On Folding Rulers in Regular Polygons

(extended abstract)

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Abstract

An *l-ruler* is a sequence of n rigid rods lying in the plane and joined consecutively at their endpoints. The endpoints are joints about which the rods may freely turn, possibly crossing over one another. Each rod has length l. An l-ruler is said to be confined inside a polygon P if each link of the ruler must remain inside the closed, bounded region bounded by P at all times. An l-ruler confined inside P is said to be *always-foldable* if, for each possible initial configuration of the ruler, the ruler can be folded onto a single segment of length l.

A study of l-rulers confined to equilateral triangles was carried out by van Kreveld, Snoeyink and Whitesides, who showed that always-foldability is a property that alternates *four* times between holding and failing as l grows from 0 to its maximum possible value. They asked whether this phenomenon occurs for l-rulers confined inside other polygons.

The present paper extends their study to regular polygons. In particular, it answers their question in the affirmative: in regular $2k$-gons, the always-foldability of l-rulers alternates between holding and failing *three* times as l grows from 0 to its maximum possible value.

1 Introduction

A *chain* Γ is a sequence of n rigid rods (also called links) joined consecutively at their endpoints (also called joints), about which they may freely rotate. An l-ruler L is a chain whose links all have the same length l.

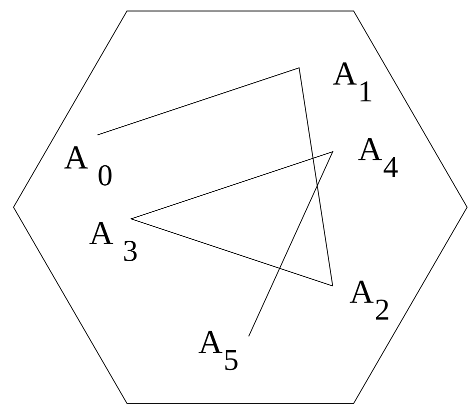

Figure 1: An l-ruler confined in a regular hexagon

A chain Γ is said to be *confined inside a polygon* P if its links must always lie inside the closed, bounded polygonal region determined by P. Two configurations of a chain Γ confined inside a closed polygonal region P are said to be *equivalent* if one can continuously move to the other while the links remain within P and while the lengths of the links maintain their initial values at all times.

Suppose an l-ruler L has just one equivalence class of configurations under the above notion of equivalence. Then in particular, every arbitrary configuration of L is equivalent to one in which all the links coincide. In this case, we say that L is *always-foldable*. Otherwise, when L has more than one equivalence class of configurations, we say that L is *not-always-foldable*.

This paper studies always-foldability for l-rulers confined inside a regular polygon P.

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Whether or not such an l-ruler is always-foldable depends in part on the relationship between l and w, the *width* of P. Here, recall that the width of any polygon, regular or not, is the minimum possible distance between two parallel lines of support for the polygon.

1.1 Summary of Main Results

Our main results are as follows. Let L be an l-ruler confined inside a regular $2k$-gon P. Then L is always-foldable for $l \leq w$; L is not-always-foldable for $w < l \leq m$, where m is the distance between a vertex of P and the midpoint of either of the two sides of P farthest from the vertex; finally, L is again always-foldable for $m < l \leq d$, where d is the diameter of P. The paper [5] proved that equilateral triangles exhibit more than one alternation of the always-foldability property and asked whether any other polygons exhibit this phenomenon. Hence, we answer this question in the affirmative, as we show regular $2k$-gons exhibit multiple alternation. Note that one would in general expect at least one transition, from always-foldable, for sufficiently small values of l, to not-always-foldable for larger values of l.

For regular $(2k+1)$-gons, we show that L is always-foldable for $l \leq b^C$, where b^C is the supremum of the radii of circles that, no matter where their centers are placed on P, cut P in exactly two places. For $w < l \leq d$, L is always-foldable. However, for $b^C < l \leq w$, we do not know for which, if any, values L is always-foldable. We leave this as an open problem.

1.2 Background

Reconfiguration properties of chains have been considered for example in [1], [2], [3], [4], [8], and [9]. In [7], the number of equivalence classes of unconfined closed chains (i.e., $A_n = A_0$) in arbitrary dimension is determined. Chains confined inside a circle and having an extremal joint anchored were studied in [1] and in [2]. Anchored and unanchored chains confined inside a square were studied in [3] and [4]. Unanchored chains whose links all have the same length (i.e., l-rulers) and that are confined inside an equilateral triangle were studied in [5]; anchored l-rulers were studied in [10]. Chains are themselves special cases of planar linkages, surveyed in [11].

