Assignment #5 Solutions
NP-completeness

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Question 1

(a) Prove 3-COL $\in$ NP

proof:

1. Clearly, the 3-COL problem is a decision problem, i.e. the output is a "yes" or "no" answer if the 3-COL is a valid coloring or not of the graph G.

2. Given a 3-coloring of the graph G as a certificate, we can verify the "yes" answer in polynomial time. The procedure is very simple. For each vertex $i \in V$, verify that the endpoint of edges e_{ij}, for all possible j, is colored different than the vertex i. This algorithm is in worst case $O(|V|^2)$, which is polynomial.

(b) Transform 3-COL into ILP in polynomial time

Variables.

For every vertex $i \in V$, we introduce six variables.

$$c_{iu} = \left\{ \begin{array}{ll} 1 & \text{if vertex i is colored with color u} \in \{1,2,3\} \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\bar{c}_{iu} = \left\{ \begin{array}{ll} 1 & \text{if vertex i is NOT colored with color u} \in \{1,2,3\} \\ 0 & \text{otherwise} \end{array} \right\}$$

Thus, we have $6|V|$ variables, which is polynomial in length of input.

Constraints.

We have three set of constraints:

1. Make sure complements agree. For all vertices $i \in V$ and for all colors $u \in \{1,2,3\}$, if $c_{iu} = 1$ than $\bar{c}_{iu} = 0$. We need the constraint,

$$c_{iu} + \bar{c}_{iu} = 1$$

   We have $3|V|$ of this type of constraints.

2. Make sure every vertex is colored by only one color. For all vertices $i \in V$, we have

$$\sum_{u=1}^{3} c_{iu} = 1$$

   We have $|V|$ of such constraints.

3. If $e_{ij} \in E$ than we must have different colors at vertex i and vertex j. Hence, for all edges $e_{ij} \in E$ and for all colors $u \in \{1, 2, 3\}$, we have

$$c_{iu} + \sum_{v \in \{1,2,3\} \setminus \{u\}} \bar{c}_{jv} = 2$$

We have $3|E|$ of such constraints. To model the same thing, we could also say that for each edge e_{ij} and each color u, we have three constraints: either i is not u or j is not u (or both), which gives constraints like

$$\bar{c}_{iu} + \bar{c}_{ju} \geq 1$$

In this case, we have $3|E|$ such constraints.

In all, this transformation is $O(|V|^2)$ in the length of the input of the 3-COL problem.

Proof of correctness

We need two parts to make this proof complete. Part 1 shows that if a "yes" instance of ILP is generated by the transformation, then the 3COL instance was a "yes" instance. Therefore, a no instance of 3-COL must map to a no instance of ILP (otherwise, we would have a contradiction). Part 2 should show that a "yes" instance of 3-COL maps to a "yes" instance of ILP. Note that it is perfectly fine to consider the contrapositive statement (sometimes easier to prove), i.e that a "no" instance of ILP maps to a "no" answer of 3-COL.

1. Suppose that G has a valid coloring $\chi : V \to \{1, 2, 3\}$. We will show that the corresponding ILP problem has a satisfying assignment. If we set our variables so that $c_{iu} = 1$ if $\chi(i) = u$ and $c_{iu} = 0$ otherwise; and also set $\bar{c}_{iu} = not(c_{iu})$ for all i, then it is easy to check that all of the constraints are satisfied. Thus true instances of VCOL are mapped to true instances of ILP.

2. On the other hand, suppose that G mapped to a positive solution to the ILP problem. Construct χ so that $\chi(i) = u$ if $c_{i,u} = 1$ (note that that $c_{i,u} = 1$ for at most one choice of u). The constraints from part 3 ensure that no edge has the same color on both vertices, thus χ is a valid coloring. Thus any VCOL problem that gets mapped to a positive ILP must have a valid vertex coloring. In other words, any VCOL problem that does *not* have a valid vertex coloring does *not* get mapped to a positive ILP.

This proves the correctness of the transformation. Note that only part 1 or only part 2 is not sufficient for a valid proof. You need to go both ways.

(c) Transform 3-COL into SAT in polynomial time

Transformation

- for each vertex i, we create three variables (similar as before)

$$\bar{c}_{iu} = \left\{ \begin{array}{ll} 1 & \text{if vertex i is NOT colored with color u} \in \{1,2,3\} \\ 0 & \text{otherwise} \end{array} \right\}$$

  There is $3|V|$ of these variables.

