Excursions in Computing Science:
Week 5. Particles with Periods

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I. Prefatory Notes

1. Are 2-numbers good for anything besides geometry (rotation, reflection, projection—but not shear)?

Yes. 2-numbers describe *amplitudes* perfectly.

2. Photons

What is light, anyway?

1670s	Isaac Newton	"corpuscles"	(refraction, ..)
1801	Thomas Young	"waves"	(2-slit interference)
1905	Albert Einstein	both:photons	(photoelectric effect)

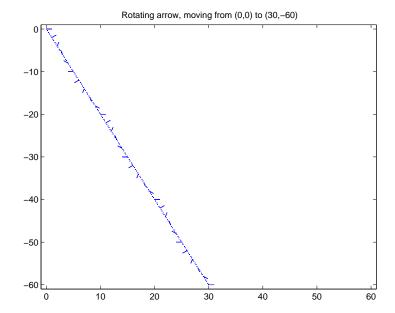How can an (indivisible) particle have a frequency?

Let's model it as a 2-number

$$\cos \omega t + i \sin \omega t$$

which is an arrow rotating with angular frequency ω radians/sec.
(i.e., frequency $\omega/2\pi$ cycles/sec.).

¡ This arrow has nothing to do with the direction of polarization, although that is another arrow. This arrow is rotating in the "unknown" space of 2-numbers. Be careful trying to picture it!

Rotating arrow, moving from (0,0) to (30,−60)

3. This 2-number (arrow) will be the *amplitude* for our photon.

Feynman's Rules

1. Add amplitudes if an event can happen in indistinguishable alternative ways.

2. Multiply amplitudes of an inseparable succession of events.

The words "indistinguishable" and "inseparable" mean that no observation is made which distinguishes alternatives or separates events.

The amplitude resulting from these combinations must be multiplied by its own conjugate

$$(c + is)(c - is) = c^2 + s^2$$

to give the positive 1-number that is the *intensity* of the process, to be compared with observation.

Well, if we are dealing with only one particle, such as a photon, which is indivisible, the result cannot be an intensity but must be interpreted as a *probability*.

4. Let's calculate probabilities for Young's double slit experiment.

2

0) $T_0' = T_0$

$$
\begin{aligned}
\text{amplitude} &= a_0(\cos \omega T_0 + i \sin \omega T_0) + a_0(\cos \omega T_0' + i \sin \omega T_0') \\
&= 2a_0(\cos \omega T_0 + i \sin \omega T_0) \\
\text{probability} &= 4a_0^2(\cos^2 \omega T_0 + \sin^2 \omega T_0) \\
&= 4a_0^2
\end{aligned}
$$

1)

$$
\begin{aligned}
\text{amplitude} &= a_0(\cos \omega T_1 + i \sin \omega T_1) + a_0(\cos \omega T_1' + i \sin \omega T_1') \\
\text{say} \omega T_1' &= \omega T_1 + \delta \\
\cos \omega T_1 + \cos \omega T_1' &= \cos \omega T_1(1 + \cos \delta) - \sin \omega T_1 \sin \delta \\
\sin \omega T_1 + \sin \omega T_1' &= \sin \omega T_1(1 + \cos \delta) + \cos \omega T_1 \sin \delta \\
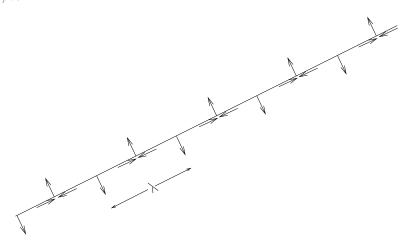\text{so amplitude} &= a_0(1 + \cos \delta + i \sin \delta)(\cos \omega T_1 + i \sin \omega T_1) \\
\text{probability} &= \text{amplitude}(i)\text{amplitude}(-i) \\
&= a_0^2((1 + \cos \delta)^2 + \sin^2 \delta) \\
&= 2a_0^2(1 + \cos \delta)
\end{aligned}
$$

Interpretation: the amplitudes are moving, rotating arrows. Two of them may wind up pointing in the same direction at the screen, hence reinforcing each other and making a bright spot. Or two of them may wind up pointing in opposite directions at the screen, cancelling and making a dark spot. Or they may wind up in arbitrary orientations to each other, depending on the travel times (or distances), and producing some intermediate brightness.

