Probabilistic Graphical Models

Conditional & Local Probability Models

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Fall 2019

Learning Objective

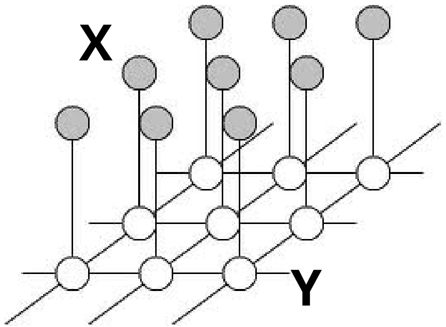
- conditional random fields
- local probability models:
  - deterministic CPDs
  - noisy-OR model
  - generalized linear model

Conditional Random Fields: Motivation

structured prediction: output labels are structured

X is always observed

Y is structured

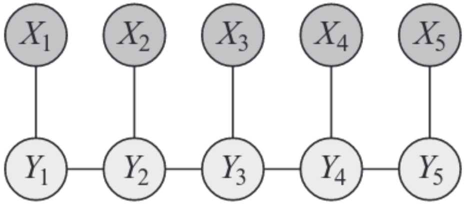


Examples:

image segmentation

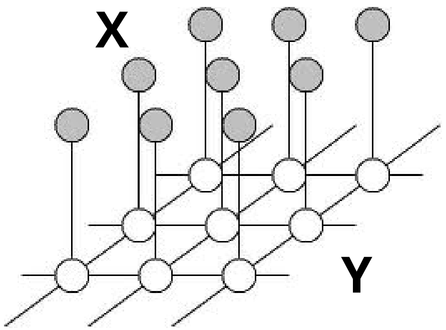
part of speech tagging

optical character recognition



Conditional Random Fields (CRF)

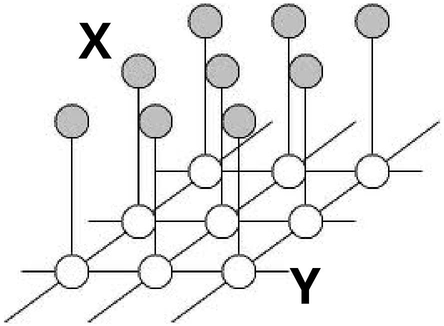
- a conditional graphical model P(**Y** | **X**)
- **first attempt:** $P(\mathbf{Y} \mid \mathbf{X}) = \frac{P(\mathbf{X},\mathbf{Y})}{P(\mathbf{X})}$
  - for prediction, no need to model P(**X**)
    - may not have enough data
    - **X** could be high-dim and P(**X**) may be complex



Conditional Random Fields (CRF)

second attempt:

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \tilde{P}(\mathbf{X}, \mathbf{Y}) = \frac{1}{Z(\mathbf{X})} \prod_k \phi_k(\mathbf{D}_k)$$



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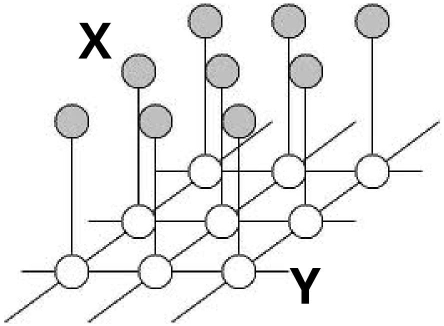
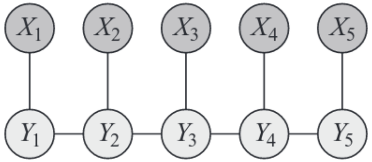


- differs from MRF in the partition function
  - *input-dependent* $Z(\mathbf{X}) = \sum_{\mathbf{Y}} \tilde{P}(\mathbf{Y},\mathbf{X})$

Conditional Random Fields: a running example

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$

$$Z(\mathbf{X}) = \sum_{\mathbf{Y}} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$



