Excursions in Computing Science: Week 2 Operators

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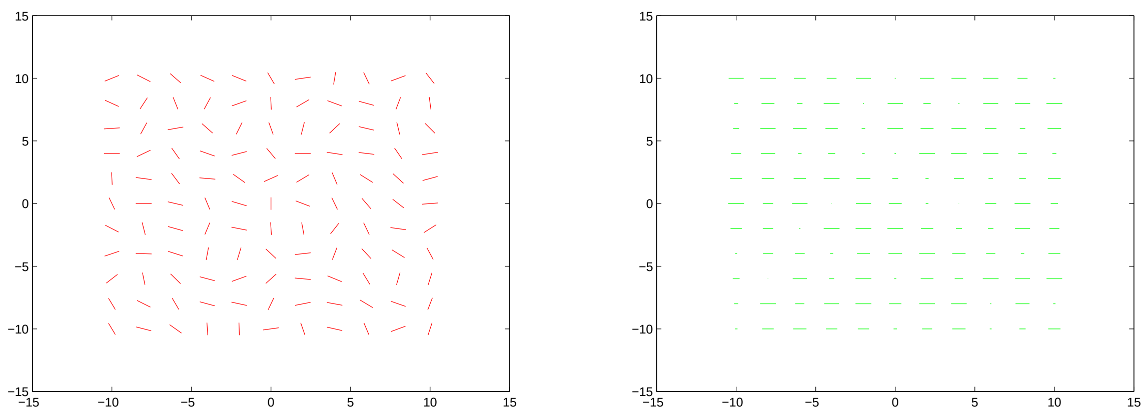
January 29, 2007

1. A polarizing filter does someting to light.

We should think of filtering as an operation and the filter as an operator.

Unpolarized light $\Rightarrow$ Polarized light

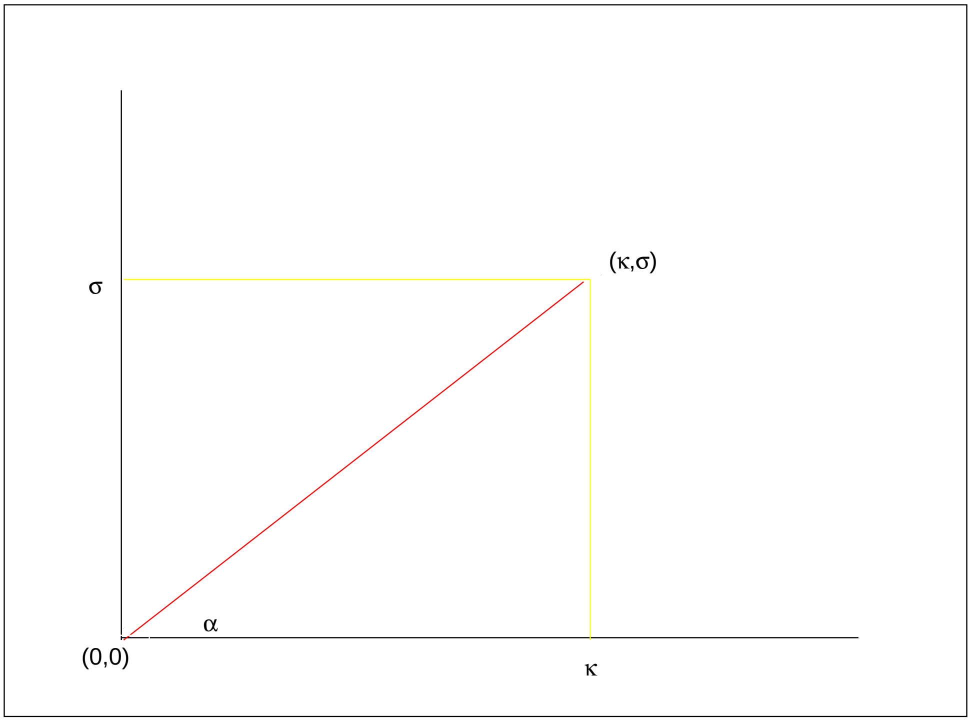


 The author gratefully acknowledges support from the taxpayers of Québec and of Canada who have paid his salary and research grants while this work was developed at McGill University, and from his students (who built the implementations and investigated the data structures and algorithms) and their funding agencies.