1.3 Terminology

We say that a chain Γ is *bounded* by b, denoted by $\Gamma \prec b$, if no link has length greater than b.

A convex obtuse polygon is a convex polygon with all internal angles measuring $\pi/2$ or more. We denote by S a square with unit side length. We denote the distance between two points p, q by $|pq|$.

We regard a polygon P as a closed, polygonal curve bounding a two-dimensional, region of finite area. When we are referring to the closed curve and not to the region it bounds, we use the notation ∂P for emphasis.

For a chain Γ confined inside a polygon P, we say that Γ is in *Rim Normal Form* (denoted RNF), if all joints of Γ lie on ∂P.

In addition to the width w and diameter d of a polygon, we also make use of the lengths b^C (defined for arbitrary polygons) and m (defined for regular $2k$-gons); recall the definitions for m and b^C from section 1.1.

2 l-Rulers in a Square

In this section, we consider the case of a square S of unit side. For $2k$-gons with $k > 2$, covered in a later section, the proofs will be essentially the same.

2.1 Short Links

Here we prove that any l-ruler L with $l \leq 1$ is always-foldable. The key idea is to begin by moving L to Rim Normal Form (RNF). The following Fact is from [4].

Fact 2.1 *If $l \leq 1$, then L inside S can be moved to RNF*

Always-foldability for $l \leq 1$ is an immediate consequence, as described below.

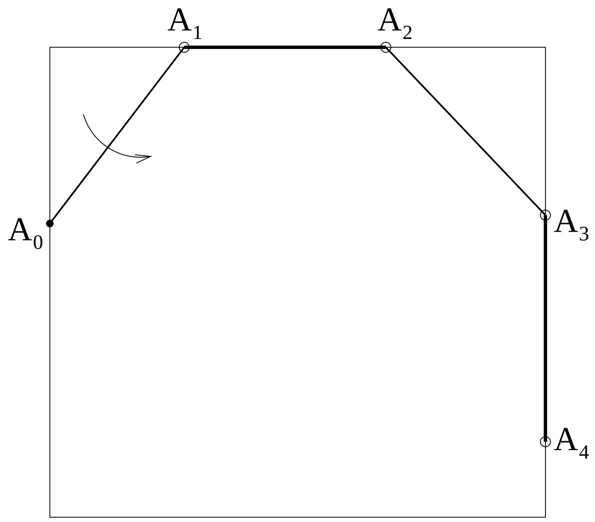

Figure 2: Always-foldable short-link rulers

Theorem 2.1 *If $l \leq 1$, then L inside S is always-foldable.*

Proof: Move L to RNF in accordance with Fact 2.1. Then fold L inductively as follows.

Fixing $A_1, A_2, \ldots, A_n$, rotate $[A_0, A_1]$ about A_1 until A_0 and A_2 coincide, as Figure 2 shows. This is possible since $b^C = w = 1$ and $l < 1$. Continue this process until Γ folds. $\square$

2.2 Midrange Links

This subsection shows that l-rulers inside S with $1 < l \leq \sqrt{5}/2$ are not-always-foldable. Note that $\sqrt{5}/2$ is the distance between a vertex of S and the midpoint of either of the two sides of S that are non-incident with the vertex (note the dashed line in Figure 3).

Theorem 2.2 *If $1 < l \leq \sqrt{5}/2$, then L inside S is not-always-foldable.*

Proof: Let s_1, s_2, s_3, s_4 be the sides of S, let v_1, v_2, v_3, v_4 be the vertices of S, and let u_1, u_2, u_3, u_4 be the midpoints of s_1, s_2, s_3, s_4, respectively. Suppose that α, β are angles between $[A_0, A_1]$, $[A_1, A_2]$ and s_1, respectively, as shown in Figure 3.

Initially we put L in the following configuration: A_0 lies at v_1, A_1 lies on s_3, A_2 lies at a point on s_1 but different from v_1. See Figure 3.

