Probabilistic Graphical Models

Monte-Carlo Inference

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Learning objectives

- the relationship between sampling and inference
- sampling from univariate distributions
- Monte Carlo sampling in graphical models

Mote Carlo inference

- calculating marginals $p(x_1 = \bar{x}_1) = \sum_{x_2, \ldots, x_n} p(\bar{x}_1, x_2, \ldots, x_n)$

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- inference in exponential family $p_\theta(x) = \exp(\langle \theta, \psi \rangle - A(\theta))$
  - is about finding the mean parameters $\mu = \mathbb{E}_{p_\theta}[\psi(x)]$
  - using L samples (particles) $\mu \approx \frac{1}{L} \sum_l \psi(X^{(l)})$

Sampling from categorical dist.

- access to *pseudo* random number generator for $X \sim U(0,1)$
- given $p(X = d) = p_d \quad \forall 1 \leq d \leq D$

- generate $X \sim U(0,1)$ and see where it falls

use binary search $\mathcal{O}(\log(D))$

Transforming probability densities

- given a random variable $X \sim p_X$
- what is the prob. density of $Y = \phi(X)$?

$$Y \sim p_Y(y) = p_X(\phi^{-1}(y)) \left| \frac{\mathrm{d}\phi^{-1}(y)}{\mathrm{d}y} \right|$$

corresponding x

how ϕ changes the volume around each point y

in multivariate case:

- determinant of the Jacobian matrix

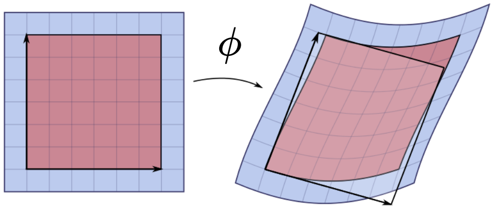

image: wikipedia

Inverse transform sampling

- let X be uniform $p_X = U(0, 1)$
- given a density p_Y

images: work.thaslwanter.at, Murphy's book

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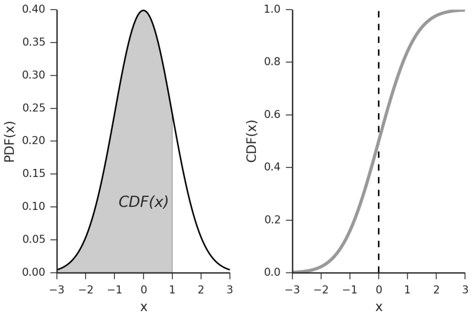
- let F_Y be its CDF $F_Y(y) = P(Y < y)$



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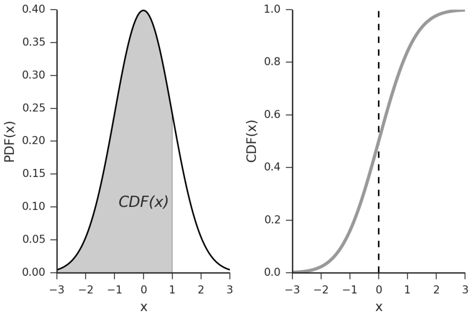
- what is the density of $Y = \phi(X)$?



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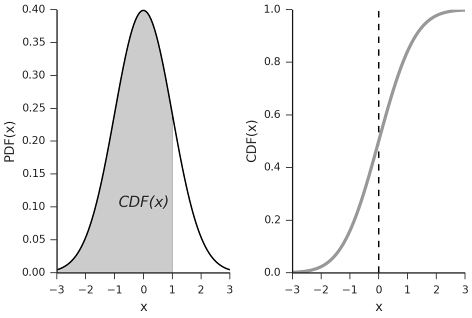
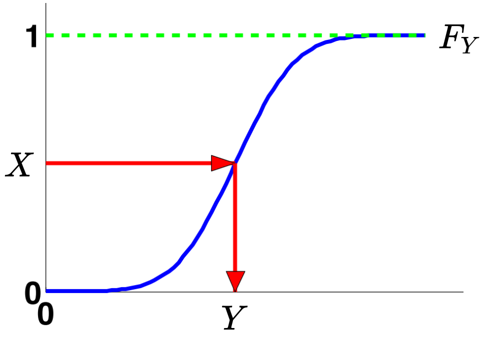
$$Y \sim p_X(\phi^{-1}(y))\left|\frac{\mathrm{d}\phi^{-1}(y)}{\mathrm{d}y}\right| = p_X(F(y))\left|\frac{\mathrm{d}F(y)}{\mathrm{d}y}\right|$$