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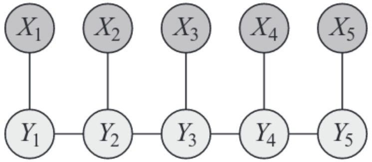
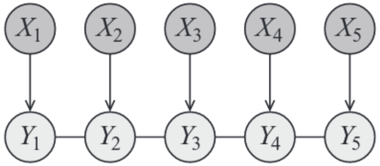
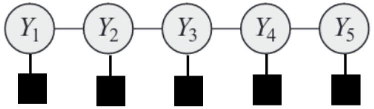


practically the same as $\longrightarrow$



- *e.g., in speech recognition (what do potentials encode?)*

for each **X=x**, we have a different MRF



Conditional Random Fields: another benefit

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$

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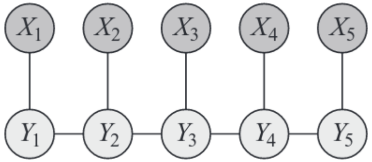
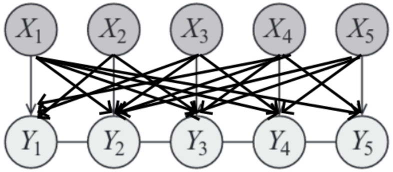
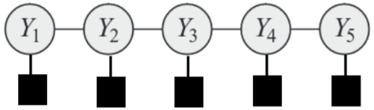


what if $\phi_i(\mathbf{X}, Y_i)$ instead of $\phi_i(X_i, Y_i)$?



sparse structure after conditioning on **X=x**



- *learning needs inference on this structure (discussed later)*
- *not true for the corresponding MRF*

Conditional Random Fields: input structure

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$

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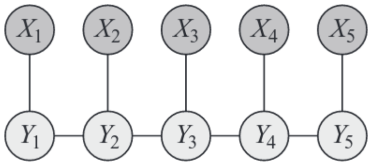
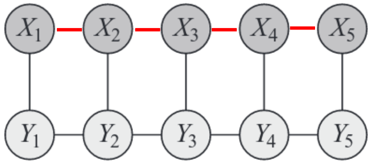


How about the **structure of the input** ?



$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1}) \gamma_i(X_i, X_{i+1}) =$$

$$\frac{1}{Z'(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$

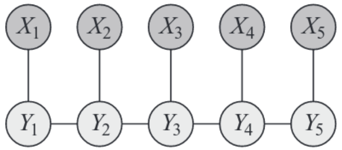
I.e., input structure can be ignored

(already accounted for in the observations)

Conditional Random Fields: parametrization

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$

$$Z(\mathbf{X}) = \sum_{\mathbf{Y}} \prod_{i=1}^{5} \phi_i(X_i, Y_i) \prod_{i=1}^{4} \psi_i(Y_i, Y_{i+1})$$



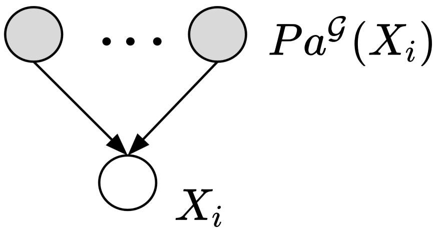
- In practice we need to learn the potentials
- parameterize them and learn the parameters (*e.g., a neural network*)
  - traditionally: a log-linear model: $\phi_i(X_i, Y_i; w_i) \triangleq \exp(\sum_k w_{i,k}\, f_{i,k}(X_i, Y_i))$
    - *E.g.*, for binary input/output:

$$\phi_i(X_i, Y_i; w_i) = \exp(w_i \mathbb{I}(X_i = 1, Y_i = 1)) = \exp(w_i X_i Y_i)$$

Local probabilistic models

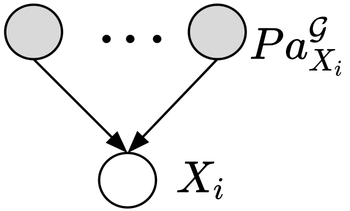
Local probabilistic models

- conditional probability distributions (**CPDs**)
  - in prediction $P(Y \mid X_1, \ldots, X_n)$
  - in Bayes-nets $P(X \mid Pa_X)$
    - discrete variables (**CPTs**)
      - exponential in $|Pa_{X_i}|$
- how to **represent** these efficiently? exploit some sort of structure

