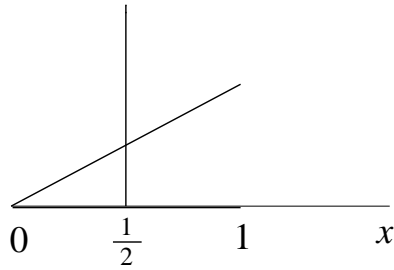
Use the slope of the bisector (it's the same as the slope of the edge of the rotated triangle in Note 8) and Pythagoras to show that its length from $x = 0$ to $x = 1$ is $2/\sqrt{3}$. Since the *vertical* bisector cuts this bisector in half, show that the common point where all bisectors meet is $1/\sqrt{3}$ from each vetex and $1/(2\sqrt{3})$ from each opposite edge.

b) Now rise straight up in the third dimension from this common point to the apex of the tetrahedron, and note where it meets the 2-D "height" of the triangular face shown and use Pythagoras to show that the 3-D height is $\sqrt{2/3}$.
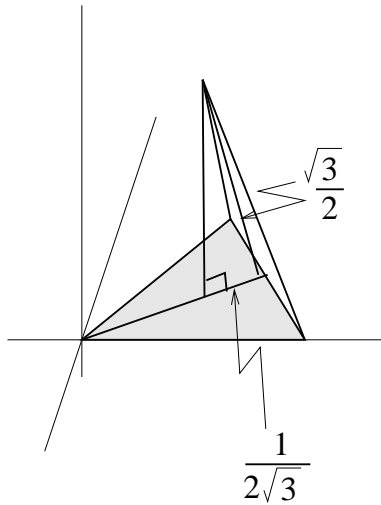
9. The algebraic approach finds the height of the tetrahedron much more quickly than the geometrical approach of the last excursion.
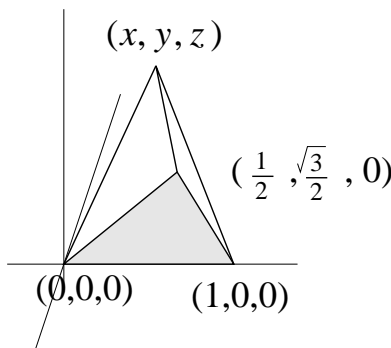
Requiring the apex, (x, y, z), to be unit distance from the three corners $(0,0,0)$, $(1,0,0)$ and $(1/2,\ \sqrt{3}/2,\ 0)$ of the base means

$$x^2 + y^2 + z^2 \;=\; 1$$

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$$(x-1)^2 + y^2 + z^2 \;=\; 1$$
$$(x - \frac{1}{2})^2 + (y - \frac{\sqrt{3}}{2})^2 + z^2 \;=\; 1$$

Show $(x, y, z) = \pm(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}})$

10. Go on from the previous excursion to find the "height" in "the 4th dimension" of the 4-D regular simplex with tetrahedral "base"

$$(0,0,0,0), (1,0,0,0), (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0) \text{ and } (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}, 0)$$

11. Any part of the Preliminary Notes that needs working through.