Excursions in Computing Science: Week iv. Space Math

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I. Prefatory Notes

1. Matrix multiplication. Teacher, help your grade scholar master the multiplication of 2×2 matrices outlined below and then encourage hem to invent a few 2×2 matrices to exercise on. Try 3×3, 2×3, and other $n \times m$ matrices as well. A grade scholar who enjoys calculating will like this work for a while and will appreciate all the more the revelations later in these Notes of what matrices mean and how they can be applied.

Polynomials in Week iii add and subtract in fairly straightforward ways. They become more intriguing when multiplied, divided and factored. In these Notes we look at a quite different assemblage of numbers, the *matrix*.

A matrix is a rectangular array of numbers. We will focus on 2×2, square rectangles.

Here are two 2×2 matrices multiplied together.

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 73 & 32 \\ 96 & 57 \end{pmatrix}$$

Here is how we get this answer.

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 \times 12 + 5 \times 5 & 4 \times 3 + 5 \times 4 \\ 3 \times 12 + 12 \times 5 & 3 \times 3 + 12 \times 4 \end{pmatrix}$$

A picture will help even more.

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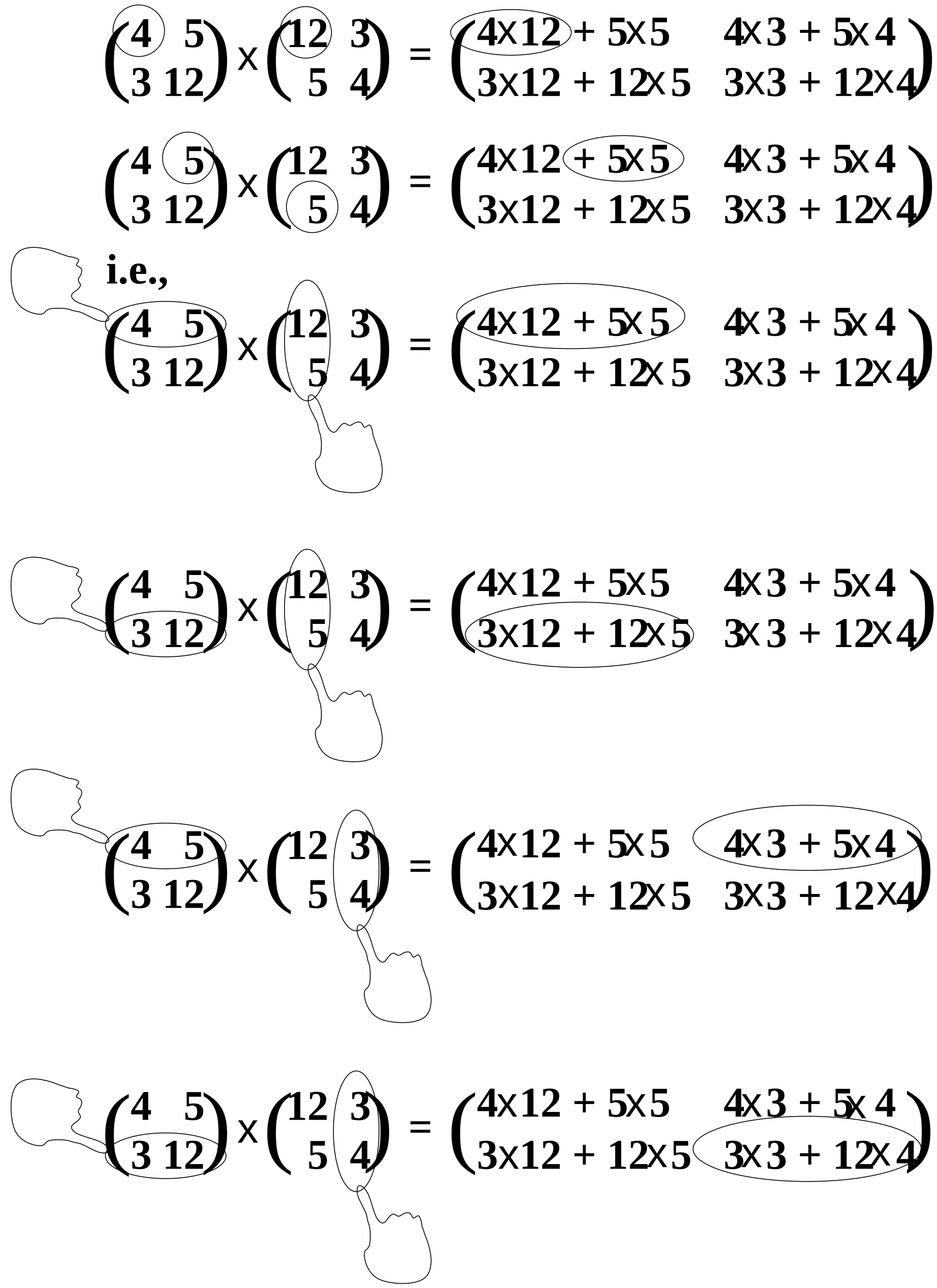


