Graphical Models

Clique trees & Belief Propagation

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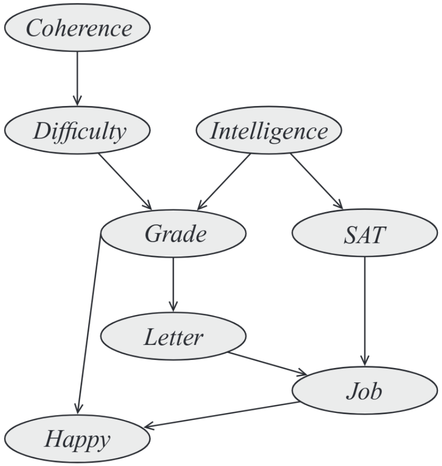
Learning objectives

- message passing on clique trees
- its relation to variable elimination
- two different forms of belief propagation

Recap: variable elimination (VE)

- marginalize over a subset - e.g., $P(J) =$
- expensive to calculate (why?) $\sum_{C,D,I,G,S,L,H} P(C, D, I, G, S, L, J, H)$
- use the factorized form of P

$$\sum_{C,D,I,G,S,L,H} P(D|C)P(G|D, I)P(S|I)P(L|G)P(J|L, S)P(H|G, J)$$



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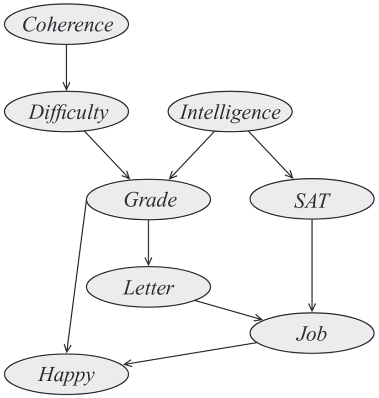
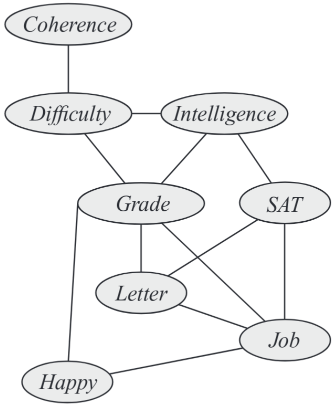
$$\downarrow$$

$$\phi_2(H,G,J)$$





note that they do not encode the same CIs

think of this as a factor/potential
same treatment of

- Bayes-nets
- Markov nets

for inference

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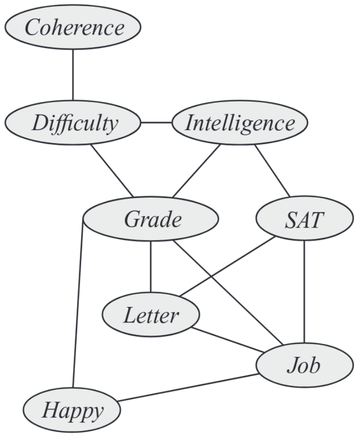
$$\sum_{C,D,I,G,S,L} \phi_1(D,C)\phi_2(G,D,I)\phi_3(S,I)\phi_4(L,G)\phi_5(J,L,S)\phi_6(H,G,J)$$