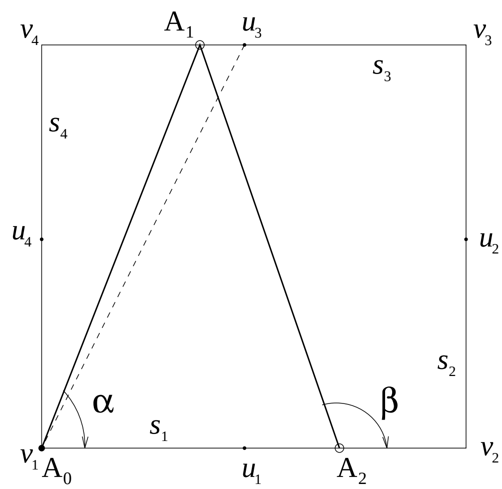

Figure 3: Not-always-foldable rulers

We show that L cannot be folded from this configuration.

Since $1 < l \leq \sqrt{5}/2$, initially A_1 lies between v_4 and u_3. Therefore $\beta > \pi/2$ initially. Clearly $\alpha < \pi/2$ initially. If L is foldable, let γ be the angle between s_1 the line through some segment onto which L can be folded. Without loss of generality, assume $r \leq \pi/2$. Since initially $\beta > \pi/2$, there exists some intermediate configuration in which $\beta = \pi/2$, i.e., $[A_1, A_2]$ is perpendicular to s_1. This contradicts $l > 1$. $\square$

2.3 Long Links

This subsection shows that any l-ruler inside S with $l > \sqrt{5}/2$ is always-foldable. We obtain this by proving that such rulers initially have to lie in "nearly-folded" configurations. In the extreme case of $l = \sqrt{2}$, the diameter of S, L has to exhibit an already folded configuration.

Before proceeding to the foldability result, we give some preliminaries. Again, let s_1, s_2, s_3, s_4 be the sides of S, let v_1, v_2, v_3, v_4 be the vertices of S, and let u_1, u_2, u_3, u_4 be the midpoints of s_1, s_2, s_3, s_4, respectively. Define C_1 as the area delimited by $v_1u_3, u_3v_3, v_3u_2, u_2v_1$, as shown in Figure 4. C_2, C_3 and C_4 are similarly defined. Then we have the following.

Lemma 2.1 *If $l > \sqrt{5}/2$, then L inside S falls completely inside exactly one of C_1, C_2, C_3, C_4.*

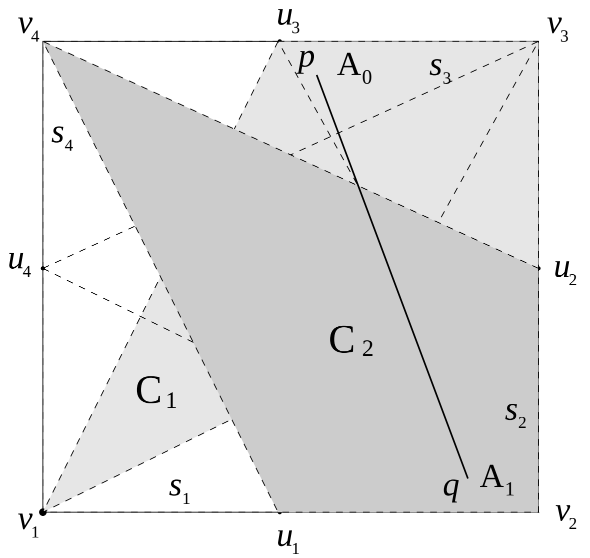

Figure 4: L lies inside one of C_1, C_2, C_3, C_4.

Proof: Since $C_1 \bigcup C_2 \bigcup C_3 \bigcup C_4 = S$, L lies inside $C_1 \bigcup C_2 \bigcup C_3 \bigcup C_4$. Next we show that L lies inside exactly one of C_1, C_2, C_3, C_4.

Note that for any i, j with $1 \leq i, j \leq 4, i \neq j$, $\forall p \in C_i, \forall q \in C_j$, $d(p,q)$ achieves its maximum at some vertex and the midpoint of some side. Therefore,

$$\max_{p \in C_i, q \in C_j} d(p,q) \leq \sqrt{5}/2.$$

Thus,

$$\max_{p \in C_i, q \in C_j - C_i} d(p,q) \leq \sqrt{5}/2. \qquad (*)$$

If there is a configuration in which L does not lie in any single one of C_1, C_2, C_3, C_4, then by convexity there exists i, j with $1 \leq i, j \leq 4, i \neq j$ and a link of L, say L_1, such that A_0 lies at some $p \in C_i$ and A_1 lies at some $q \in C_j - C_i$. See Figure 4. By $(*)$, $d(p,q) \leq \sqrt{5}/2$. This contradicts $l > \sqrt{5}/2$. □

Now we are ready to give the following foldability result.