Matrix multiplication is *not* necessarily commutative.

$$\begin{pmatrix} 12 & 3 \\ 5 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 57 & 96 \\ 32 & 73 \end{pmatrix}$$

2. Vectors. Matrices do not have to be square. Here are two rather special 2×1 matrices.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using the matrix multiplication rule

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

2×1 matrices are called *vectors*. (So are 1×2 matrices.)

The two vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are special because any other vector can be made up from them.

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For example,

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4 \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This introduces two new operations on matrices: scalar multiplication and addition, both easy.

Scalar multiplication

$$2 \times \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Addition

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

3. Identity matrix. Notice how the first multiplication in Note 2 “selects” the first column of the matrix, and the second multiplication “selects” the second column.

We can actually lump together these two multiplications.

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}$$

And, swapped

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}$$

So we have a special square matrix, called the *identity*.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The identity matrix plays the same role in matrix multiplication that 1 does in number multiplication.

4. Matrix inverse. Given a matrix, what matrix multiplied by it gives the identity? This will be the *inverse* of the given matrix.

A fairly simple rule gives the inverse for a 2×2 matrix. The rule starts: Swap the diagonal elements and change the signs of the off-diagonal elements.

Try

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 12 & -5 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 33 & 0 \\ 0 & 33 \end{pmatrix}$$

This is almost the identity: we must just divide by 33.

Before we say what this 33 is, notice carefully just why the swap and the sign change give the off-diagonal zeroes in the result.

Try multiplying the two diagonal elements of the original matrix, then subtracting the product of the off-diagonal elements. This is called the *determinant* of the 2×2 matrix and in this case it is 33.

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \frac{\begin{pmatrix} 12 & -5 \\ -3 & 4 \end{pmatrix}}{4 \times 12 - 5 \times 3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So the rest of the inversion rule is: Divide the new matrix by the determinant of the original matrix. Now you have the inverse of the original.

The convention is to use an exponent -1 to signify the inverse.

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} \times \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We do not usually talk about matrix division because the important operation is inversion, and inversion is enough to give us division.

$$\begin{pmatrix} 12 & 3 \\ 5 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} 135/33 & -48/33 \\ 48/33 & -9/33 \end{pmatrix}$$

is what we would mean if we could say

$$\begin{pmatrix} 12 & 3 \\ 5 & 4 \end{pmatrix} \div \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix} = \begin{pmatrix} 135/33 & -48/33 \\ 48/33 & -9/33 \end{pmatrix}$$