$$= \ldots \sum_I \phi_3(S|I) \sum_D \phi_2(G,D,I) \sum_C \phi_1(D,C)$$

$\psi'_1(D)$ $\psi_1(D,C)$

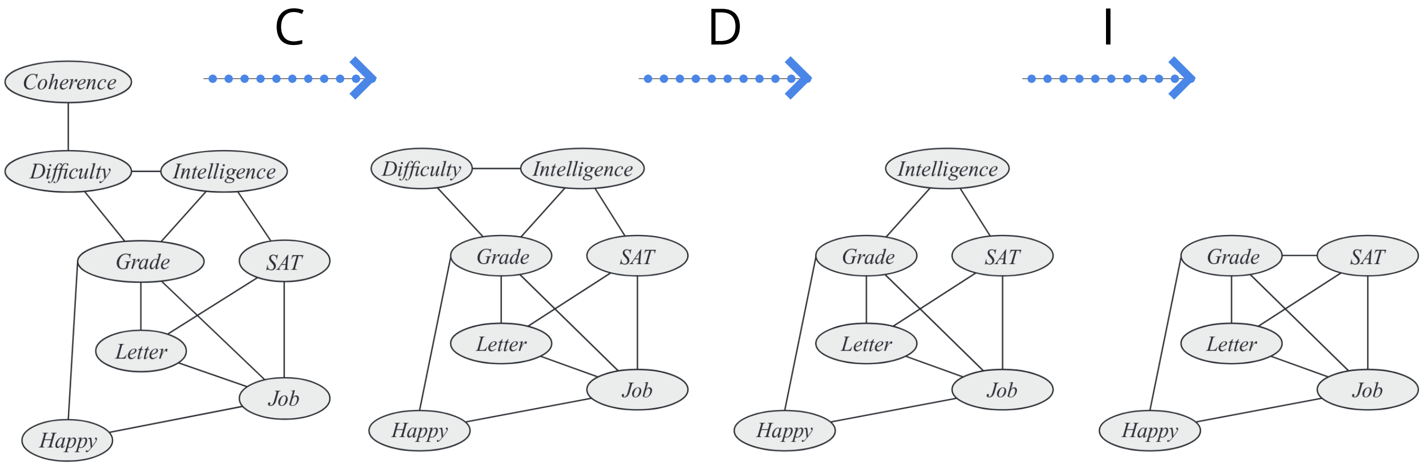
- repeat this

$$= \ldots \sum_I \phi_3(S,I) \sum_D \phi_2(G,D,I)\psi'_1(D)$$

$\psi'_2(G,I)$ $\psi_2(G,I,D)$

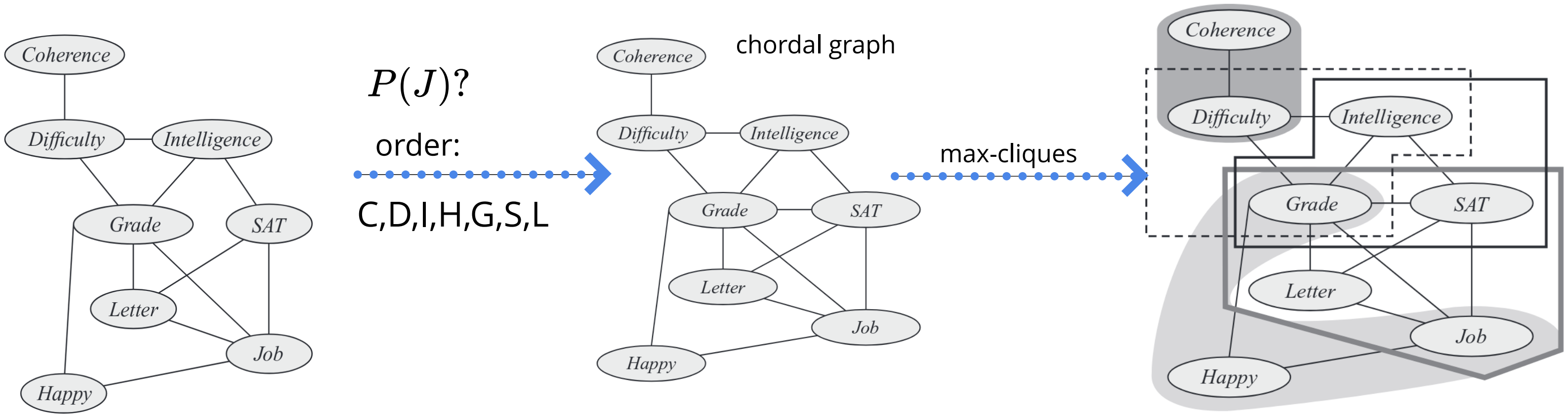
Recap: variable elimination (VE)

- marginalize over a subset - e.g., $P(J) = \sum_{C,D,I,G,S,L,H} P(C,D,I,G,S,L,J,H)$
- expensive to calculate (why?)
- eliminate variables in some order (order of factors in the summation)



