# Applied Machine Learning

Perceptron and Support Vector Machines

**Siamak Ravanbakhsh**

COMP 551 (winter 2020)

# Learning objectives

geometry of linear classification

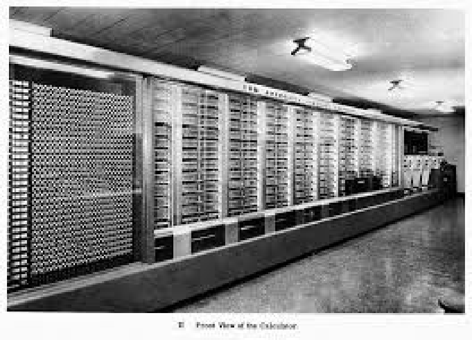
Perceptron learning algorithm

margin maximization and support vectors

hinge loss and relation to logistic regression

# Perceptron

old implementation (1960's)



historically a significant algorithm

(first neural network, or rather just a neuron)

biologically motivated model

simple learning algorithm

convergence proof

beginning of *connectionist* AI

it's criticism in the book  "Perceptrons" was a factor in AI winter

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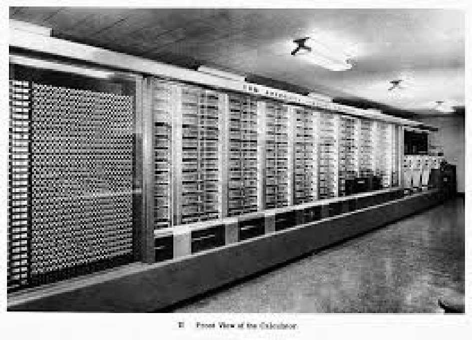

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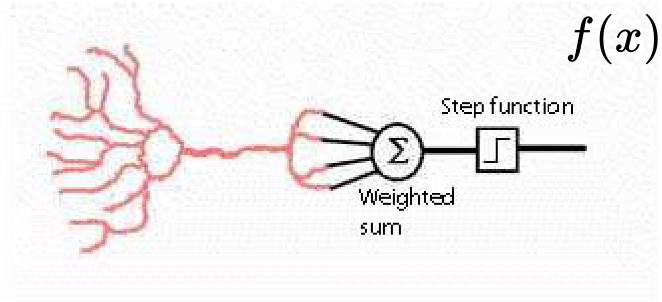


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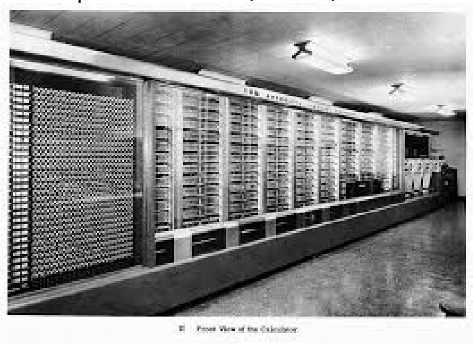

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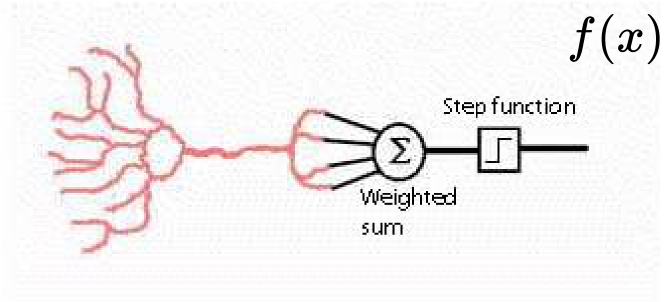


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note that we're using +1/-1 for labels rather than 0/1.