5. "Wavelength", λ

A travelling rotating arrow periodically points in the same direction as when it started.

The interval between successive such pointings is, if measured in time, the *period* and, if measured in space, the *wavelength*.

If the distance from one slit to a point on the screen differs by a whole number of wavelengths from the distance from the other slit to the same point, then the intensity|probability at that point is the same as at point 0, to which these two distances are the same.

At point 0, we just now saw that the intensity|probability is $4a_0^2$ for some a_0^2 which gives the intensity|probability of the light at the source. We also saw that the intensity|probability in general

on the screen is $2a_0^2(1 + \cos\delta)$ for some δ whose exact relationship with the position on the screen we haven't figured out yet.

What this $2a_0^2(1 + \cos\delta)$ tells us already before figuring out anything more is that $4a_0^2$ is the *maximum* intensity|probability. This maximum intensity occurs periodically, just as we thought. In between the maxima, there are minima of 0 intensity.

6. Now let's relate δ to the vertical distance, y, from the central point, 0, on the screen.

δ is the *phase difference* between the two paths, and equals what is shown as $d\sin\theta$ in the figure, where d is the separation between the two slits.

$$\delta = d\sin\theta$$

The vertical distance, y, is shown as $L\sin\theta$, where L is distance between the plane of the slits and the plane of the screen.

$$y = L\sin\theta$$

So from these two relations we can get rid of the $\sin\theta$

$$\delta = dy/L$$

We can also find the separation between bright spots by putting
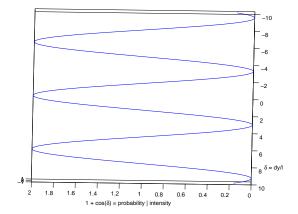
$$\delta = m\lambda$$

for any whole number, m.

$$y = Lm\lambda/d$$

And so the y-separation between two successive bright spots, corresponding to m and $m+1$, say, is

$$\Delta y = L\lambda/d$$

Here is how the intensity|probability varies, with 0 on the vertical axis marking the central bright spot, 0, on the double-slit diagram. (a_0^2 is set arbitrarily to 1/2 in this plot.)

4

So if we know how far apart the bright spots are on the screen (Δy), and also d and L, then we can find λ, the wavelength of the light.
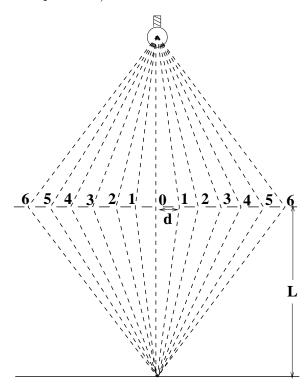
Thinking of a photon as a moving, rotating 2-number shows that light is wavelike. It predicts the result of the double slit experiment.

7.   *Lord Ronald said nothing; he flung himself from the room,*
flung himself upon his horse and rode madly off in all directions.
Stephen Leacock, *Gertrude the Governess*

Another thing we can show with these ideas is that light travels in straight lines.

Let's start by thinking about a whole bunch of slits, a source, and a single target (corresponding to point 0 on the double-slit experiment).
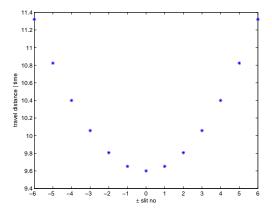

Here the travel distances are proportional to the following.

$$T_0 \quad \propto \quad L$$

$$T_1 \quad \propto \quad \sqrt{L^2 + d^2}$$
$$\vdots$$
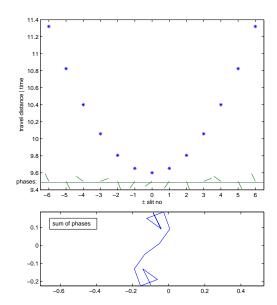$$T_k \quad \propto \quad \sqrt{L^2 + (kd)^2}$$
$$\vdots$$

(Note that two slits are each labelled 1, 2, .., because we are interested only in the lengths of travel.)