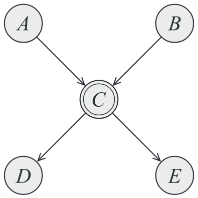


Deterministic CPDs

$$P(X \mid Pa_X) \triangleq \mathbb{I}(X_i = f(Pa_{X_i}))$$



determinism produces **additional independencies**:



without determinism: $(D \perp E \mid A, B) \notin \mathcal{I}(\mathcal{G})$

with determinsim: $(D \perp E \mid A, B) \in \mathcal{I}(\mathcal{G})$

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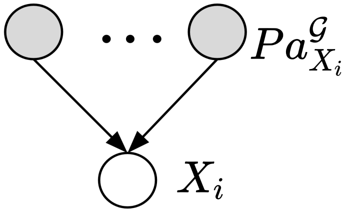

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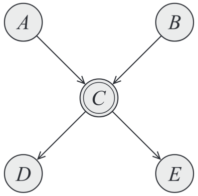


without determinism: $(D \perp E \mid A, B) \notin \mathcal{I}(\mathcal{G})$

with determinsim: $(D \perp E \mid A, B) \in \mathcal{I}(\mathcal{G})$

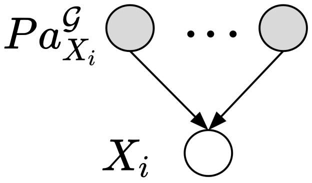
deterministic d-separation: $(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z})$?

- add all the variables that deterministically follow $\mathbf{Z}$ to define $\mathbf{Z}^+$
- run d-separation for $(\mathbf{X}, \mathbf{Y} \mid \mathbf{Z}^+)$

Noisy-OR model

- for **binary** variables only
- number of parameters is linear in $|Pa^{\mathcal{G}}_{X_i}|$
- each parent ($X_j = 1$) is an **independent cause**
- each cause is observed with prob $P(X'_j = 1) = \lambda_j X_j$

noise parameter



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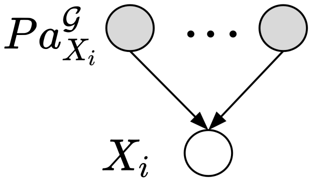
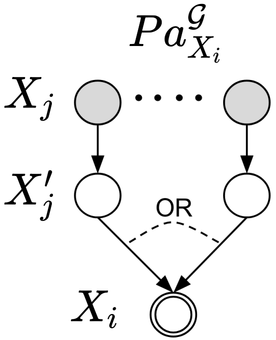




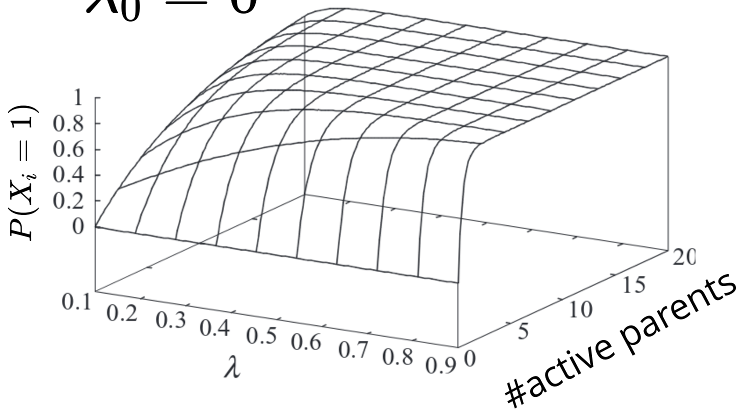
$$p(X_i = 0 \mid Pa_{X_i}) = (1 - \lambda_0) \prod_{X_j \in Pa_{X_i}} (1 - \lambda_j X_j)$$

leak parameter (role of a bias term)