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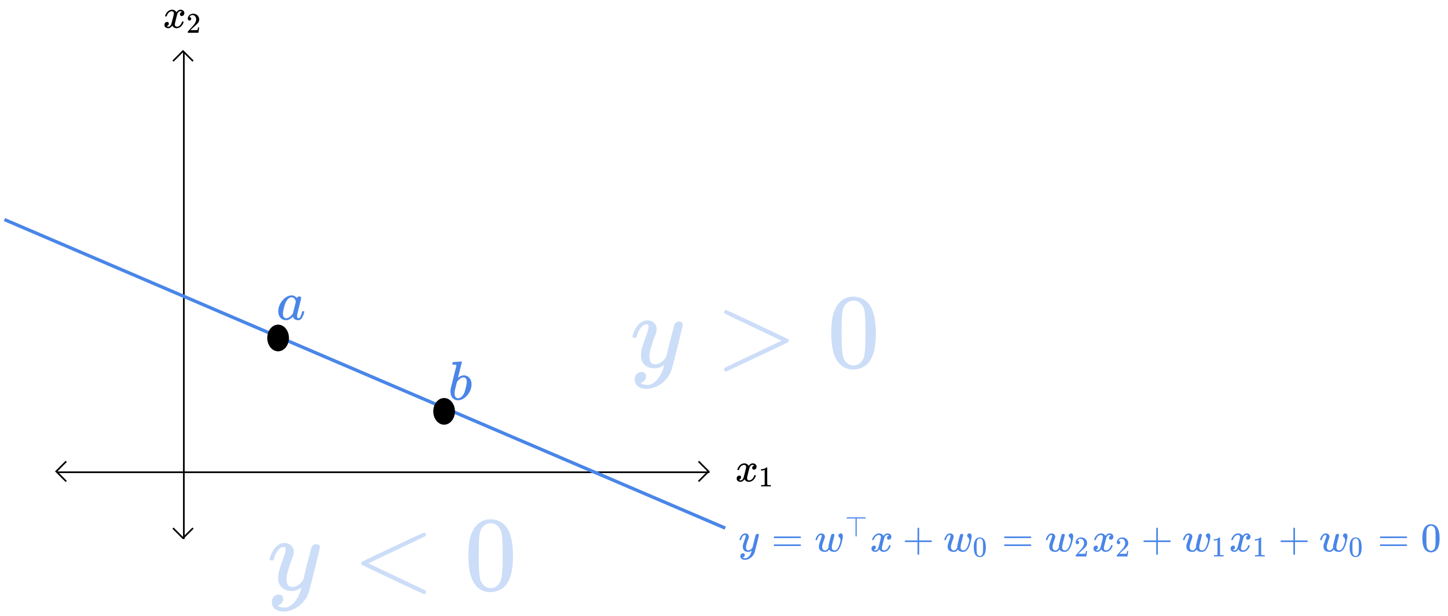
this hyperplane has one dimension lower than D (number of features)



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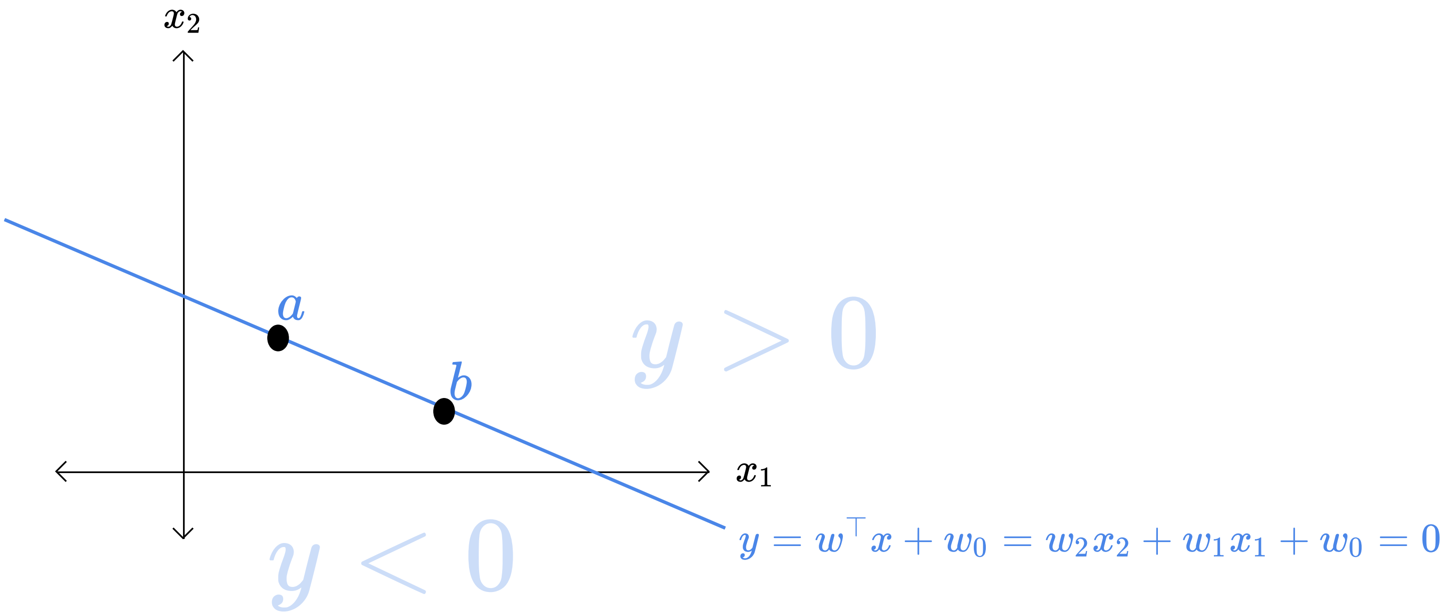