2. We'll call a light beam a *vector*

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

(using κ, the Greek k, for $\cos(\alpha)$, and σ, the Greek s, for $\sin(\alpha)$).



3. A horizontal polarizing filter extracts the κ component.
(Let's work with amplitudes: we can always square them at the end to get intensities.)

For a vector ($\vec{u}$ for "unpolarized"),

$$\vec{u} = \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

what operator finds the κ component?

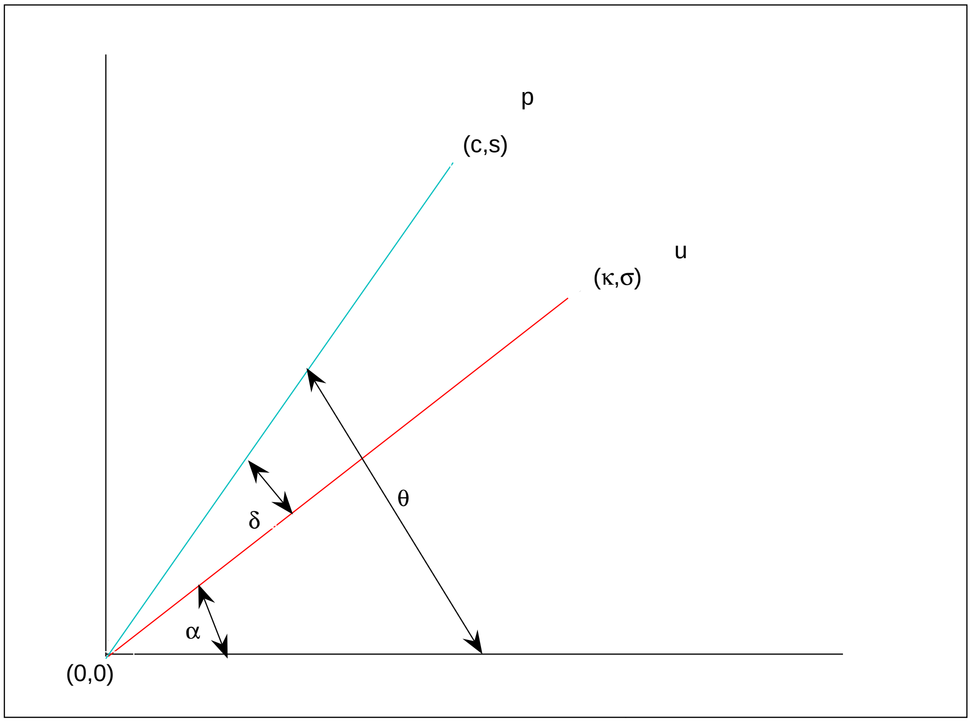
Easy. Just set $\sigma \to 0$.

4. What if the polarizing filter is at angle θ?

Describe the filter as c, s: $c = \cos\theta, s = \sin\theta$.

So the result will be along the vector ($\vec{p}$ for "polarized")

$$\vec{p} = \begin{pmatrix} c \\ s \end{pmatrix}$$



$$\begin{aligned} \kappa &= \cos\alpha \\ \sigma &= \sin\alpha \\ c &= \cos\theta \\ s &= \sin\theta \\ \delta &= \theta - \alpha \end{aligned}$$