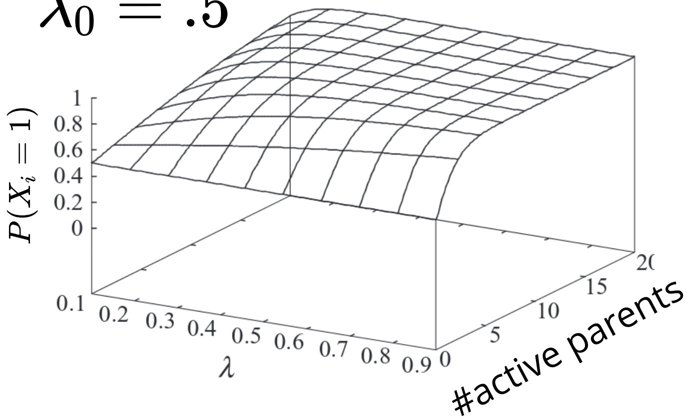
prob. of *no cause observed*

Noisy-OR model: visualization

$\lambda_0 = 0$



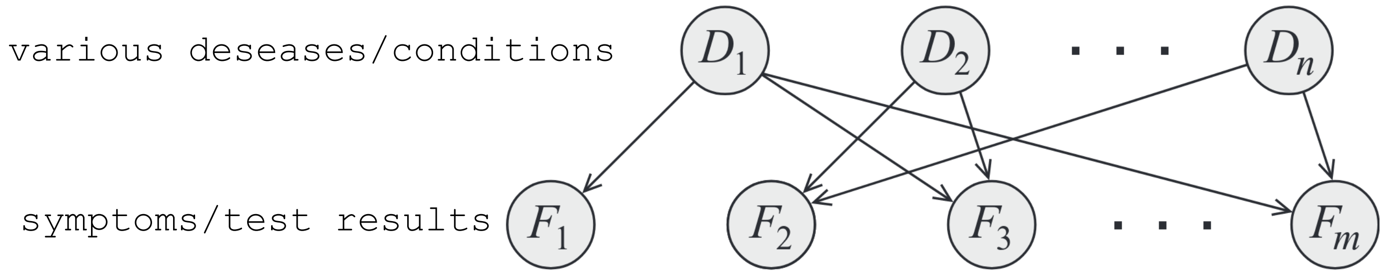
$\lambda_0 = .5$



$$p(X_i = 0 \mid Pa_{X_i}) = (1 - \underbrace{\lambda_0}_{\text{leak parameter (role of a bias term)}}) \underbrace{\prod_{X_j \in Pa_{X_i}} (1 - \lambda_j X_j)}_{\text{prob. of no cause observed}}$$

Noisy-OR model: example

Medical diagnosis (*BN2O network*)

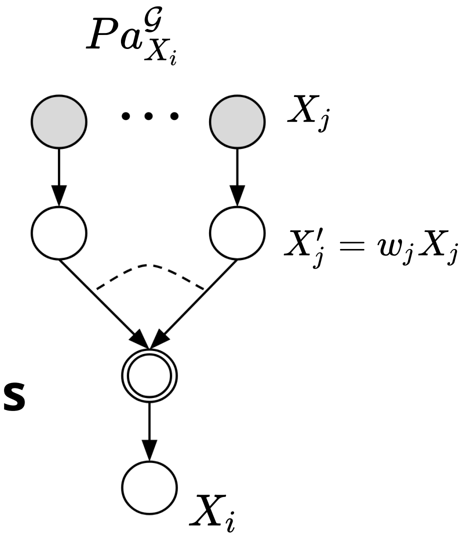


CPDs: $p(F_i = 0 \mid Pa_{F_i}) = (1 - \lambda_{i,0}) \prod_{D_j \in Pa_{F_i}} (1 - \lambda_{i,j} D_j)$

Logistic CPD

for **binary output** variables

$$P(X_i = 1) = \frac{\exp(\sum_j w_j X_j)}{1+\exp(\sum_j w_j X_j)}$$



logistic aggregation function

generally, the input can be **discrete** or **continuous**

- *E.g.,* $X_j = 2$ or $\underbrace{X_j, \ldots, X_{j+n} = 0, 1, \ldots, 0}_{\text{one-hot coding}}$

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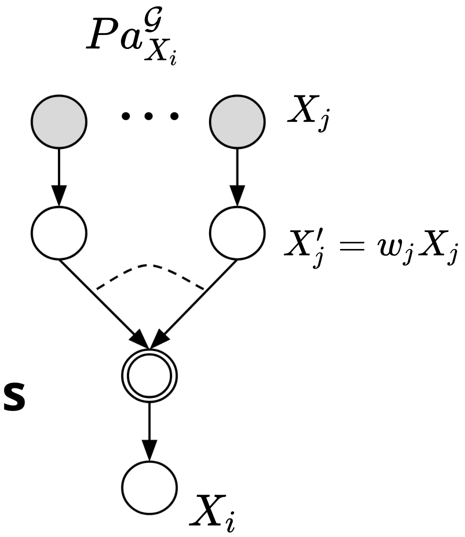