- Now, for each edge $e_{ij} \in E$, we create three clause (one for each color $u \in \{1, 2, 3\}$) with two literals in them. It is of the form,

$$(\bar{c}_{iu} \quad \text{OR} \quad \bar{c}_{ju})$$

  We take the AND operation of clauses of that form for each edge and each color. Hence, we have $3|E|$ clauses and $3|V|$ variables. Thus, the transformation is polynomial with respect to the input length.

Proof of correctness

1. Suppose a "yes" instance is generated by the transformation, i.e the AND of all clauses returns a "true" or "yes" answer. This implies that every clause is true. Thus, for each edge e_{ij} and for each color u, either i is not u or j is not u or both. This means that vertex i and vertex j have different colors. Therefore, we have a valid 3-coloring and a 'yes" instance of 3-COL.

2. Now suppose a "no" instance is generated by the transformation, i.e. some clause is false. This implies that
$$(\bar{c}_{iu} \quad \text{OR} \quad \bar{c}_{ju}) = 0 = \text{FALSE}$$
for some i, j, u. The OR operation returns 0 only if $\bar{c}_{iu} = 0$ and $\bar{c}_{ju} = 0$. This means that the color at vertex i and the color at vertex j is the same. Therefore, we have an invalid 3-coloring and we have a "no" instance of 3-COL.

(d) Prove that 4-COL is NP-complete, assuming 3-COL is NP-complete

We want to prove that 4-COL is NP-complete. First, we need to show that it is in NP and then must reduce a known NP-complete problem to it. Since, 3-COL is NP-complete by assumption, if we can reduce 3-COL to 4-COL then we can conclude that 4-COL is as well.

4-COL $\in$ NP

The proof is the same as Question 1(a). The only difference is that we now have four colors.

Reduction of 3-COL to 4-COL

The trick to transform 3-COL to 4-COL is adding a new vertex of the fourth color that gets joined to all the other vertices.

Proof of correctness

The new graph is 4 colourable if and only if the original graph is 3-colourable.

1. Suppose the original graph is 3-colourable. Then, by construction, the new graph is trivially 4-colourable as we have only added a single vertex of the fourth color connected to all other vertices. Since no other vertex than the one added is colored with the 4th color, we have a valid 4-coloring.

2. Suppose the original graph is NOT 3-colourable. The new constructed is clearly NOT 4-colourable as the original graph is a subgraph of the new and the transformation does not modify any property of the original graph.

   Therefore, because 4-COL is in NP and that we have found a reduction from a NP-complete problem to it, we conclude that 4-COL is NP-complete.

Question 2. Prove that Independent Set (IS) is NP-complete, assuming Vertex Cover (VC) is NP-complete

Independent Set $\in$ NP

1. IS is a decision problem. *Is there a set of k vertices which have no edges between them?* always has a "yes" or "no" answer.

2. Given an independent set of the graph as a certificate, we can verify the "yes" in polynomial time. The procedure is simple, for all possible pair of vertices (i, j) in the independent set, verify that they have no edge between them in the graph. If the independent set has n vertices, the running time of this procedure is

$$\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2} = O(n^2)$$

Thus, in worst case, the verification of the certificate is $O(|V|^2)$, which is polynomial.

Reduction of Vertex Cover to IS

Given an instance of Vertex Cover, a graph G and an integer k, we want to transform it into an instance of IS by keeping the same graph but with integer $|V| - k$.

Proof of correctness

I claim that there is a VC in G of size k if and only if there is a IS in G of size $|V| - k$.

1. Suppose there is a vertex cover V' of size k in G. I claim that $U = V - V'$ is an independent set. Suppose not for a contraction. This means there exists vertices i and $j \in U$ which have an edge $e_{ij} \in E$ between them. But vertices i and j are not in V', which implies that the edge e_{ij} is not covered by the vertex cover V'. CONTRADICTION. Hence, U is an independent set of size $|V| - k$.

2. Suppose there is an independent set U of size $t = |V| - k$ in G. I claim that $V' = V - U$ is a vertex cover in G. Suppose not for a contradiction. This means there exists an edge $e_{ij} \in E$ which is not covered by V'. This implies vertex i and j are neither in V'. Thus, they have to be in U since U contains all other vertices. But this is a CONTRADICTION because U is an independent set by assumption (there cannot be an edge between i and j). Thus, V' is a vertex cover of size k.

Since IS is in NP and we have found a NP-complete problem that reduces to it, we can conclude that IS is an NP-complete problem.