5. Now some new math. Two kinds of "vector multiplication":

i) "inner product" (or "dot product"), $\vec{p}^{\,T}\vec{u}$, (1 by n) $\times$ (n by 1):

$$\begin{pmatrix} c \\ s \end{pmatrix} \underline{(c,s)\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}} \quad = \quad \begin{pmatrix} c \\ s \end{pmatrix}(c\kappa + s\sigma) \qquad\qquad (1 \text{ by } 2) \times (2 \text{ by } 1) \rightarrow (1 \text{ by } 1)$$

ii) "outer product" (or "tensor product"), $\vec{p}\vec{p}^{\,T}$, (n by 1) $\times$ (1 by n):

$$\underline{\begin{pmatrix} c \\ s \end{pmatrix}(c,s)}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \quad = \quad \begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \qquad\qquad (2 \text{ by } 1) \times (1 \text{ by } 2) \rightarrow (2 \text{ by } 2)$$

iii) There is also a "scalar product", (m by n) $\times$ "1 by 1" or "1 by 1" $\times$ (m by n):

$$\begin{pmatrix} c \\ s \end{pmatrix}(c\kappa + s\sigma) \quad = \quad \begin{pmatrix} c^2\kappa + cs\sigma \\ sc\kappa + s^2\sigma \end{pmatrix} \qquad\qquad (2 \text{ by } 1) \times (1 \text{ by } 1) \rightarrow (2 \text{ by } 1)$$

$$= \quad (c\kappa + s\sigma)\begin{pmatrix} c \\ s \end{pmatrix} \qquad\qquad (1 \text{ by } 1) \times (2 \text{ by } 1) \rightarrow (2 \text{ by } 1)$$

iv) And the general "matrix product", (k by m) $\times$ (m by n):

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \quad = \quad \begin{pmatrix} c^2\kappa + cs\sigma \\ sc\kappa + s^2\sigma \end{pmatrix} \qquad\qquad (2 \text{ by } 2) \times (2 \text{ by } 1) \rightarrow (2 \text{ by } 1)$$

In MATLAB:

```
alpha = pi/6;
kappa = cos(alpha);
sigma = sin(alpha);
u = [kappa;sigma]

u =

    0.8660
    0.5000
theta = pi/5;
c = cos(theta);
s = sin(theta);
p = [c;s]

p =

    0.8090
    0.5878
```

```
p'

ans =

    0.8090    0.5878

pu = p' * u

pu =

    0.9945

pp = p * p'

pp =

    0.6545    0.4755
    0.4755    0.3455
```

```
p * pu

ans =

    0.8046
    0.5846

pu * p

ans =

    0.8046
    0.5846

pp * u

ans =

    0.8046
    0.5846
```

6. So the polarizing filter is the operator

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}$$

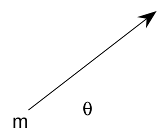
Let's try this for the bottom, middle, and top filters, **b**, **m**, **t**.

b: $\theta = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

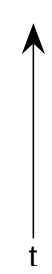
m: $\theta =$ any angle

$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}$$



t: $\theta = \pi/2$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



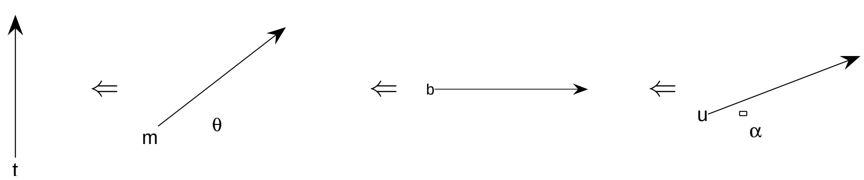
7. We filter

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

by **b** then **m** then **t**:

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^2 & 0 \\ sc & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ sc & 0 \end{pmatrix}\begin{pmatrix} \kappa \\ \sigma \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ sc\kappa \end{pmatrix}
\end{aligned}$$

where sc is $\sin\theta\cos\theta$ as before.