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binary input: *each cause has a multiplicative effect on the ratio* $\frac{P(X_i=1)}{P(X_i=0)}$

Softmax CPD

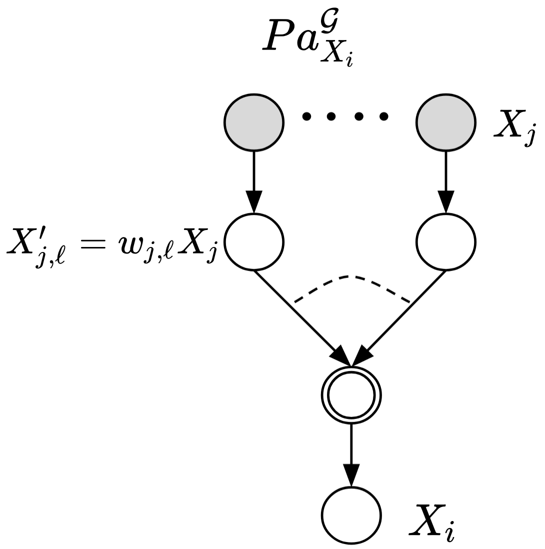
extension for categorical outputs

softmax function for aggregation:

$$f(z_\ell) = \frac{\exp(z_\ell)}{\sum_{\ell'} \exp(z_{\ell'})}$$

functional form of the CPD:

$$P(X_i = \ell) = \frac{\exp(\sum_j w_{j,\ell} X_j)}{\sum_{\ell'} \exp(\sum_j w_{j,\ell'} X_j)}$$



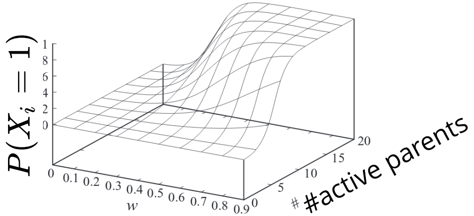
Independence of causal influence

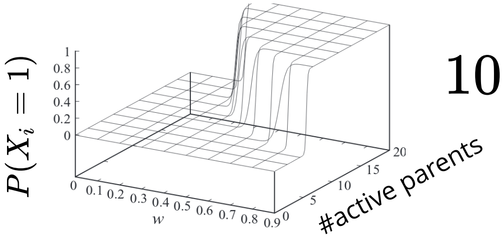
Commutative and associative aggregation

Logistic CPD

transformation $X'_j = w_j X_j$

aggregation logistic function



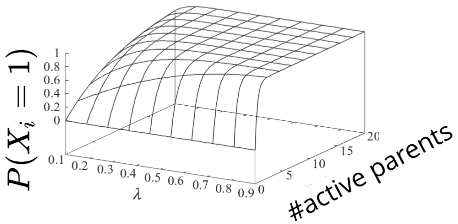


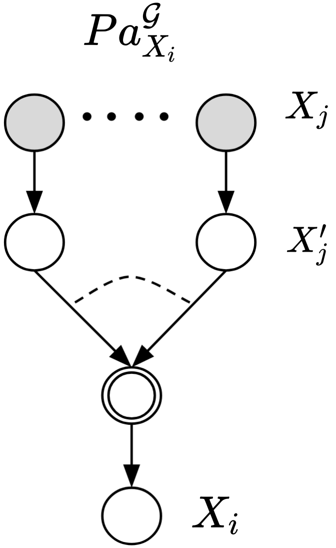
$10 \times w$

Noisy-OR

$P(X'_j = 1) = \lambda_j X_j \quad 0 \leq \lambda \leq 1$

OR /Max/...