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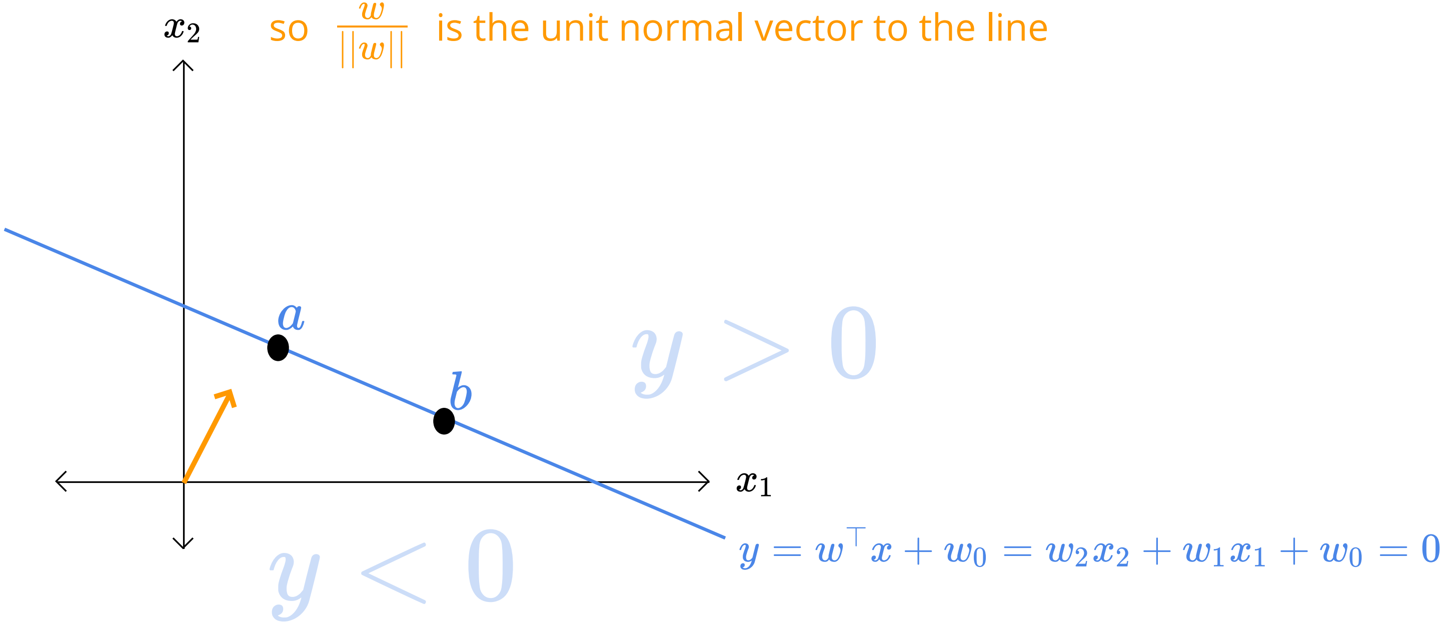