constant: $p_X = U(0,1)$

$p_Y(y)$





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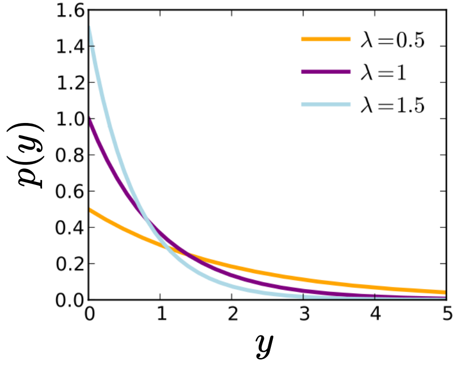
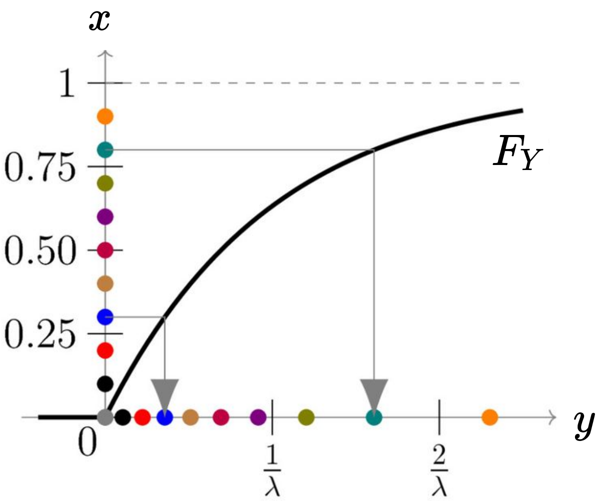
Inverse transform sampling: example



Expoenential distribution

$$p(y) = \lambda e^{-\lambda y}$$

$$F_Y(y) = 1 - e^{-\lambda y}$$



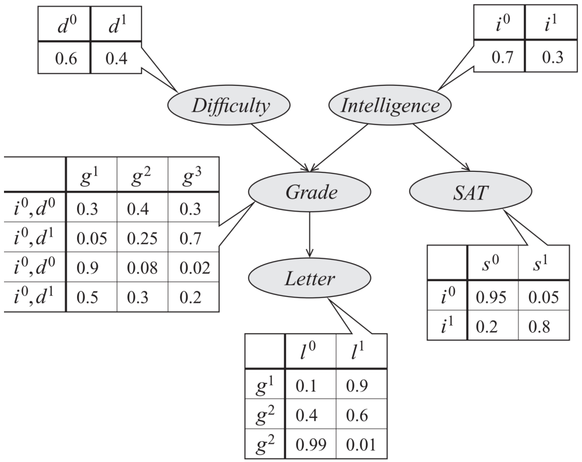
calculate the inverse CDF:

$$F_Y^{-1}(x) = -\frac{1}{\lambda}\ln(1-x)$$

image:wikipedia

Sampling in graphical models

ancestral sampling for Bayes-nets

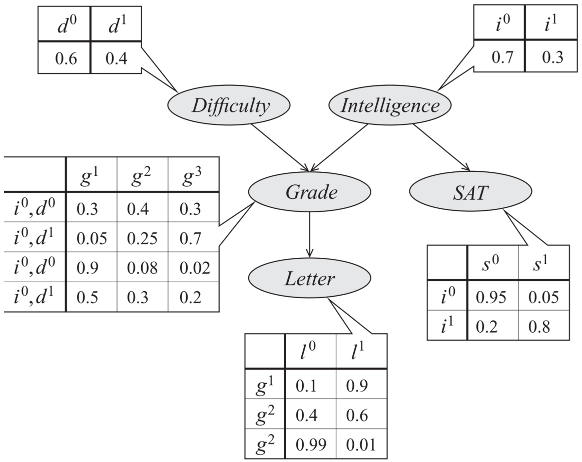


Sampling in graphical models

ancestral sampling for Bayes-nets

- find a *topological ordering*
  - e.g., D,I,G,S,L or I,S,D,G,L
- sample by conditioning on parents