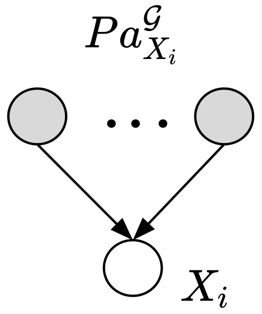




Linear Gaussian CPD

for **continuous** input/output variables

$$P(X_i) = \mathcal{N}(\sum_j w_j X_j; \sigma^2)$$



X_d

Linear Gaussian CPD

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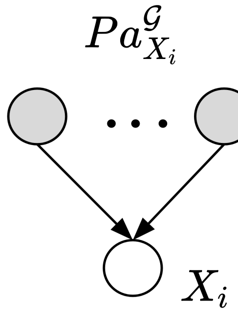
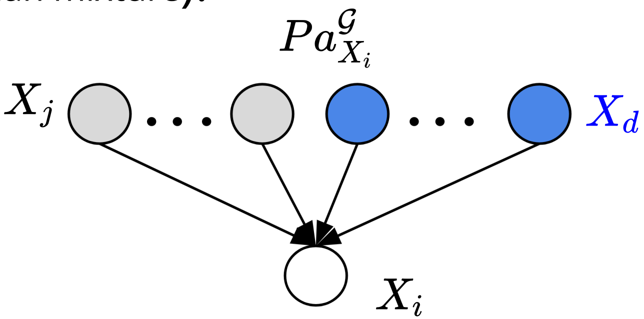


alternatively, a **discrete** input selects among **continuous** coefficients (produces a Gaussian mixture):

$$P(X_i) = \mathcal{N}(\sum_j w_{j,X_d} X_j; \sigma^2_{X_d})$$



conditional linear Gaussian CPD:

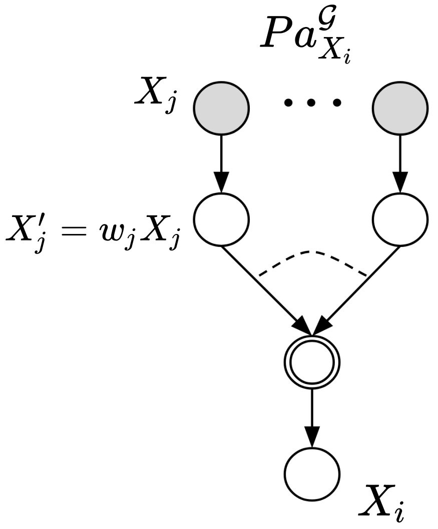
one Gaussian mixture for each discrete assignment

Generalized linear models

$$\mathbb{E}[X_i] = \underbrace{f}_{\text{mean function}}(\mathbf{w}^\top Pa_{X_i})$$

Logistic CPD: ***f*** *is the logistic function*

Gaussian CPD: ***f*** *is the identity function*



Generalized linear models

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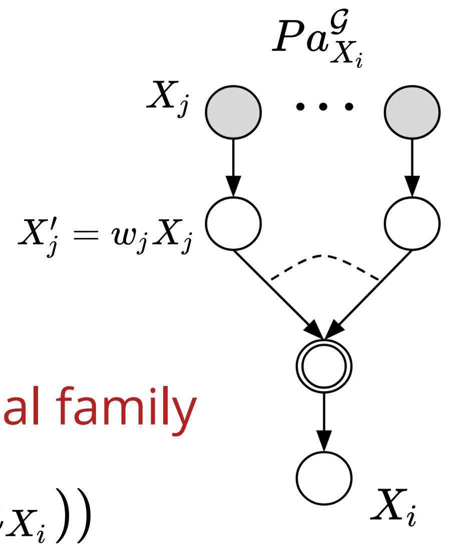


*Logistic CPD: **f** is the logistic function*

*Gaussian CPD: **f** is the identity function*

conditional dist. is a member of the exponential family

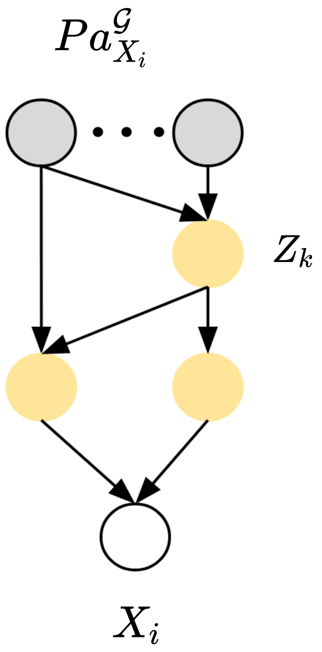
$$p(x_i \mid Pa_{X_i}) = \underbrace{h}_{\text{base measure}}(x_i)\exp(\mathbf{w}^\top Pa_{X_i} - \underbrace{F}_{\text{integral of } f}(\mathbf{w}^\top Pa_{X_i}))$$