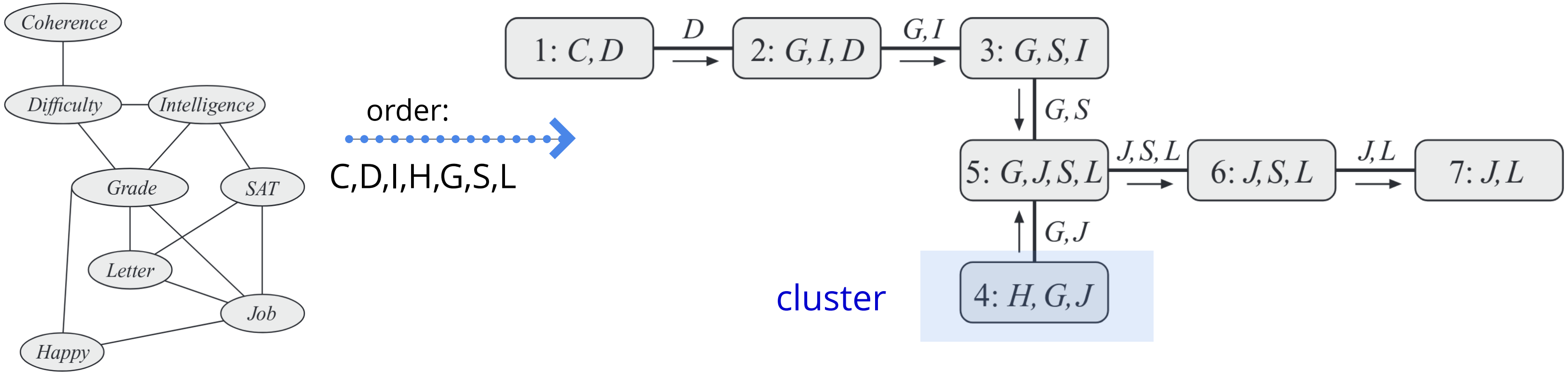
Recap: variable elimination (VE)

- eliminate variables in some order
- creates a chordal graph
- maximal cliques are factors created during VE (ψ_t)



Clique-tree

- summarize the *VE computation* using a clique-tree



- clusters are maximal cliques (scope of factors created during VE) $C_i = Scope[\psi_i]$

Clique-tree

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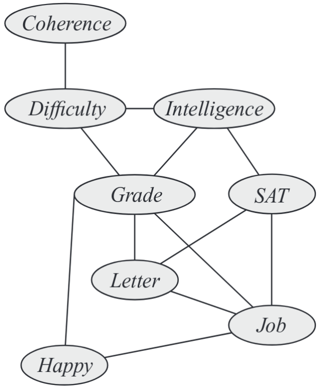
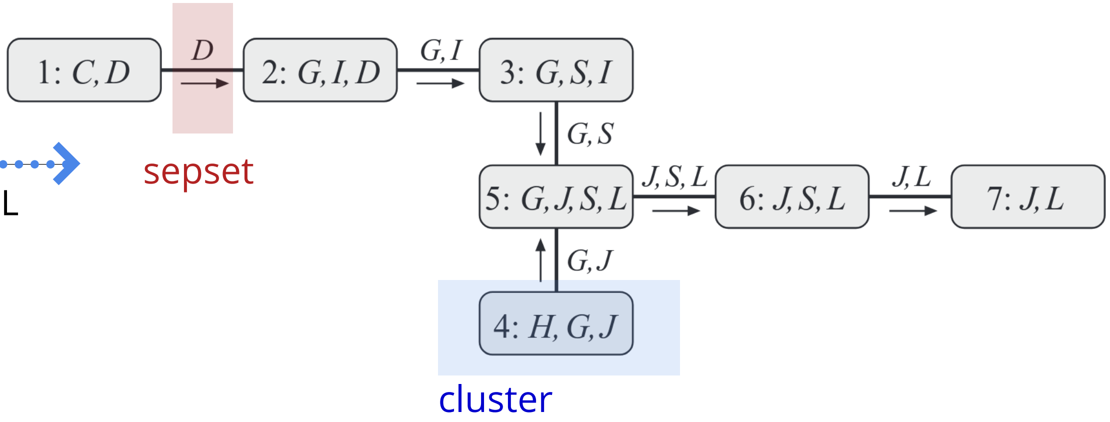


order:
C,D,I,H,G,S,L



- clusters are maximal cliques (factors that are marginalized)