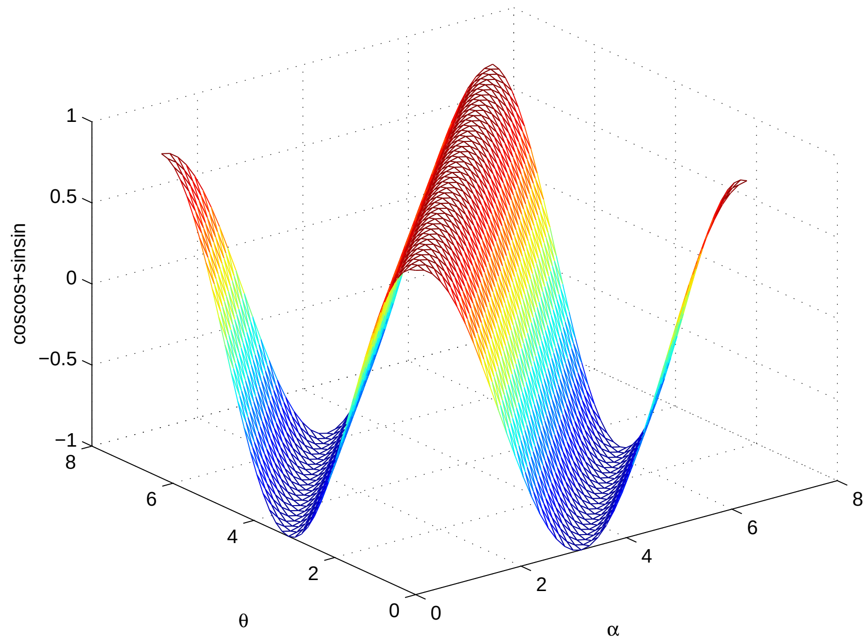


8. We keep seeing c^2, cs, s^2, i.e., $\cos^2(), \cos()\sin(), \sin^2()$, and even $c\kappa + s\sigma$.

What do these combinations of trig. functions mean?

Let's try MATLAB

```
alpha = 0:pi/20:2*pi;
theta = 0:pi/20:2*pi;
mesh(theta',alpha,cos(theta')*cos(alpha) + sin(theta')*sin(alpha))
xlabel('\alpha'),ylabel('\theta'),zlabel('coscos+sinsin')
```



Look carefully! This goes through two cycles as α and θ each go through one.

We can get at this by looking at another operator, *rotate*.

Rotate: $(x, y) \to (x', y')$

So let's consider rotation to be an operator, say

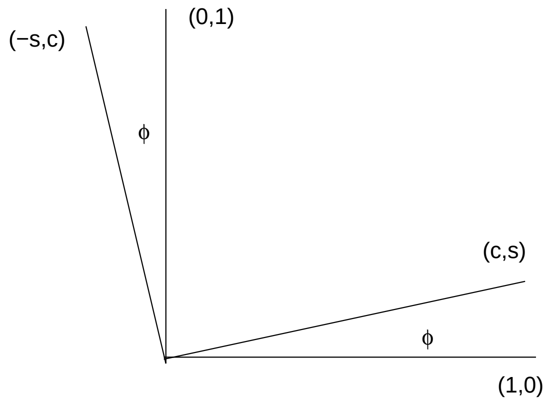
$$R = \begin{pmatrix} p & u \\ q & v \end{pmatrix}_\phi$$

where ϕ is the angle of rotation.

Rotating the x-axis, $R\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, should give the line oriented at angle ϕ:

$$R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ s \end{pmatrix}_\phi$$

This means $p = c$ and $q = s$. (Why?)



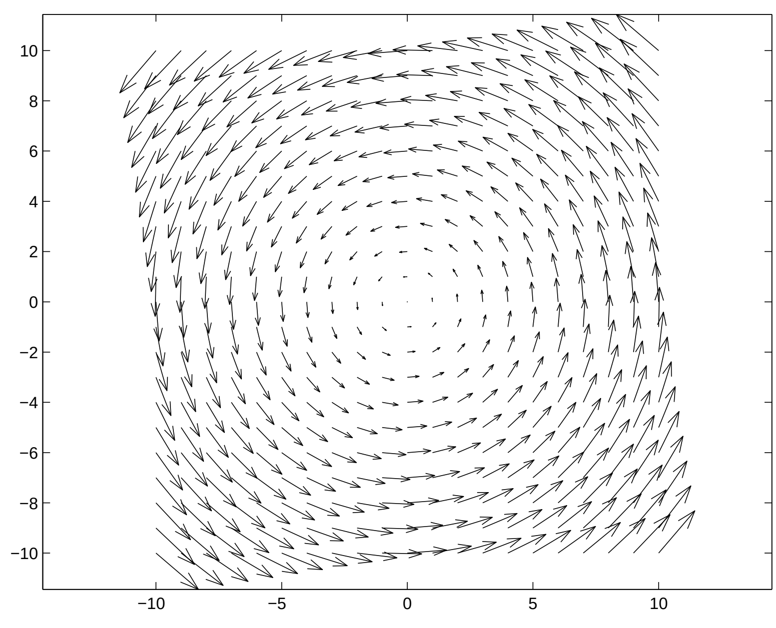
Similarly, rotating the y-axis, $R\begin{pmatrix}0\\1\end{pmatrix}$, should give the line oriented at angle ϕ *to the y axis*:

$$R\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-s\\c\end{pmatrix}_\phi$$

This means $u = -s$ and $v = c$. (Why?)

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}c & -s\\ s & c\end{pmatrix}_\phi \begin{pmatrix}x\\y\end{pmatrix} \quad \text{i.e.,}$$

$$\begin{aligned} x' &= x\cos\phi - y\sin\phi \\ y' &= x\sin\phi + y\cos\phi \end{aligned}$$



9. What is the operator for *two* rotations?

ϕ then ψ is $\phi + \psi$

$$\begin{pmatrix}c' & -s'\\ s' & c'\end{pmatrix}_\psi \begin{pmatrix}c & -s\\ s & c\end{pmatrix}_\phi \rightarrow \begin{pmatrix}C & -S\\ S & C\end{pmatrix}_{\phi+\psi}$$

$$
\begin{aligned}
C &= c'c - s's \\
(\cos(\phi+\psi) &= \cos\psi\cos\phi - \sin\psi\sin\phi) \\
S &= s'c + c's \\
(\sin(\phi+\psi) &= \sin\psi\cos\phi + \cos\psi\sin\phi)
\end{aligned}
$$

Try $\psi = \phi$

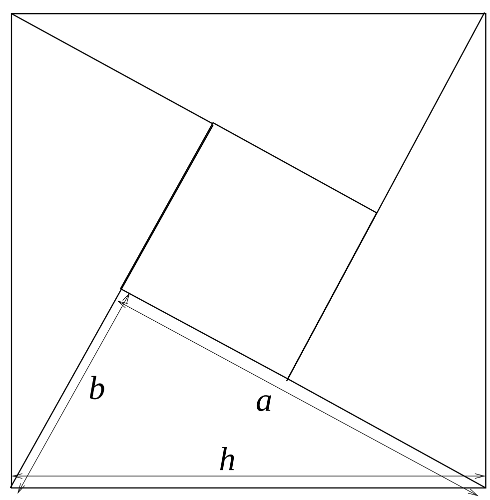
$$
\begin{aligned}
\cos(2\phi) &= \cos^2\phi - \sin^2\phi \\
\sin(2\phi) &= 2\sin\phi\cos\phi
\end{aligned}
$$

Try $\psi = -\phi$

$$
\begin{aligned}
0 = \sin(0) &= -\sin\phi\cos\phi + \cos\phi\sin\phi \\
1 = \cos(0) &= \cos^2\phi + \sin^2\phi
\end{aligned}
$$

NB cos() is even, sin() is odd.

Pythagoras!



$h^2 = (a-b)^2 + 2ab$

Again, try $\psi = -\phi$

$$
\begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix}_\psi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi = \begin{pmatrix} c' & s' \\ -s' & c' \end{pmatrix}_\psi \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

The Identity operator.