(will come back to this in exp. family lecture)

Conditional Bayesian networks

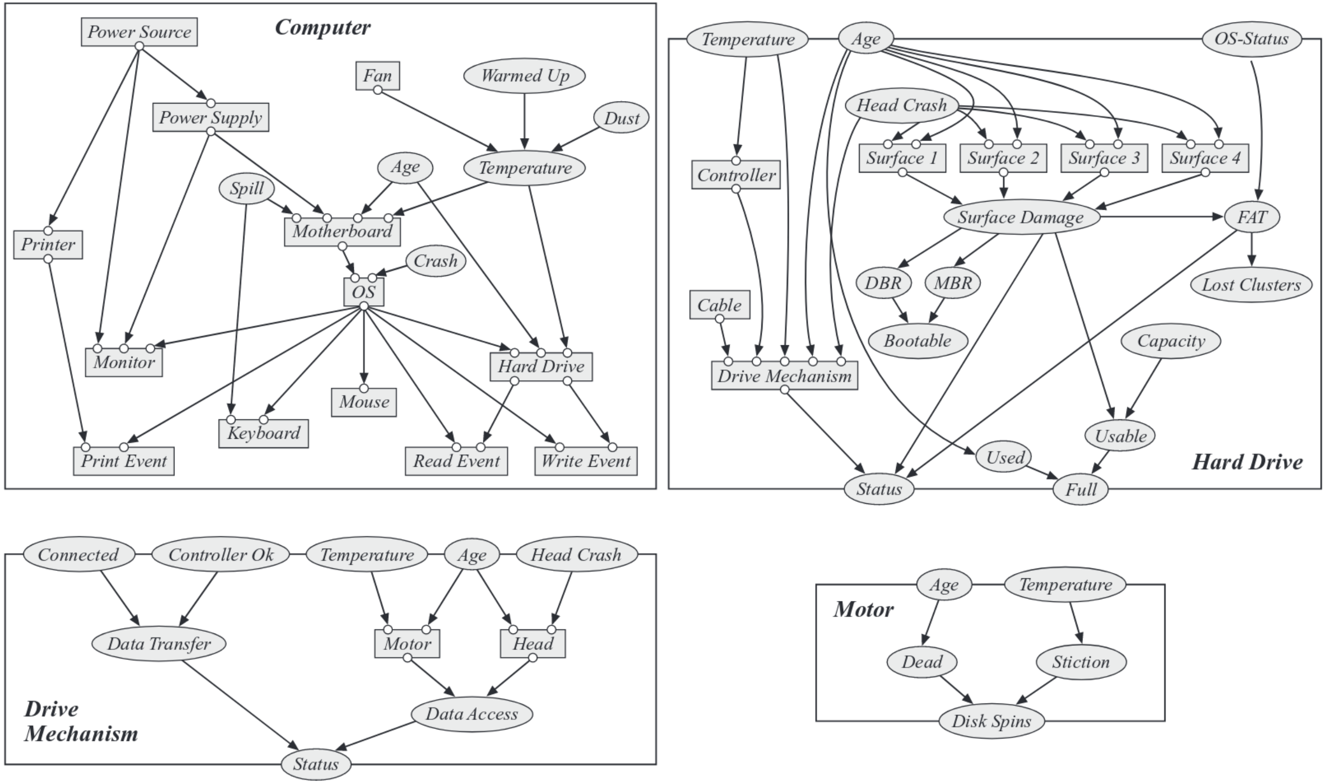
use an entire Bayes-net to represent a CPD