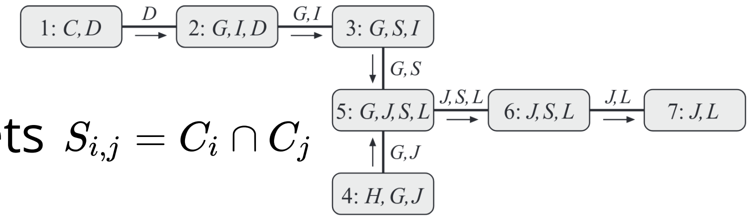
$$C_i = Scope[\psi_i]$$

- sepsets are the result of marginalization over cliques

$$S_{i,j} = Scope[\psi_i']$$

$$S_{i,j} = C_i \cap C_j$$

Clique-tree: properties



- a tree $\mathcal{T}$ from clusters C_i and sepsets $S_{i,j} = C_i \cap C_j$

- **family-preserving property:** $\alpha(\phi) = j$
  - each factor ϕ is associated with a cluster C_j s.t. $Scope[\phi] \subseteq C_j$

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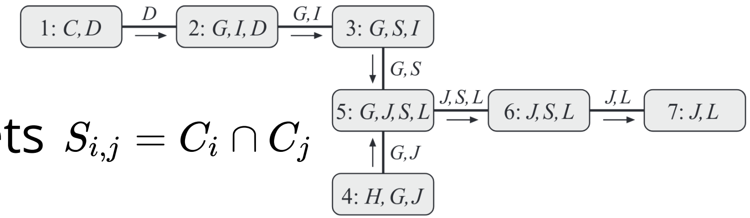

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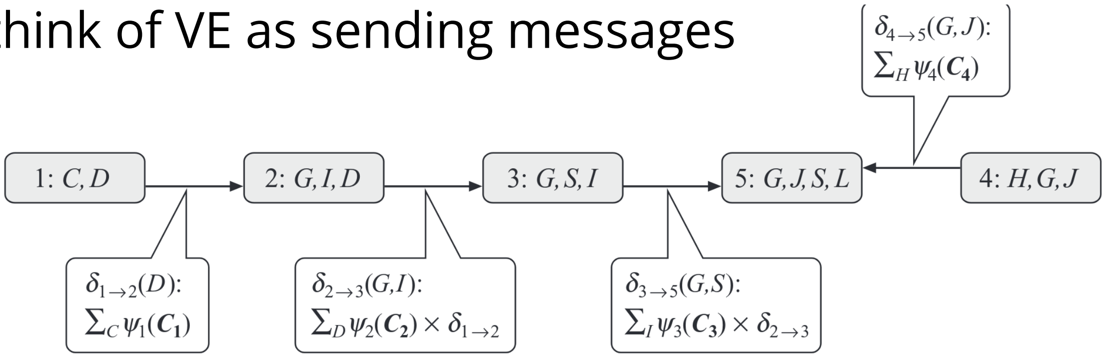
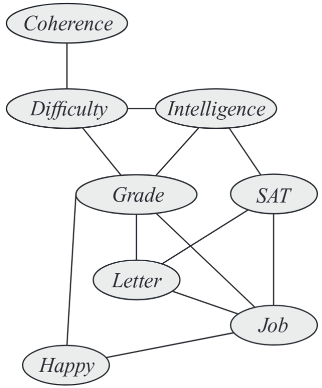
- **running intersection property:**
  - if $X \in C_i, C_j$ then $X \in C_k$ for C_k in the path $C_i \to \ldots \to C_j$