$$G \sim P(g \mid I, D)$$

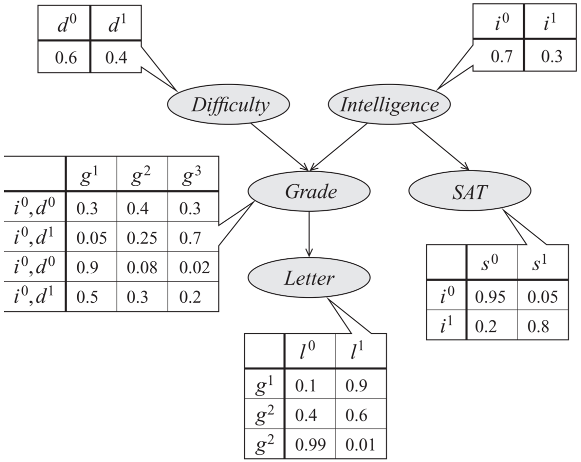


Introducing evidence

what if we have an evidence

- E.g., how to sample from the posterior?

$$p(D, I, S, L \mid G = g^0)$$



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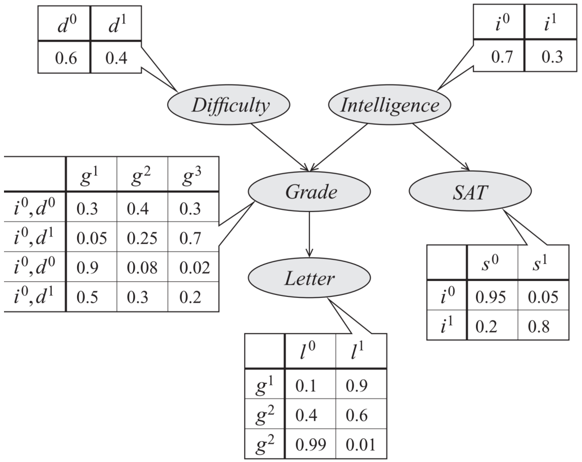


d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^0,d^0	0.9	0.08	0.02
i^0,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

rejection sampling

- find a topological ordering
- sample by conditioning on parents
- only keep samples compatible with evidence $(G = g^0)$
  - wasteful if evidence has a low probability

Rejection sampling

general form

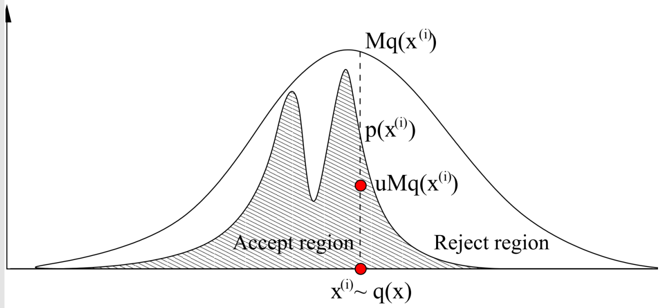
to sample from $p(x) = \frac{1}{Z}\tilde{p}(x)$

use a **proposal** distribution $q(x)$

such that $Mq(x) > \tilde{p}(x)$ everywhere

sample $X \sim q(x)$

accept the sample with probability $\frac{\tilde{p}(x)}{Mq(x)}$



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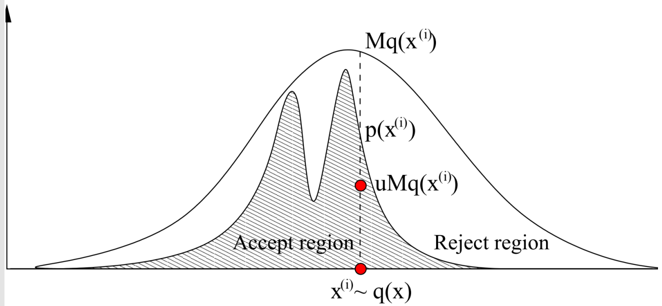


what is the probability of acceptance? $\int_x q(x)\frac{\tilde{p}(x)}{Mq(x)}\mathrm{d}x = \frac{Z}{M}$

for high-dimensional dists. $\frac{Z}{M}$ becomes small!