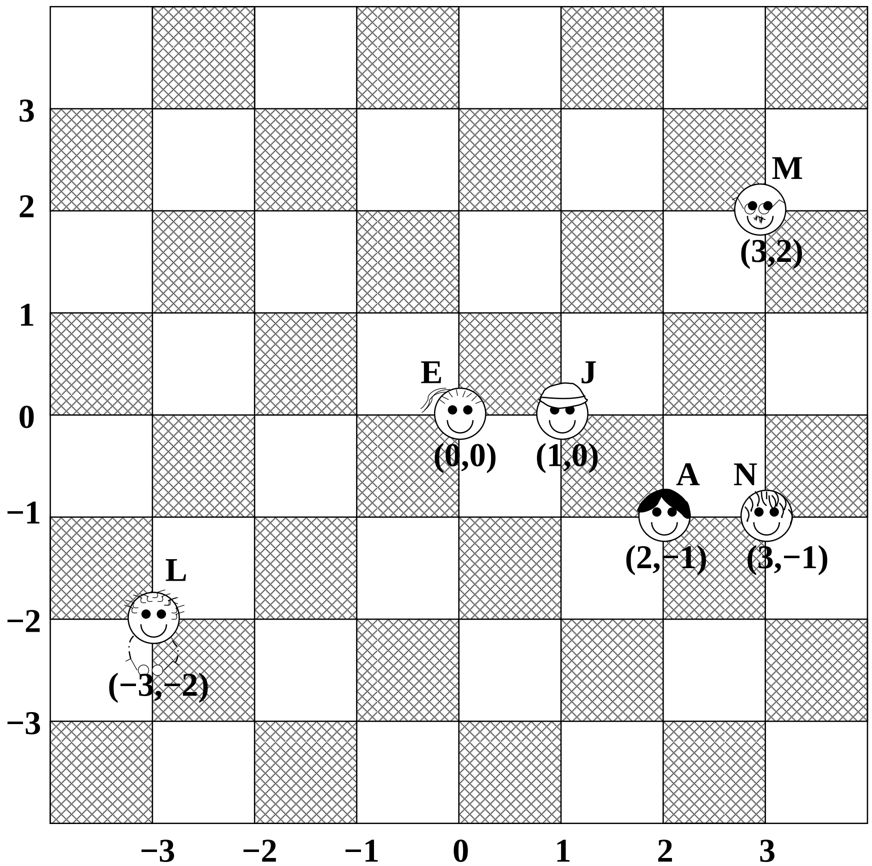
(By the way, the matrix we got here is in a special class, called "antisymmetric" matrices: the off-diagonal element(s) above the diagonal differ only in their sign from their counterpart(s) below the diagonal. Can you see why this result had to turn out anti-symmetric?)

If its determinant is zero, a matrix is not invertible. (Why?). Such a matrix is called *singular*. Singular matrices play the role in matrix "division" that 0 plays in number division. But note that there will be more than one singular matrix.

5. Vectors in space. Now let's see what all these matrices and their strange operations might mean and might be useful for.

We start with vectors, specifically the "column vectors" (2×1 matrices) we have been using. These are just pairs of numbers, and so are useful for working with two-dimensional space.

Here is a view from the ceiling of a classroom with a floor tiled with large dark and light linoleum tiles, and of the six people currently in the classroom. (It's not that they are all looking at the ceiling and not paying attention, but that I couldn't draw both the floor and the faces of the people at their desks in any other way.)



Everybody's name (one letter each) is also shown, and so are their positions (two numbers—a vector—each).

Positions must be measured from some starting point, and by convention they are all measured from the *origin*, the point (0,0).

So we had to show the origin and, for symmetry, it appears in the centre of the picture. It could be anywhere else, such as the bottom left-hand corner (a frequent scientific convention) or the top left-hand corner (the usual computer graphics convention) or somewhere completely outside the picture.

Wherever it has been put, the origin is the point of reference for all positions, hence its name.

Putting the origin at the centre of the picture allows us to show negative numbers on the same footing as positive.

Note how "M" is positioned 3 tile widths right of "E" (who happens to be sitting at the origin) and 2 tile heights above. So these two numbers form the two components of the position of "M".

"L" on the other hand is directly opposite "M" relative to the origin. By convention (again) rightwards and upwards are indicated by positive numbers and leftwards and downwards by negative. So "L"'s position consists of two numbers which are the respective negatives of "M"'s numbers, -3 and -2.

Vectors can be written either horizontally as 1×2 matrices or vertically as 2×1 matrices. It was convenient to write them horizontally in the picture but in the text we will stick to column vectors. These are more common than row vectors, and I myself have some trouble with left and right which I do not have with top and bottom. It is important to distinguish the first from the second element since the first element of a vector conventionally describes the left-right direction in space while the second describes the up-down direction. In row vectors the first element is the left one. In column vectors it is the top one.