10. Note that $R(\psi)R(\phi) = R(\psi+\phi)$ for rotations $R()$

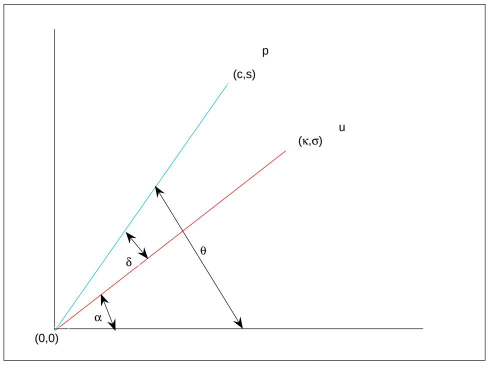
What other kind of function, *multiplied* by itself, gives itself on the *sum* of its arguments?

11. **Summary**

(These notes show the trees. Try to see the forest!)

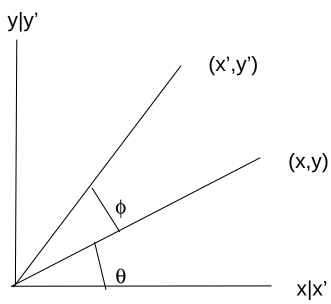
- Vector and matrix products: inner, outer, scalar.

- Projection operator, $\vec{u}$ on $\vec{p}$



$$\begin{pmatrix} c^2 & cs \\ sc & s^2 \end{pmatrix}_\theta \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}$$

- Polarizing filter is a projection.
- Rotation operator by ϕ



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}_\phi \begin{pmatrix} x \\ y \end{pmatrix}$$

- Identity operator.
- Multiplying cos and sin by each other adds angles.

12. **Excursions for Friday and beyond.**

You've seen lots of ideas. Now *do* something with them!

1. Show that $\cos^2(\theta) = (1 + \cos(2\theta))/2$ and that $\sin^2(\theta) = (1 - \cos(2\theta))/2$. "Squaring is doubling": discuss!
2. What does MATLAB give for

```
u = 0:2
v = (0:2)'
uv
vu
```

3. A "normalized" vector has length 1, and so (in two dimensions) can be written as $(\cos(\theta),\sin(\theta))$, where ($\theta$ is its angle with the x-axis. Show that the scalar product of two such vectors

$$(c, s)\begin{pmatrix} C \\ S \end{pmatrix} = (C, S)\begin{pmatrix} c \\ s \end{pmatrix} = \cos(\theta_1 - \theta_2)$$

where $c = \cos(\theta_1), s = \sin(\theta_1), C = \cos(\theta_2), S = \sin(\theta_2)$ and $\theta_1 - \theta_2$ is the angle between the two vectors.

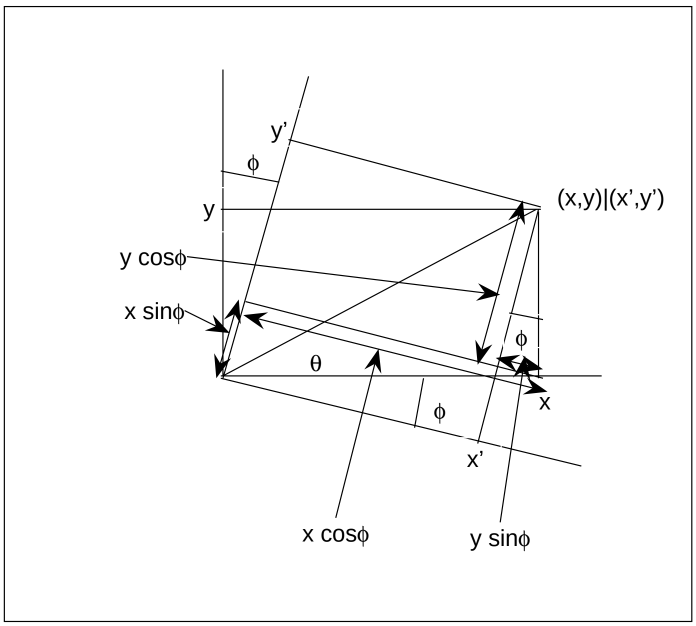
Show that the vector can also be written as (cos(θ),cos(θ')) where (θ is its angle with the x-axis, as before, and θ' is its angle with the y-axis.