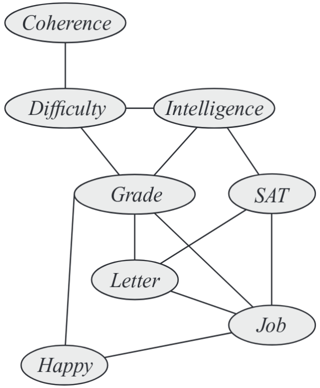
VE as **message passing**

think of VE as sending messages





VE as message passing



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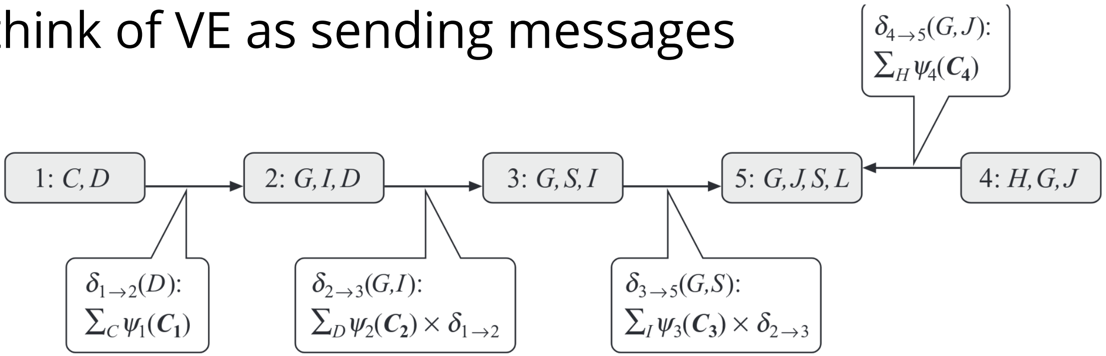


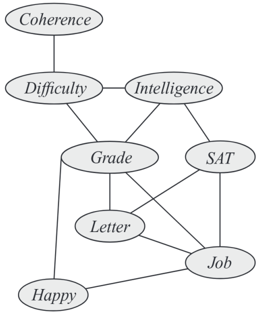
calculate the product of factors in each clique $\psi_i(C_i) \triangleq \prod_{\phi:\alpha(\phi)=i} \phi$

send messages from the leaves towards a root:

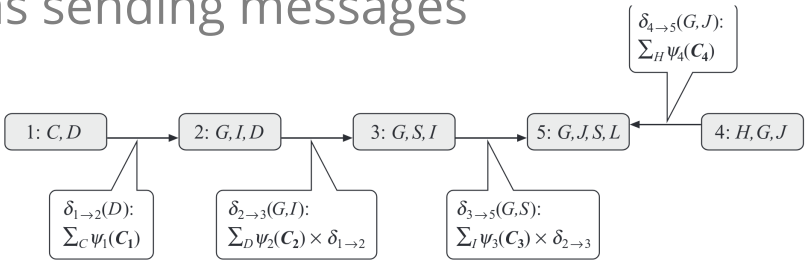
$$\delta_{i\to j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k\in Nb_i - j} \delta_{k\to i}(S_{i,k})$$

neighbours

message passing



- think of VE as sending messages



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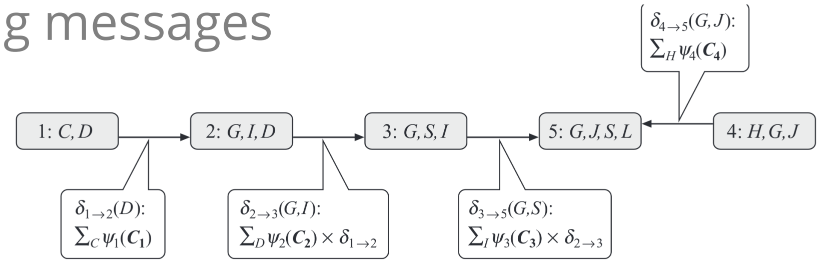
$$= \sum_{\mathcal{V}_{\prec(i\to j)}} \prod_{\phi\in\mathcal{F}_{\prec(i\to j)}} \phi$$

all variable on **i** side of the tree — all the factors on **i** side of the tree

- the message is the marginal from one side of the tree

message passing

- think of VE as sending messages



- send messages from the leaves towards a root:

$$\delta_{i \to j}(S_{i,j}) \triangleq \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k \in Nb_i - j} \delta_{k \to i}(S_{i,k})$$