Here are the vectors corresponding to the positions of the six people in the classroom. One other vector is added because we do not have anybody seated at the position given by the second special vector.

$$\begin{matrix} & \text{J} & \text{E} & \text{N} & \text{A} & \text{L} & \text{M} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 \\ -1 \end{pmatrix} & \begin{pmatrix} 2 \\ -1 \end{pmatrix} & \begin{pmatrix} -3 \\ -2 \end{pmatrix} & \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{matrix}$$

These seven vectors can also be lumped into a single 2×7 matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 3 & 2 & -3 & 3 \\ 1 & 0 & 0 & -1 & -1 & -2 & 2 \end{pmatrix}$$

6. Positions and intervals. So far the vectors just stand for positions in space. They can also stand for intervals.

For example, the interval from "A" to "N" is N − A:

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note that this is the same vector as the position of "J".

So vectors representing intervals also represent them relative to the origin: they don't start at the first position.

We would really need four numbers to give both the interval and its starting point. But we already have these four numbers in the two vectors N and A. So it is economical just to take the two numbers in N − A as the interval. But this can be a confusing convention and takes getting used to.

A similar convention also holds when we interpret ordinary numbers as positions along a line (such as the Celcius temperature scale) or as intervals on the line (such as how much the temperature went up today (positive interval) or down last night (negative interval)).

Thus we can interpret addition and subtraction of vectors. Two vectors representing positions can be subtracted to give the vector representing the interval between. Two vectors representing position and interval respectively can be added to give the new position (again a vector) that is the given interval away from the first position.

7. Transforming space. How can we interpret multiplication? By the rule for matrix multiplication we cannot multiply two vectors (except only if the first is a row vector and the second a column vector, but we are sticking to column vectors): why?

So we must return to multiplying 2×2 matrices and column vectors.

Recall from Note 2 that the two special column vectors "select" the two columns of the matrix when multiplied by the matrix. But these special vectors just describe the intervals of one step (tile) rightward and one step upward in the classroom space. So we can easily read the effect of multiplying these special vectors by the matrix.

For an example, I'm going to modify the matrix a little from

$$\begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}$$

by dividing the first column by 5 and the second by 13. (What is special about the triplets 3,4,5 and 5,12,13? The answer may give a hint about why I am making this change, but it will not become clear until Week 2.)

$$\begin{pmatrix} 4/5 & 5/13 \\ 3/5 & 12/13 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.38 \\ 0.6 & 0.92 \end{pmatrix}$$

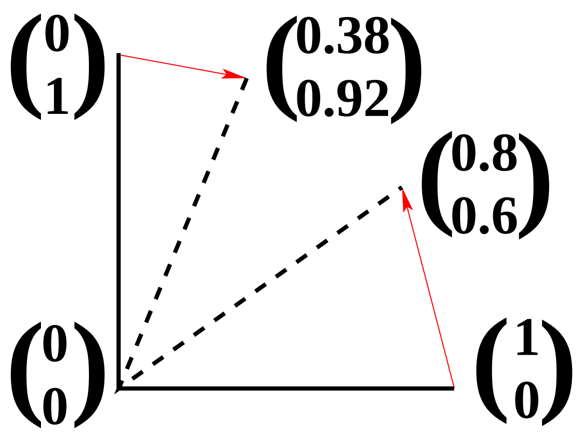
Here is the effect of this matrix on the left-right unit vector (a "unit" vector has length 1)

$$\begin{pmatrix} 0.8 & 0.38 \\ 0.6 & 0.92 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$$

and here is the effect on the up-down unit vector

$$\begin{pmatrix} 0.8 & 0.38 \\ 0.6 & 0.92 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.38 \\ 0.92 \end{pmatrix}$$

and we see that multiplying by the matrix has had the effect of bending the left-right unit vector upwards to a new vector, and bending the up-down unit vector rightwards to another new vector.



The red arrows show the changes to the special vectors.