- rejection sampling becomes wasteful

image: Murphy's book

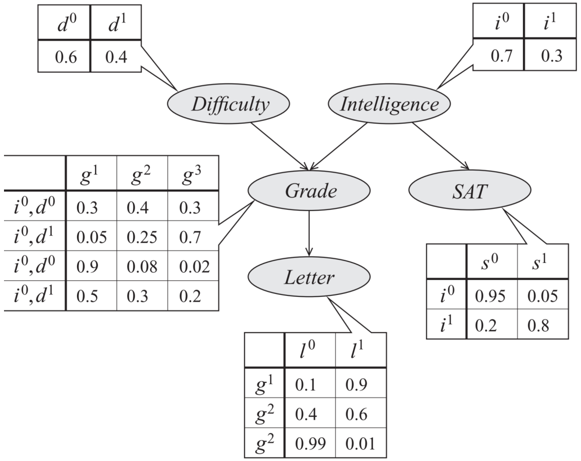
Likelihood weighting

what if we have an evidence?

- E.g., how to sample from the posterior?
  $p(D, I, S, L \mid G = g^1)$

- find a topological ordering
- assign a weight to each particle $w^{(l)} \leftarrow 1$
- sample by conditioning on parents
- when sampling an observed variable
  - set it to its observed value $G = g^1$
  - update the sample's weight $w^{(l)} \leftarrow w^{(l)} \times p(G = g^1 \mid D = d^{(l)}, I = i^{(l)})$

current assignments to parents



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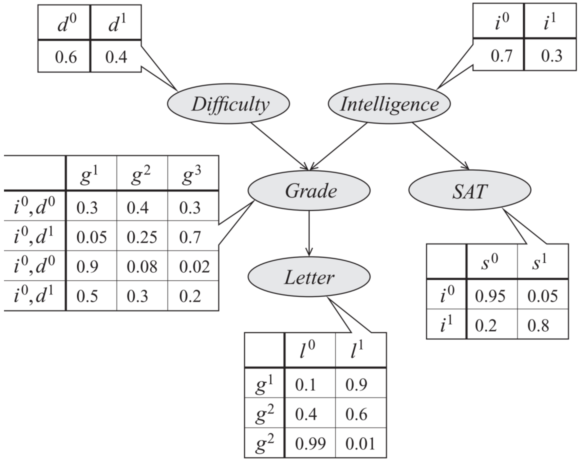

now, we have weighted particles

answering inference queries:

$$p(S = s^0 \mid G = g^1) = \frac{\sum_l w_l \mathbb{I}(S^{(l)}=s^0)}{\sum_l w_l}$$

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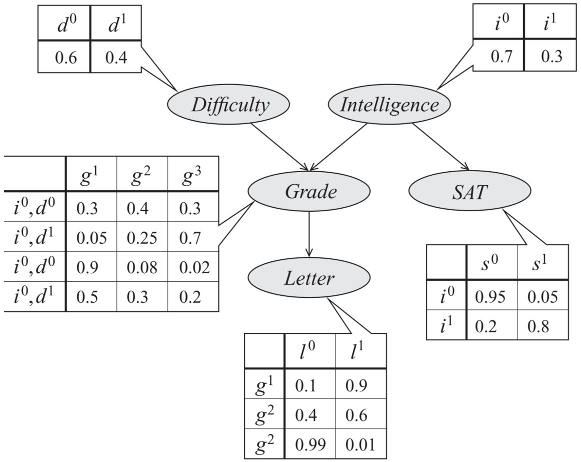

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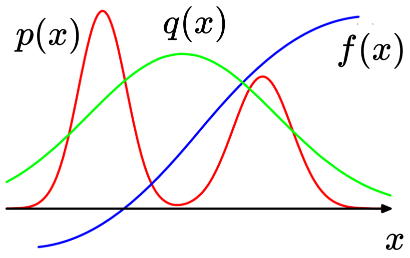
special case of importance sampling