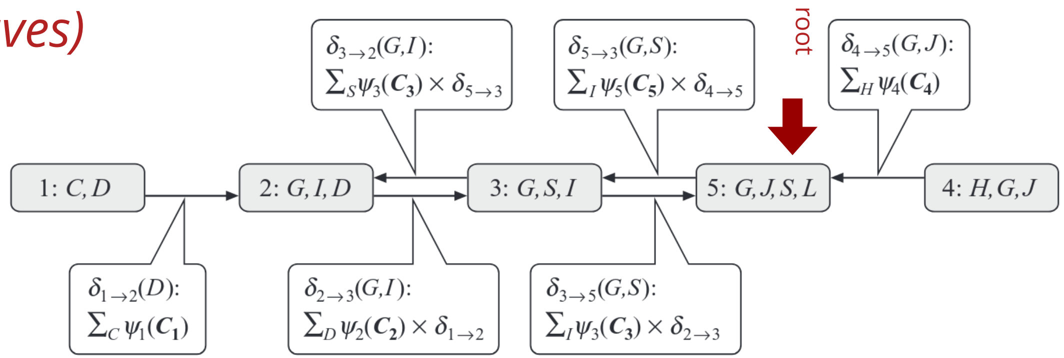
- the belief at the root clique is $\beta_r(C_r) \triangleq \psi_r(C_r) \prod_{k \in Nb_r} \delta_{k \to r}(S_{r,k})$

  proportional to the marginal $\beta_r(C_r) \propto \sum_{\mathbf{X}-C_i} P(\mathbf{X})$

message passing: root-to-leaves

- what if we continue sending messages?

  (from the root to leaves)



- clique **i** sends a message to clique **j** when received messages from all the **other** neighbors **k**

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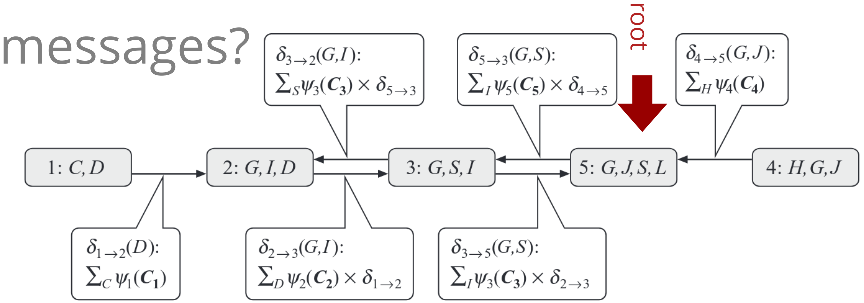

- sum-product **belief propagation** (BP)

$$\delta_{i\to j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \psi_i(C_i) \prod_{k \in Nb_i - j} \delta_{k\to i}(S_{i,k})$$

$$\mu_{i,j}(S_{i,j}) \triangleq \delta_{i\to j}(S_{i,j})\delta_{j\to i}(S_{i,j})$$

$$\beta_i(C_i) \triangleq \psi_i(C_i) \prod_{k \in Nb_i} \delta_{k\to i}(S_{i,k})$$

marginals

for any clique (not only root)

Summery so far...

- VE creates a chordal induced graph
- maximum cliques in this graph: clusters
- message passing view of VE:
  - send messages between clusters towards a root
- going beyond VE:
  - send messages back from the root
  - produce marginal over all clusters

Clique-tree: calibration

represent P using marginals: $\frac{\prod_i \beta_i}{\prod_{i,j \in \mathcal{E}} \mu_{i,j}} = \frac{\prod_i \psi_i \prod_{k \to i} \delta_{k \to i}}{\prod_{i,j \in \mathcal{E}} \delta_{i \to j} \delta_{j \to i}} = \prod_i \psi_i = \tilde{P}$

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an arbitrary assignment to all $\beta_i, \mu_{i,j}$ is calibrated iff

BP produces calibrated beliefs

$$\mu_{i,j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$$

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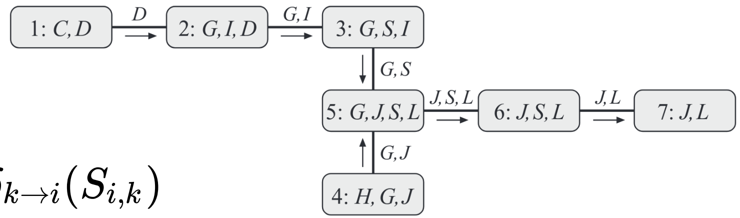
being calibrated and ★ means that all $\beta_i, \mu_{i,j}$ are marginals

$$\tilde{P}(\mathbf{X}) \propto \frac{\prod \beta_i(C_i)}{\prod_{i,j \in \mathcal{E}} \mu_{i,j}(S_{i,j})} \quad \Leftrightarrow \quad \beta_i(C_i) \propto P(C_i)$$