Theorem 2.3 *If $l > \sqrt{5}/2$, then L inside S is always-foldable.*

Proof: By Lemma 2.1, L lies inside exactly one of C_1, C_2, C_3, C_4. Without loss of generality, assume that L lies inside C_1. We fold L inductively as follows. Fixing $A_1, A_2, \ldots, A_n$, rotate $[A_0, A_1]$ about A_1 until A_0 and A_2 coincide, as Figure 5 shows. We claim that A_0 will not hit the boundary ∂C_1 of C_1 during this reconfiguration.

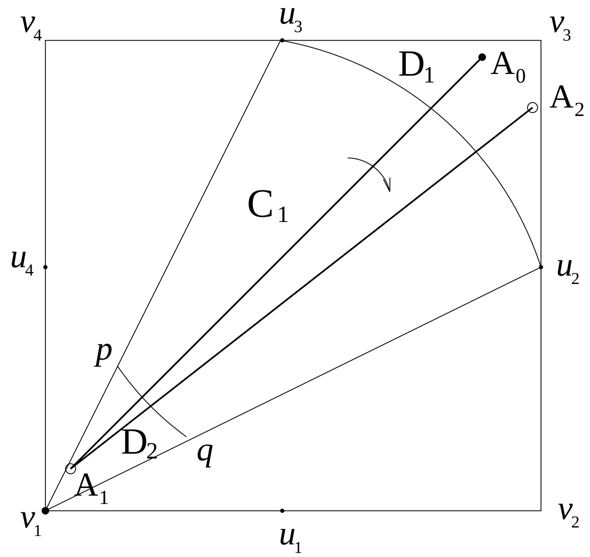

Figure 5: Folding long-link rulers

This can be seen as follows. Suppose that $\{p, q\} = C(v_3, \sqrt{5}/2) \bigcap \partial C_1$ and note that $\{u_3, u_2\} = C(v_1, \sqrt{5}/2) \bigcap \partial C_1$. We define cone D_1 as the area delimited by $v_3u_3, \widehat{u_3u_2}, u_2v_3$ and cone D_2 as the area delimited by $v_1p, \widehat{pq}, qv_2$, as shown in Figure 5. Since $l > \sqrt{5}/2$, each joint of L has to lie inside D_1 or D_2.

If A_0 lies in D_1, then A_1 lies in D_2 and A_2 lies in D_1. If A_0 lies in D_2, then A_1 lies in D_1 and A_2 lies in D_2. In both cases, A_0 will not hit ∂C_1 before A_0 and A_2 coincide. Hence the claim.

Continue this process until Γ folds. □

3 l-Rulers in a Regular $2k$-gon

This section generalizes the foldability result of an l-ruler within squares to arbitrary regular $2k$-gons.

3.1 Foldability within Regular $2k$-gons

The foldability result of an l-ruler within a square can be readily extended to any regular $2k$-gon, based on the following results from [9].

Fact 3.1 *Let Γ be an n-link chain confined within a convex obtuse polygon P. If $\Gamma \prec b^C$, then Γ can be moved to RNF.*

Fact 3.2 *Let P be a regular $2k$-gon. Then $b^C = w$.*

We thus have the following, which immediately implies the foldability of rulers with short links inside a regular $2k$-gon.

Fact 3.3 *Let P be a regular $2k$-gon. If $l \leq w$, then L can be moved to RNF.*

Theorem 3.1 *Let P be a regular $2k$-gon. If $l \leq w$, then L is foldable.*

The foldability of rulers with midrange and long links within a regular $2k$-gon is completely similar to that within a square. In the following, let P be a regular $2k$-gon, let v be a vertex of P, let u be the midpoint of an opposite side of v, and let $m = |uv|$.