Here is a plot of these distances for the multislit experiment shown. (Note that the vertical axis does not start at 0.)

The distance travelled does not vary much for light passing through slit 0 and nearby slits. As the slits get further away from 0, the distance travelled varies (increases) much more rapidly.

What happens to the arrows, moving along with the photons and rotating as they move?

The *phases* of the arrows do not differ much for slits close to slit 0, but they can differ wildly for slits far away from 0.
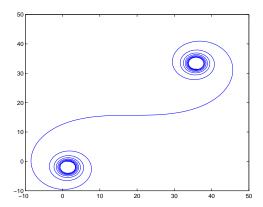

The wild differences tend to cause the phases to cancel each other out—they interfere destructively, just as phases in the double-slit experiment interfered to cancel all the light in the dark areas on

the screen.

The consequence is that only light travelling very close to slit 0, i.e. a straight line, gets through to the target uncancelled.

If we think of only one photon, we can say that it *does* go madly off in all directions—but the amplitude for the straight-line path from source to destination so dominates that we see light travelling in straight lines.

Here is the cumulative sum of all the arrows from a more refined multi-slit configuration, as a turtle would draw them. The first arrow is the orientation of the photon on arrival at the destination if it were to pass through the leftmost slit, and is shown starting at (0,0). The arrow from the slit to its immediate right is added, and so on.

We see that the arrows to the extreme left almost form a circle, thus cancelling each other. As the paths get closer to slit 0 the arrows start adding constructively, making the large net contribution across the middle of the figure. Over on the extreme right, however, the arrows add up to form another circle of cancellations.
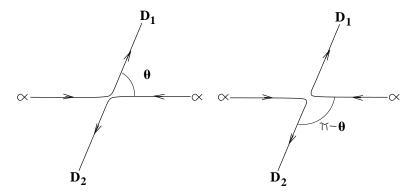
The next step is to consider a "barrier" with so many slits that there is no barrier at all. The same considerations apply, only the sum of all the paths now becomes an integral over all the paths. Furthermore, these paths can wander anywhere in space, leftwards or rightwards off to infinity, behind the source, around the final destination and back, .., anywhere. The integrals are the source of "Feynman diagrams". The most accessible book on this subject [Zee03] is considerably beyond our present skills (requiring calculus of variations and the principle of least action [FLS64a, Chap. 19]). You can come back to it after mastering Week 8c part IV.

Thinking of a photon as a moving, rotating 2-number shows that light is particle-like. It predicts that light will appear to travel in straight lines, because all other paths cancel out.

8. Adding amplitudes in this way works for all sorts of particles, not just photons (light).

Feynman's Rules (again)

1. Add amplitudes if an event can happen in indistinguishable alternative ways.

2. Multiply amplitudes of an inseparable succession of events.

The words "indistinguishable" and "inseparable" mean that no observation is made which distinguishes alternatives or separates events.

Let's collide two α-particles.

This shows two ways to get an α-particle to D_1:

$$\text{prob} = \mid a(\theta) + a(\pi - \theta) \mid^2$$

(For example, if $\theta = \pi/2$, this probability is $4 \mid a(\pi/2) \mid^2$.)

This is because α-particles are *indistinguishable* from each other.

If we could distinguish them, say calling the α-particle from the left α_1 and the α-particle from the right α_2, then the probability that some α-particle gets to D_1 is

$$\text{prob} = \mid a(\theta) \mid^2 + \mid a(\pi - \theta) \mid^2$$

(For example, if $\theta = \pi/2$, this probability is $2 \mid a(\pi/2) \mid^2$.)