BP: an alternative update

approach 1. **message update**



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calculate the beliefs **in the end** $\beta_i(C_i) = \psi_i(C_i) \prod_{k\in Nb_i} \delta_{k\to i}(S_{i,k})$

Update the beliefs so that:

- they are calibrated
- they satisfy

BP: an alternative update

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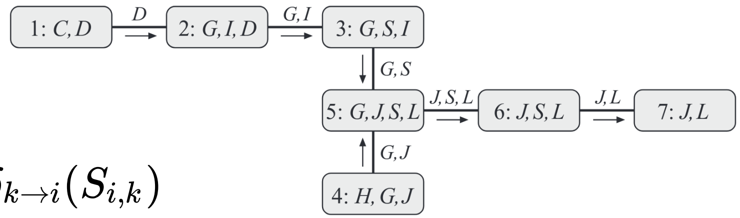

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calculate the beliefs **in the end** $\beta_i(C_i) = \psi_i(C_i) \prod_{k \in Nb_i} \delta_{k\to i}(S_{i,k})$

approach 2. **belief update idea**

Update the beliefs so that:

- they are calibrated $\mu_{i,j}(S_{i,j}) = \sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$
- they satisfy $\frac{\prod_i \beta_i}{\prod_{i,j \in \mathcal{E}} \mu_{i,j}} = \prod_i \psi_i$

BP: an alternative update

belief update

```
initialize
```
$\beta_i \leftarrow \psi_i = \prod_{\phi:\alpha(\phi)=i} \phi, \quad \mu_{i,j} \leftarrow 1$

```
until convergence:

        pick some
```
$(i,j) \in \mathcal{E}$

$\hat{\mu}_{i,j} \leftarrow \sum_{C_i - S_{i,j}} \beta_i$ // $\hat{\mu}_{i,j} = \delta^{new}_{i\to j}\delta_{j\to i}$

$\beta_j \leftarrow \beta_j \frac{\hat{\mu}_{i,j}}{\mu_{i,j}}$ // $\frac{\hat{\mu}_{i,j}}{\mu_{i,j}} = \frac{\delta^{new}_{i\to j}\delta_{j\to i}}{\delta^{old}_{i\to j}\delta_{j\to i}} = \frac{\delta^{new}_{i\to j}}{\delta^{old}_{i\to j}}$

$\mu_{i,j} \leftarrow \hat{\mu}_{i,j}$

at convergence, beliefs are calibrated $\quad \sum_{C_i - S_{i,j}} \beta_i(C_i) = \sum_{C_j - S_{i,j}} \beta_j(C_j)$

and so they are $\propto$ marginals

Clique-tree & queries

What **type of queries** can we answer?

- marginals over subset of cliques $P(A) \quad A \subseteq C_i$

Clique-tree & queries

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- updating the beliefs after new evidence $P(A \mid E^{(t)} = e^{(t)}) \quad A \subseteq C_i, E \subseteq C_j$
  - multiply the *(previously calibrated)* beliefs $\beta(C_i)\mathbb{I}(E^{(t)} = e^{(t)})$
  - propagate to recalibrate (belief update procedure)

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  - define a super-clique that has both A,B
  - a more efficient alternative?

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- partition function $Z = \sum_{C_i} \beta_i(C_i)$

Building a clique-tree

how to create it for a given graphical model:

Building a clique-tree

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1. triangulation: make a chordal graph
    - e.g. *induced graph* in VE
    - finding the chordal graph with min max-clique is NP-hard (heuristics we discussed)

Building a clique-tree

how to create it for a given graphical model:

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2. find maximal cliques (clusters in the clique-tree)
    - in general graphs NP-hard, but easy for chordal graphs
    - assign each factor to a clique

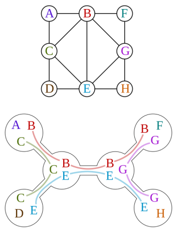

image: wikipedia

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3. use max. spanning-tree to build a tree (edge-cost $|C_i \cap C_j|$)