Using the scalar product, above, show that $\cos\theta_1 \cos\theta_2 + \cos\theta'_1 \cos\theta'_2 = \cos(\theta_1 - \theta_2)$ where $\theta'_1 = \pi/2 - \theta_1$ and $\theta'_2 = \pi/2 - \theta_2$.

Show that a three-dimensional normalized vector has components (cos(α),cos(β),cos(γ)), where α is its angle with the x-axis, β its angle with the y-axis, and γ its angle with the z-axis. What does this say about $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$? What expression do you think gives the cosine of the angle between two 3-D unit vectors?

4. In MATLAB write the operator that projects onto orientation $\pi/4$, and try it on half a dozen vectors at different angles. (At least one of these angles should be a multiple of one of the others, say twice it.) Describe your experiments in terms of a polarizing filter and light beams. With more projections, model the 2- and 3-filter polarizers discussed in class.

5. Using the diagram



derive the rotation operator

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_\phi \begin{pmatrix} x \\ y \end{pmatrix}$$

The diagram views the vector at angle θ staying still and the *axes* rotating through the negative angle, $-\phi$. Since the rotation operator just changes the coordinates describing the vector, it is the same thing if the vector rotates one way or if the axes rotate the other way.

6. In MATLAB write the operator that rotates by angle $\pi/4$, and try it on half a dozen vectors at different angles. (At least one of these angles should be a multiple of one of the others, say twice it.) Write other rotations by other angles, such as $-\pi/4, pi/2$, and 0, and try them on your vectors.

7. Use MATLAB to generate the 25 coordinate pairs

$$(x, y) = (-2, -2), (-2, -1), .., (0, 0), .., (2, 2)$$

to rotate all through $\pi/20$

$$(u, v) = R(x, y)$$

and to display the rotation as arrows from each (x, y) to the corresponding (u, v). (Try `help quiver`)

8. Do the same for a projection, say to orientation $\pi/5$.

9. What is the inverse of a projection?

10. How does the expression $\cos^2() + \sin^2() = 1$ relate to Pythagoras?

11. Why does the square-within-a-square diagram in lecture 4 prove Pythagoras' theorem, $a^2 + b^2 = h^2$, for h the hypotenuse of a right-triangle with other sides a and b?

12. A matrix is an array of numbers laid out in two "dimensions". The positions of its elements might be given as the pairs of integers (0,0), (0,1), (1,0), (0,2), (1,1), (2,0), .. Computer memory is usually organized along only one "dimension": a sequence of locations numbered 0, 1, 2, ... Give formulas which map the array pairs into memory locations: a) running through all of row 0, then all of row 1, and so on; b) running through all of column 0, then all of column 1, and so on. You will need to use w, the width of the array or h, its height.

13. Matrix multiplication is made easy for us by MATLAB: `A*B`. Under the hood, a computer program to multiply the l-by-m matrix `A` by the m-by-n matrix `B` must evaluate the sum of products

$$\Sigma_{j=1..m} A_{ij} B_{jk}$$

for each $i = 1..l$ and $k = 1..n$, where A_{ij} is the element of `A` in row i and column j, and B_{jk} is the element of `B` in row j and column k. Do this in MATLAB using the for statement. How many element multiplications will be done in this whole matrix product?

14. a) An operator, L, is *linear* if it obeys two rules (called "axioms"):
    Axiom 1 $L(x + y) = L(x) + L(y)$ and
    Axiom 2 $L(ax) = aL(x)$.
    Show that multiplication by matrix, M, is linear, if x and y, above, are matrices or vectors, and a, above, is a scalar (an ordinary number).
    b) Translation by an amount (tx, ty) has the result of adding (tx, ty) to every vector being translated. Show that this is not linear. Which axiom is violated? What operation would violate the other axiom but not this one?
    c) Translation can be expressed by matrix multiplication by adding a third dimension, z, mapping the original two-dimensional space to the plane $z = 1$, and doing a shear operation which changes the z axis but not the x or y axes. Figure out how to do this.

15. Look up Arthur Cayley, 1821-1895. How did he come up with matrices?

16. Any part of the lecture that needs working through.