$$P(X_i \mid Pa_{X_i}) = \sum_{\mathbf{Z}} P(X_i, \mathbf{Z} \mid Pa_{X_i})$$



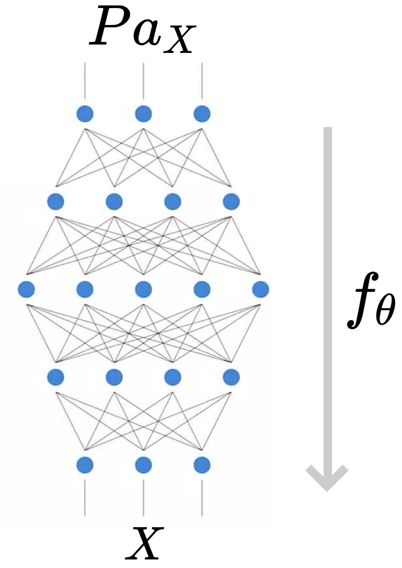
Conditional Bayesian networks: example

can be used for encapsulation in complex models



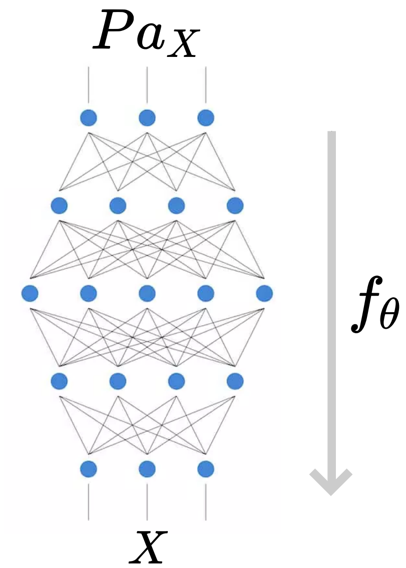
Neural networks defining CPDs

- this idea is extensively used in *deep generative models*
- alternative strategies:



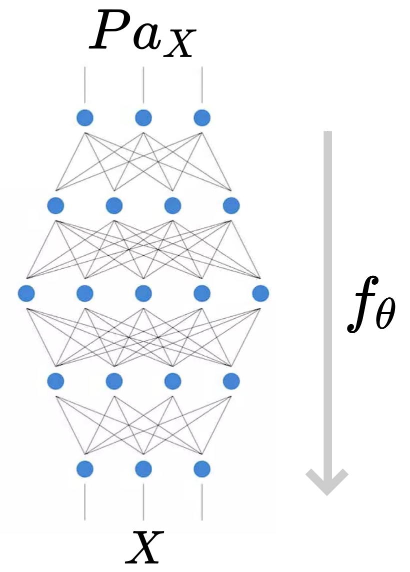
Neural networks defining CPDs

- this idea is extensively used in *deep generative models*
- alternative strategies:
  - ***f*** is a deterministic CPD
    - in Generative Adversarial Networks (GANs)
    - in Normalizing Flows *(special family of functions f)*



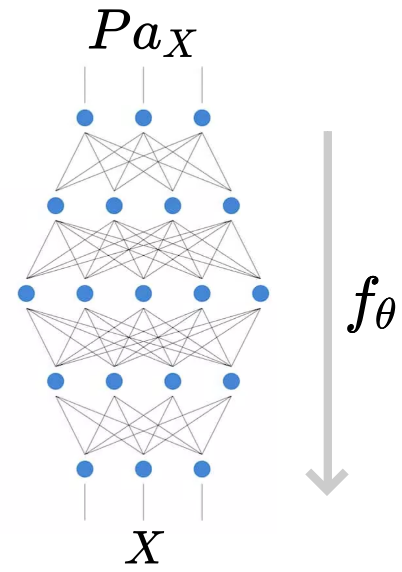
Neural networks defining CPDs

- this idea is extensively used in *deep generative models*
- alternative strategies:
  - ***f*** is a deterministic CPD
    - in Generative Adversarial Networks (GANs)
    - in Normalizing Flows *(special family of functions f)*
  - ***f*** defines parameters in a parametric distribution $X \sim p_{f_\theta(Pa_X)}(x)$
    - In Variational Auto Encoders (VAEs) and Auto-Regressive models



Neural networks defining CPDs

- this idea is extensively used in *deep generative models*
- alternative strategies:
  - ***f*** is a deterministic CPD
    - in Generative Adversarial Networks (GANs)
    - in Normalizing Flows *(special family of functions f)*
  - ***f*** defines parameters in a parametric distribution $X \sim p_{f_\theta(Pa_X)}(x)$
    - In Variational Auto Encoders (VAEs) and Auto-Regressive models
  - ***f*** is a stochastic function itself
    - In energy-based models or VAEs with stochastic middle layers $X \sim f_\theta(Pa_X)$



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- Conditional Bayes-nets

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neural networks define expressive CPDs