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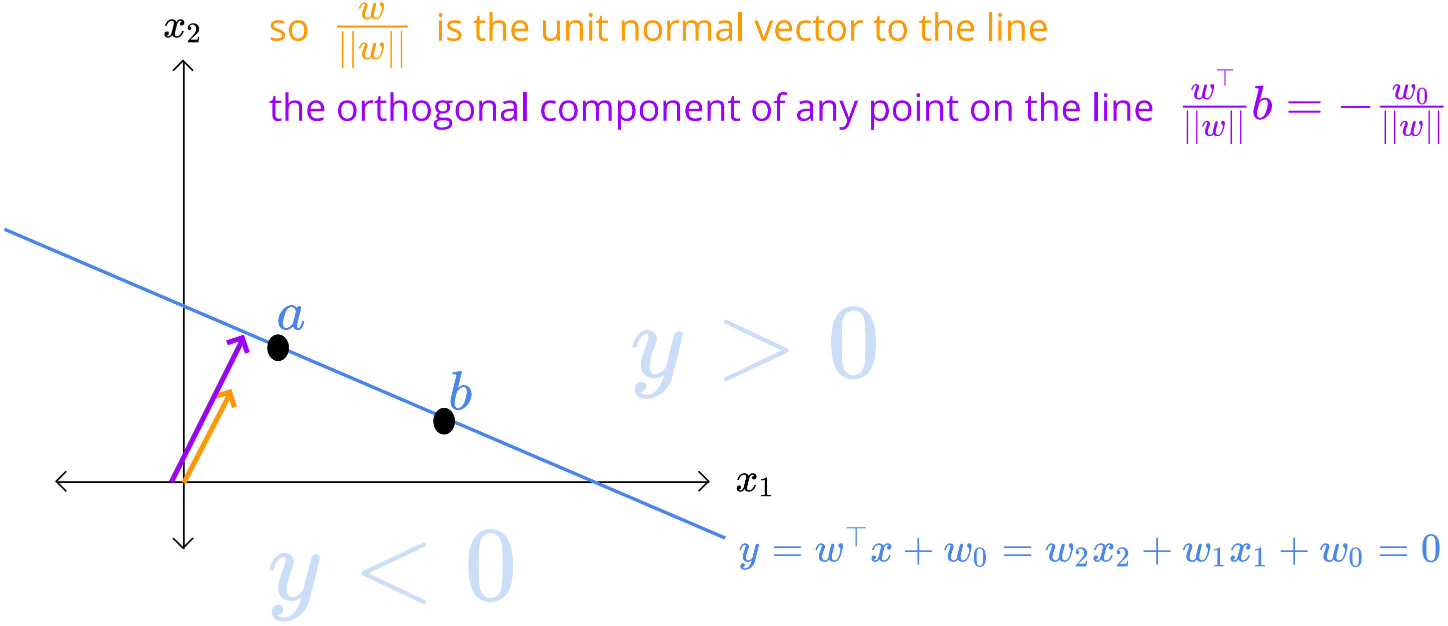

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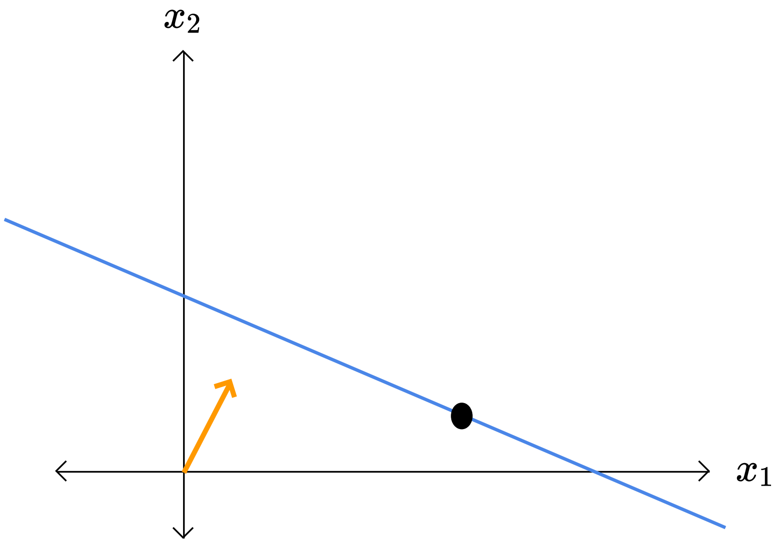