This illustrates the Feynman rule about adding amplitudes of indistinguishable alternatives and not adding them if they are distinguishable. In the second example, we added *probabilities* and got only half the probability we got in the first example.

Statistics are different for indistinguishable and distinguishable particles.

9. But for electrons, there is a new rule:

- Change sign when exchange indistinguishable electrons.

This means we *subtract* amplitudes.

(This rule will be justified in Week 7a and Week 9.)

$$\text{prob} = \mid a(\theta) - a(\pi - \theta) \mid^2$$

(For example, if $\theta = \pi/2$, this probability is 0.)

There are two kinds of particle:

- **Bosons** Add amplitudes if collision can take place in (two) alternative ways. It follows that it is OK for two bosons to be in the same state (i.e., have the same amplitudes, a_B); the probability is
$$\mid a_B + a_B \mid^2 = 4 \mid a_B \mid^2$$
  .

- **Fermions** Subtract amplitudes if collision can take place in (two) alternative ways. It follows that it is impossible for two fermions to be in the same state (i.e., have the same amplitudes, a_F); the probability is
$$\mid a_F - a_F \mid^2 = 0$$
  .

Note that this formulation is not as general as "Feynman's Rules" given above. The general statement is that *exchanging* two identical particles of either species could be considered to cause a phase change in the amplitude, $a() \rightarrow e^{i\delta}a()$ because such a phase change has no effect on the observable probabilities. Since *two* exchanges are the same as no exchange at all, $e^{2i\delta} = 1 = e^{2mi\pi}$ for some integer, m, which means effectively that $\delta =$ either 0 or π.

Because anything not ruled out by quantum mechanics as impossible actually seems to occur in nature, we find two classes of particle: the bosons, for which $\delta = 0$; and the fermions, for which $\delta = \pi$. See Week 7a and [FLS64b, p.4-1]

"Fermions" and "bosons" form the quantum-mechanical classification of all possible particles. The fermions (named after Enrico Fermi, 1901–1954) are the basis of all matter: electrons and quarks, protons, neutrons and the mesons are all fermions. The bosons (named after Satyendra Nath Bose, 1894–1974) are responsible for all forces: photons (electromagnetic), W and Z (weak force), the gluons (strong force), and the hypothetical graviton are all bosons. So are pairs of fermions, such as the coupled electrons that give superconductivity, and Helium-4 atoms, which are superfluid.

10. Summary

(These notes show the trees. Try to see the forest!)

- Particles can have frequencies because they somehow have 2-numbers associated with them: particles are moving, rotating arrows.

- Particles can have wavelengths if they are moving: light is wavelike.

- Particle paths that stray away from straight lines between source and destination cancel by interference: light is particle-like.

- Feynmans's Rules for amplitudes.

- Intensities become probabilities when there is only one particle.

- Approximate reasoning.

- Bosons (add amplitudes, basis of forces) and Fermions (subtract amplitudes, basis of matter).

II. The Excursions

You've seen lots of ideas. Now *do* something with them!

1. Run the following MATLAB function.

```
% function arrowMovie(f,vx,vy,n)              THM              061205
% animate arrow rotating freq f, velocity (vx, vy) for n cycles
% (1/f steps per cycle; f cycles per step)
function arrowMovie(f,vx,vy,n)
% set axis sizes
avx = abs(vx); avy = abs(vy);
if avx >= avy, size = 1 + avx*n; else size = 1 + avy*n; end
if vx < 0, xmin = -size; xmax = 1; else xmin = -1; xmax = size; end
if vy < 0, ymin = -size; ymax = 1; else ymin = -1; ymax = size; end
lf(1,:) = [0 0];         % show line of flight
lf(2,:) = [vx*n vy*n];
adjac = [0,1;1,0];       % adjacency matrix for gplot: single edge (1,2)
for t = 0:n;
  x = vx*t;
  y = vy*t;
```

```
        phase = f*t*2*pi;
        u = cos(phase);
        v = sin(phase);
        gplot(adjac,lf,':'), hold on  % line of flight
        xy(1,:) = [x y];
        xy(2,:) = [x+u y+v];
        gplot(adjac,xy), hold off     % animate arrow
        axis([xmin xmax ymin ymax])
        M(t+1) = getframe
    end;
    movieview(M)
```

Extend it to animate all paths in the two-slit and multi-slit experiments.