Theorem 3.2 *Let P be a regular $2k$-gon. If $w < l \leq m$, then L is not-always-foldable. Refer to (a) of Figure 6.*

Theorem 3.3 *Let P be a regular $2k$-gon. If $l > m$, then L is always-foldable. Refer to (b) of Figure 6.*

4 Regular $(2k+1)$-gons

The reason that the above approach does not apply to regular $(2k+1)$-gons is due to gap between b^C and $w > b^C$ in regular $(2k+1)$-gons. For $b^C < l \leq w$, we do not know the foldability of L.

We observe that an l-ruler remains not-always-foldable when $l > w$, for reasons similar to the non-foldability of rulers with midrange links within a square. We use Figure 7 to suggest the ideas. This phenomenon shows that $2k$-gons and $(2k+1)$-gons as confining regions exhibit different foldability properties.

Below we give the known foldability results of rulers within regular $(2k+1)$-gons and pose the foldability of an l-ruler with $b^C < l \leq w$ inside regular $(2k+1)$-gons as an open problem.

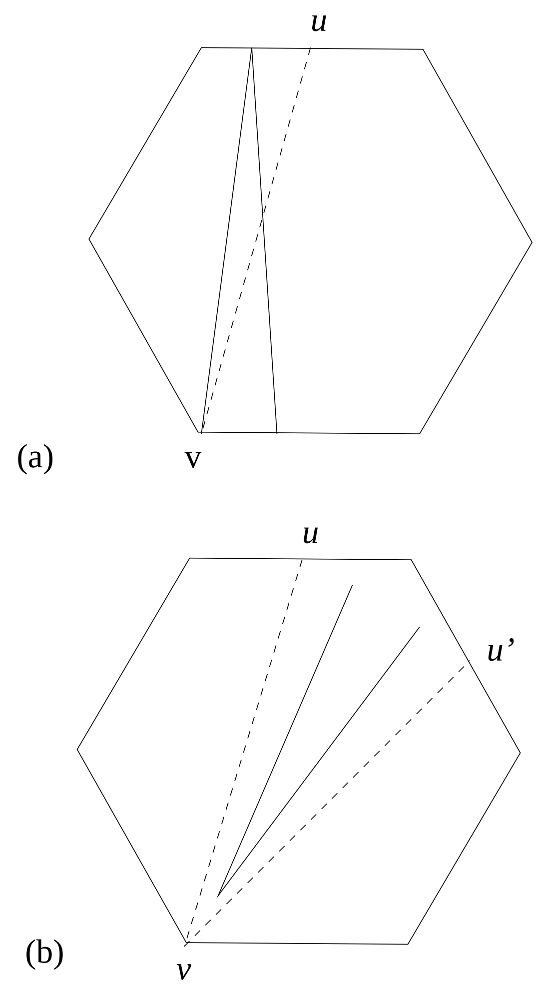

Figure 6: Foldability within regular $2k$-gons

Theorem 4.1 *Let P be a regular $(2k+1)$-gon. If $l \leq b^C$, then L is always-foldable.*

Theorem 4.2 *Let P be a regular $(2k+1)$-gon. If $l > w$, then L is not-always-foldable. Refer to Figure 7.*

5 Conclusion

As the segment length l of an l-ruler confined in a polygon P increases from 0 to its maximum value, one expects that for all sufficiently small l, the ruler is always-foldable and that for some critical value of l, the ruler become not-always-foldable. This paper has shown that in a regular $2k$-gon P, the always-foldability property alternates *three* times, from holding, to not

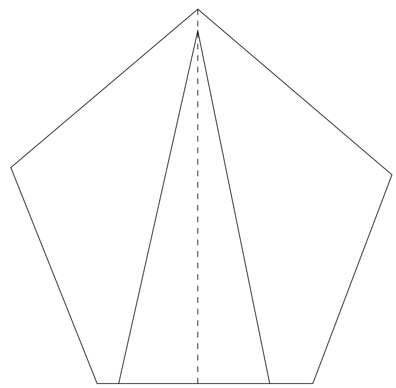

Figure 7: Foldability within regular $(2k + 1)$-gons

holding, and finally back to holding. This answers in the affirmative the question of [5] as to whether interesting alternation phenomena occur for polygons other than equilateral triangles, where alternation occurs *four* times.

For regular $2k$-gons, we exhibited the critical values of l for which the transitions between always-foldability and not-always-foldability, or the reverse, occur.

For regular $(2k + 1)$-gons, $k > 1$, we showed that all such transitions occur between b^C and w. We leave as an open problem the number of transitions that occur in this range, and their corresponding critical values.

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