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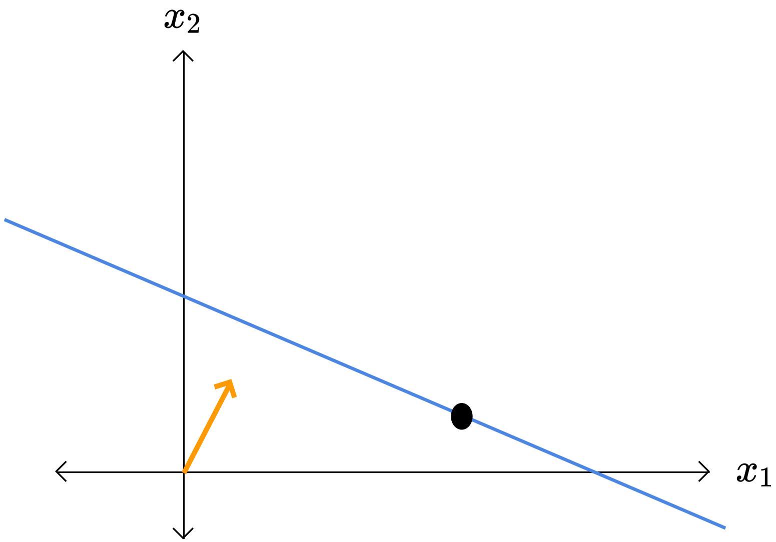


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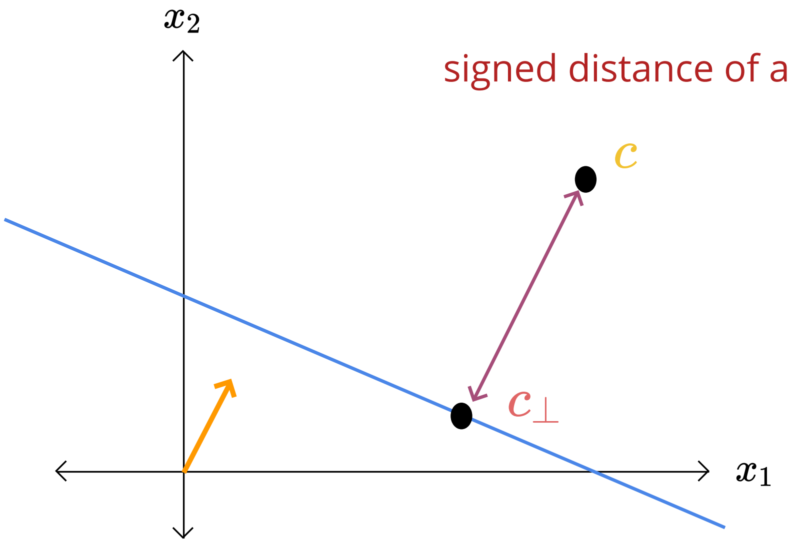