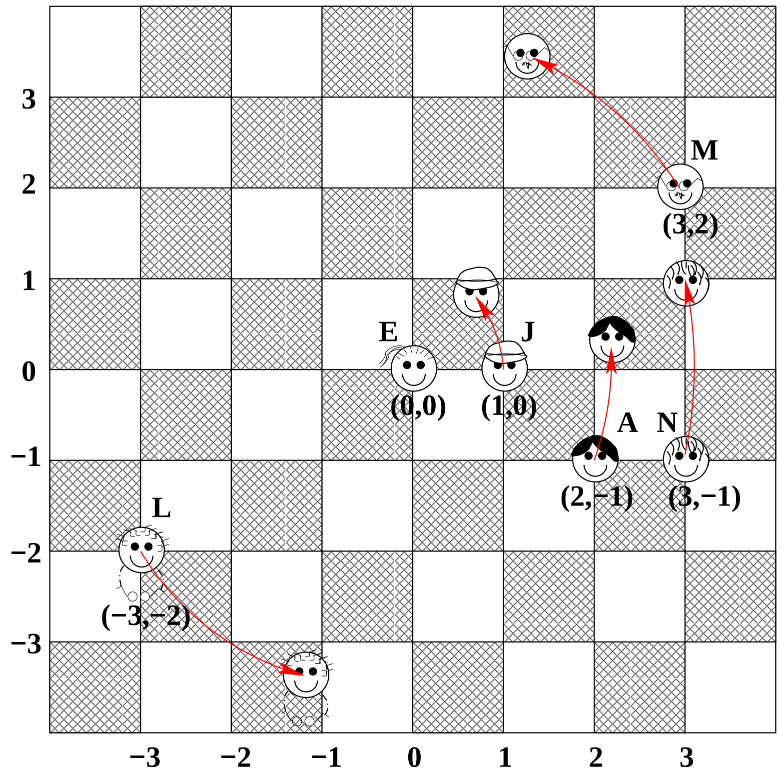
Recall also from Note 2 that any vector is a combination of the two special vectors. So any vector is part left-right unit vector and part up-down unit vector. The left-right part will be bent by the matrix multiplication in the way we have just seen. The up-down part will be bent leftwards as we also saw.

8. Rotations. Let's look at this in a special case, the matrix

$$\begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix} = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

Multiply every position in the classroom by this matrix and see where everybody moves to. (I'll use the lumped vectors, the 2×7 matrix, to write this more compactly.)

$$\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 3 & 2 & -3 & 3 \\ 1 & 0 & 0 & -1 & -1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} -.6 & .8 & 0 & 3 & 2.2 & -1.2 & 1.2 \\ .8 & .6 & 0 & 1 & 0.4 & -3.4 & 3.4 \end{pmatrix}$$



9. Shear. All rotation matrices have determinant 1. (Check this for Note 8.) We can find other matrices which also have determinant 1.

An example is a shear matrix. The matrix we started with in Note 1 shears space as we saw in Note 7: it squeezes it in one direction and lets it squirt out in another direction, like a toothpaste tube. That matrix also does other things to the space so let's see if we can purify the notion of shear.

First, we can make the distortion symmetrical. This takes a "symmetric" matrix, such as

$$\begin{pmatrix} 4/5 & 3/5 \\ 3/5 & 4/5 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

However, the determinant is no longer 1. (What is it?)

To get determinant 1 for the symmetric matrix

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

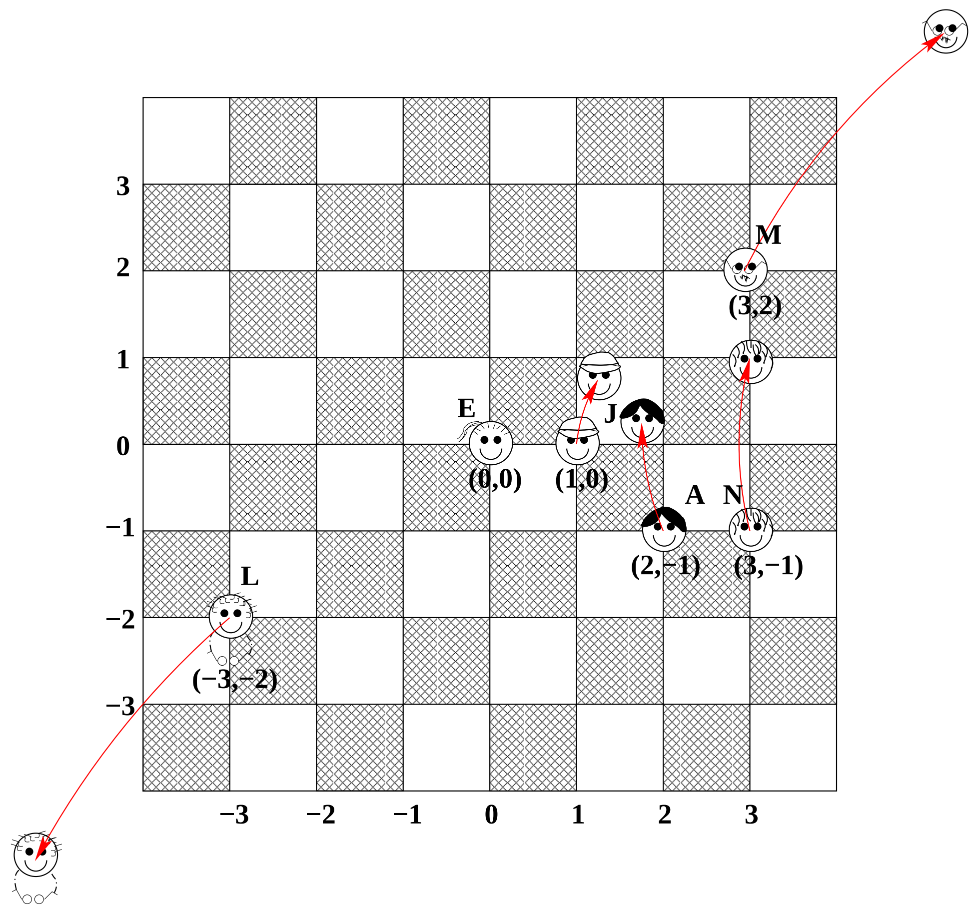
we need $a^2 - b^2 = 1$. This can also be done with a Pythagorean triple.