2. Write a MATLAB program to plot `1 + cos(x)` and use the interactive `Rotate 3D` facility to show the intensity|probability of light plotted against the vertical position, y.

3. Using 2-number exponentials, rewrite the calculations of Note 4 giving the intensity|probability of light on a screen after passing two slits from a single source.

4. Using particular values for δ, including 0 and π, draw arrows for the calculations of Note 4, showing the relationships between the pairs of arrows where they arrive at the same point on the screen. Show the result of adding the two arrows in these pairs.

5. The argument giving the separation between bright spots in the double slit experiment, $\Delta y = L\lambda/d$, has two caution signs (from American Allsafe). Explain the approximations these signs warn of.

6. In a 2-slit measurement with the slits 1 mm apart and 1 m from the screen, a bright line is seen on the screen every half mm. What colour is the light?

7. Given the wavelength, λ, of the light in the 2-slit measurement, calculate *exactly* the locations of the bright spots on the screen and see if they are really equally spaced.
Calculate the *exact* intensities | probabilities and see if they are really equal for all y.

8. Write a MATLAB program to plot the lengths of the paths from source to destination in the multi-slit diagram of Note 7. Try $L = 9.6$ and $d = 1$ to match the proportions of the diagram. Go on to calculate the directions of the arrows at the destination for each path, assuming their angle is 0 at the source and that the wavelength $\lambda = 1$. Use MATLAB's `quiver(X,Y,U,V,0,'.')` to plot these final arrows along the bottom of your first plot.

9. Use MATLAB to calculate the S-shaped accumulation of arrows for light passing from source to target 10,000 wavelengths away through a screen, halfway, of 201 slits separated from each other by one wavelength. The central slit is on the direct line from source to target, and the others are sprread out from number 1 to number 100 on each side. (No light passes around this screen.) What happens if the screen stretches further to the sides, say 1–1000 or 1–2000 or more?
Do it also for the thirteen arrows of the previous excursion.

10. Look up Isaac Newton, 1642–1727, and discuss his corpuscular theory of light and his explanation of refraction.

11. Look up Thomas Young, 1773–1829, and discuss his double slit experiment and the interference of light. Did he invent the approximate-sine argument?

12. Look up Albert Einstein, 1879–1955, and discuss his explanation of the photoelectric effect.

13. Look up Richard Phillips Feynman, 1918–1988, particularly his *QED: The Strange Theory of Light and Matter* [Fey85]. Can you relate "Feynman diagrams" to our discussion of light travelling in all directions but cancelling so that it effectively travels in straight lines? Look up his demonstration [FLS64b, p.4-6] that bosons actually *attract* other bosons. Since we found out in Note 9 that fermions repel other fermions, it is understandable that fermions provide the basis for solid matter (hard to compress). It is also understandable that many bosons will cooperate to produce forces, such as magnetic forces, which are strong enough for us to experience in the macroscopic world. Find "Feynman's rules" [Fey85, pp.31,61], [FLS64b, pp.3-3,3-7].

14. Any part of the lecture that needs working through.

References

[Fey85]   R. P. Feynman. *QED: The Strange Theory of Light and Matter*. Princeton University Press, Princeton, N.J., 1985. Alix G. Mautner Memorial Lecture, UCLA, 1983.

[FLS64a] R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume II*. Addison-Wesley, 1964.

[FLS64b] R. P. Feynman, R. B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume III*. Addison-Wesley, 1964.

[Zee03]   A. Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, Princeton, Oxford, 2003.