Unnormalized importance sampling

Objective: Monte Carlo estimate $\mathbb{E}_p[f(x)]$

- difficult to sample from p **(yet easy to evaluate)**
- use a proposal distribution q $: p(x) > 0 \Rightarrow q(x) > 0$



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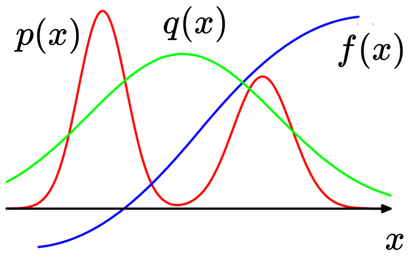

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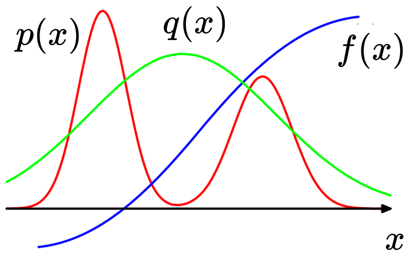

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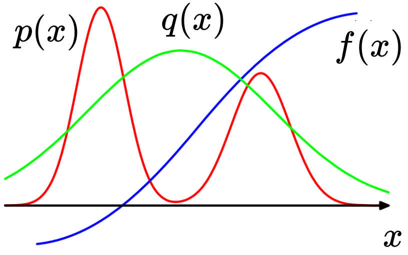

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$\mathbb{E}_p[f(x)] \approx \frac{1}{L} \sum_l w(X^{(l)}) f(X^{(l)})$ *is an unbiased estimator*

can be more efficient than sampling from p itself! (why?)

normalized importance sampling

What if we can evaluate p, up to a constant? $p(x) = \frac{1}{Z}\tilde{p}(x)$

Examples

posterior in directed models $p(x \mid E = e) = \frac{1}{p(e)} p(x, e)$

prior in undirected models $p(x) = \frac{1}{Z} \prod_I \phi_I(x_I)$

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define $w(x) = \frac{\tilde{p}(x)}{q(x)}$ then $\mathbb{E}_q[w(x)] = \int_x \tilde{p}(x)\mathrm{d}x = Z$

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sample $X^{(l)} \sim q(x)$

assign an importance sampling weight $w(X^{(l)}) = \frac{\tilde{p}(X^{(l)})}{q(X^{(l)})}$

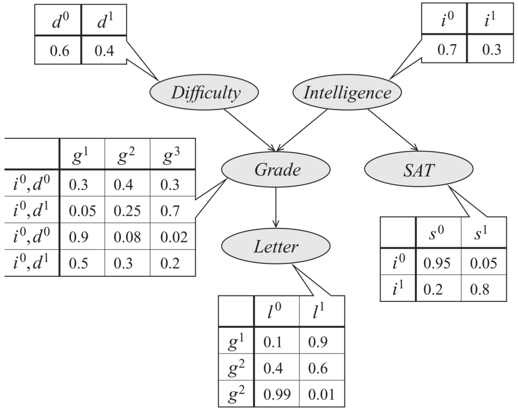
$\mathbb{E}_p[f(x)] \approx \frac{\sum_l w(X^{(l)}) f(X^{(l)})}{\sum_l w(X^{(l)})}$ *is a biased estimator (e.g., consider L=1)*

Revisiting likelihood weighting

likelihood weighting:

$$p(S = s^0 \mid G = g^2, I = i^1) = \frac{\sum_l w_l \mathbb{I}(S^{(l)} = s^0)}{\sum_l w_l}$$

equivalent to:



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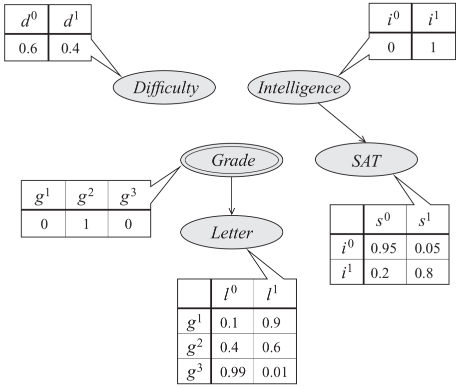