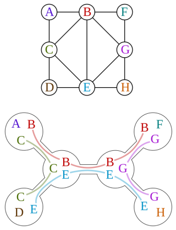
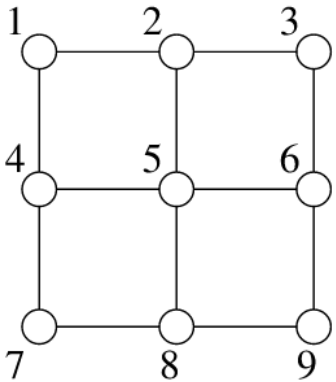


image: wikipedia

Building a clique-tree: example

input

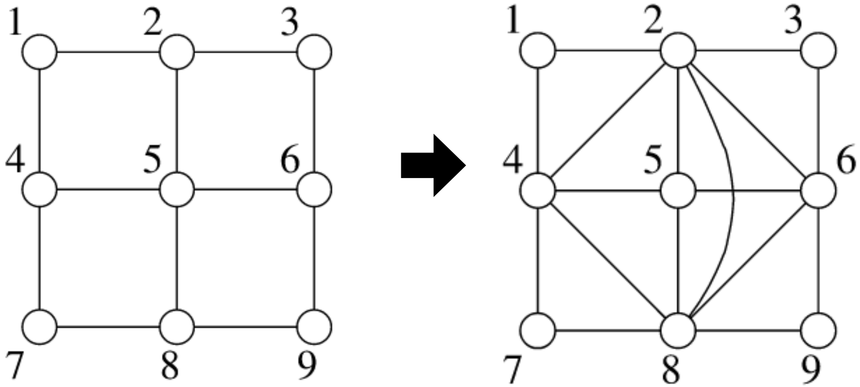


from: wainwright & jordan

Building a clique-tree: example

input

triangulated



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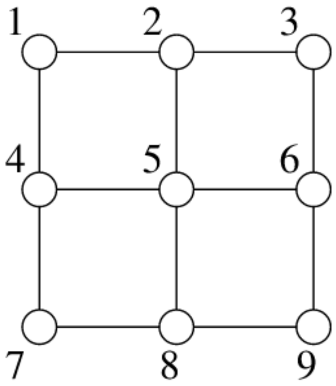
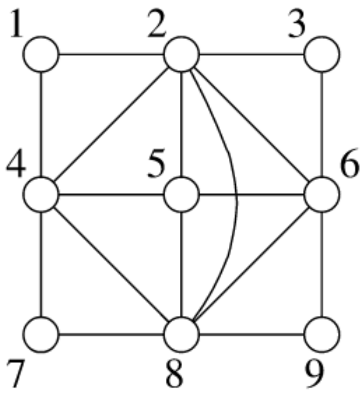
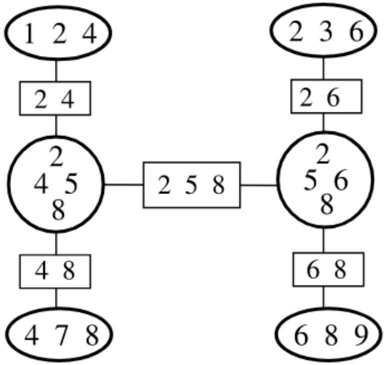


triangulated



clique-tree



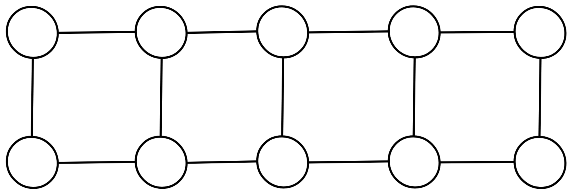
from: wainwright & jordan

clique-tree quiz

what clique-tree to use here?

what are the sepsets?

cost of exact inference?



Summary

- VE as message passing in a clique-tree
- clique-tree: running intersection & family preserving
- belief propagation updates:
  - message update
  - belief update
- types of queries
- how to build a clique-tree for exact inference

Chordal graph and clique-tree

Chordal graph = Markov $\bigcap$ Bayesian networks

convert MRF to Bayes-net (the actual procedure):

- triangulate
- build a clique-tree
- **within cliques:** fully connected directed edges
- **between cliques:** from root to leaves