For example, $(5/4)^2 - (3/4)^2 = 1$, so

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$$

is a symmetric, det=1 matrix. We call such matrices (pure) "shear" matrices.

Here is the effect on the classroom space.



10. Summary
(These notes show the trees. Try to see the forest!)

1. Matrix multiplication.
2. Vectors.
3. Identity matrix.
4. Matrix inverse.
5. Vectors in space.
6. Positions and intervals.
7. Transforming space.
8. Rotations.
9. Shear.

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. "Transpose" the operations in Note 2 by rewriting each 2×1 matrix as a 1×2 matrix. When you transpose each matrix in a multiplication, note that the multiplication rule can no longer work. So you must also exchange the two matrices. Try

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \times \begin{pmatrix} 4 & 3 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 4 & 3 \end{pmatrix}$$

Using these two ideas, rework all the matrix calculations in these Notes into their transposes.

2. What is

$$\frac{\begin{pmatrix} 12 & -5 \\ -3 & 4 \end{pmatrix}}{4 \times 12 - 5 \times 3} \times \begin{pmatrix} 4 & 5 \\ 3 & 12 \end{pmatrix}$$

3. Are the following matrices singular?

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$$

   What is the pattern? Do all singular matrices obey this pattern? Can any non-singular matrix obey it? How does this pattern transform space? (Draw the effect on the two special unit vectors.)

4. What is the condition that the determinant of the antisymmetric matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

   be 1? How can this be achieved by Pythagorean triples?

5. a) Write down a rotation matrix based on the Pythagorean triple 5, 12, 13.
   b) Multiply this both ways with the rotation matrix from Note 8: does matrix multiplication commute for rotation matrices? Does this make sense?
   c) What are the inverses of these rotation matrices?
   d) What is the vector that is twice the angle from horizontal as that made by

$$\begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

   What is the corresponding Pythagorean triple?

6. Calculate the effect of the first symmetric matrix of Note 9 on the seven vectors of the classroom in Note 8 and compare this with the shear matrix by drawing the transformed space.

7. a) How do

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

   transform under the shear matrix of Note 9? Draw the new space.
   b) Which of these vectors are appropriately called "invariant" vectors of the matrix?

8. a) Write down a shear matrix based on the Pythagorean triple 5, 12, 13.
   b) Multiply this both ways with the shear matrix from Note 9: does matrix multiplication commute for shear matrices?
   c) What are the inverses of these shear matrices?
   d) What are the invariant vectors of your new shear matrix?

9. Draw the classroom of Note 8 as transformed by any of the 2×2 matrices discussed in these Notes or that you have invented yourself.

10. The "MAT" in the MATLAB programming language stands for matrices. The TI81 calculator and its successors can also do matrix operations. Learn how to use these or equivalent software to check the calculations in these Notes and your own exercises.

11. Any part of the lecture that needs working through.