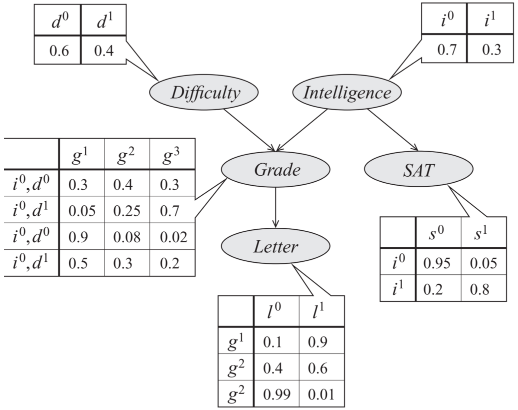

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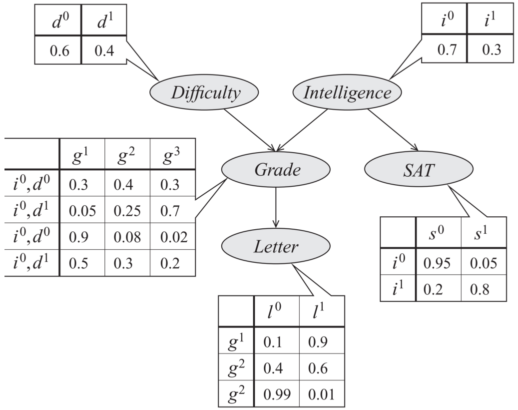

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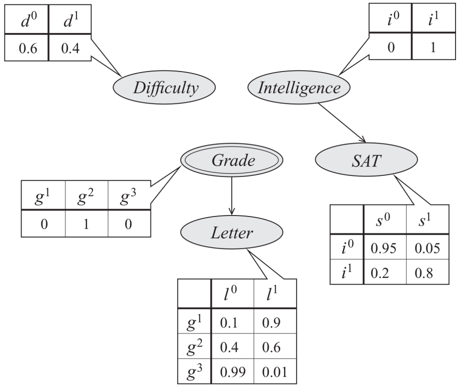

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similar to initial algorithm for likelihood weighting

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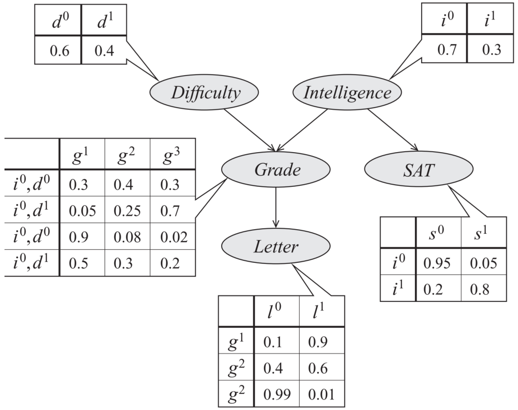

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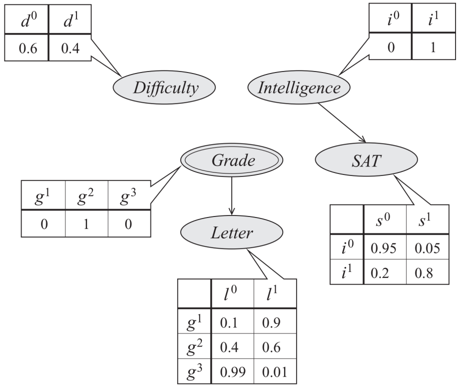

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similar to initial algorithm for likelihood weighting

- evidence only affects sampling for the descendants
- what if all evidence appears at leaf nodes?

Summary

Monte-carlo sampling for approximate inference:

- sampling from univariates:
  - categorical distribution
  - inverse transform sampling
- marginals in directed models:
  - ancestral sampling
- more sophisticated: (incorporating evidence)
  - rejection sampling
  - importance sampling (likelihood weighting)