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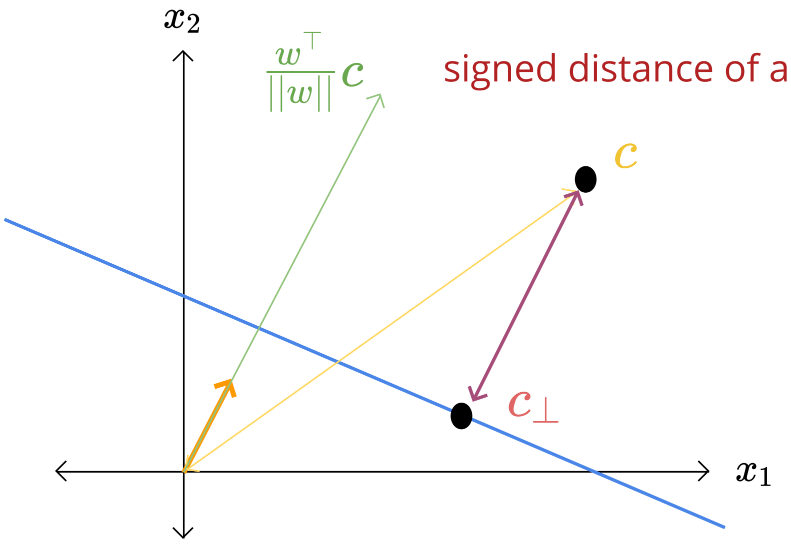


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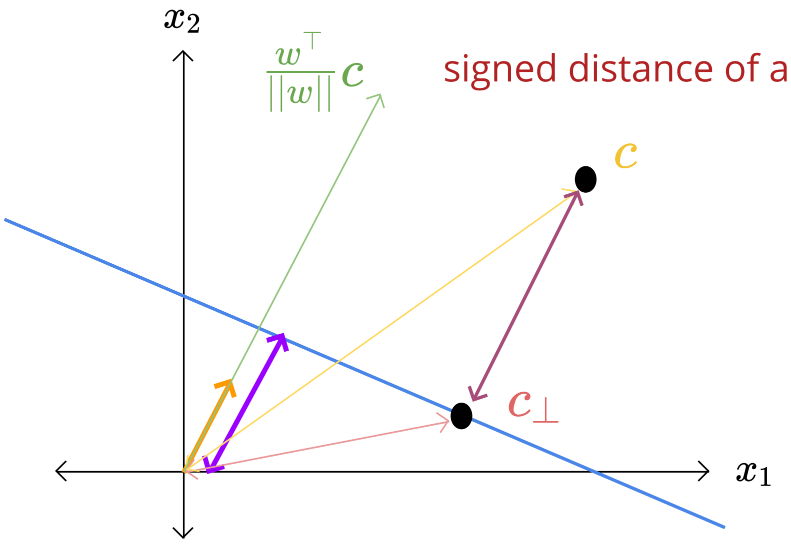


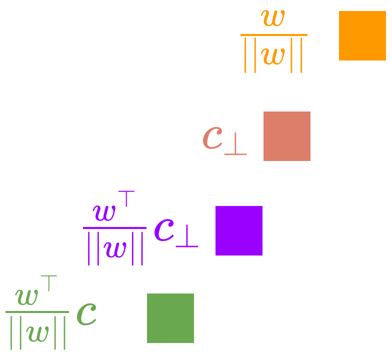

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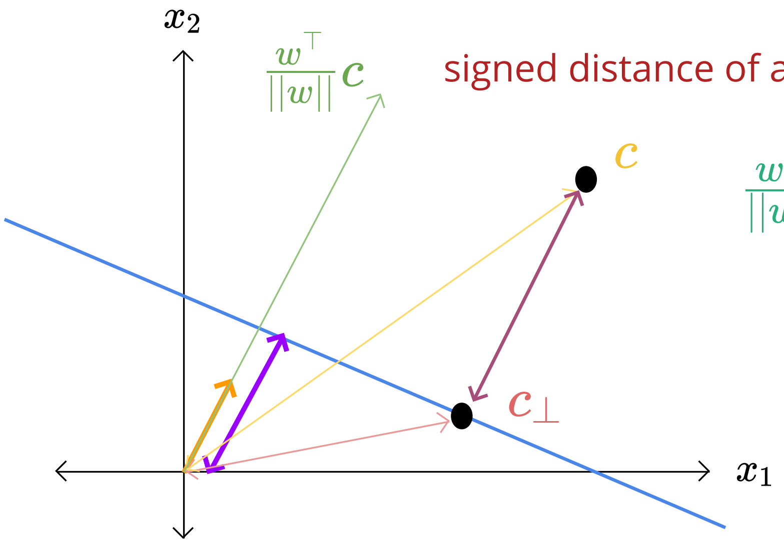


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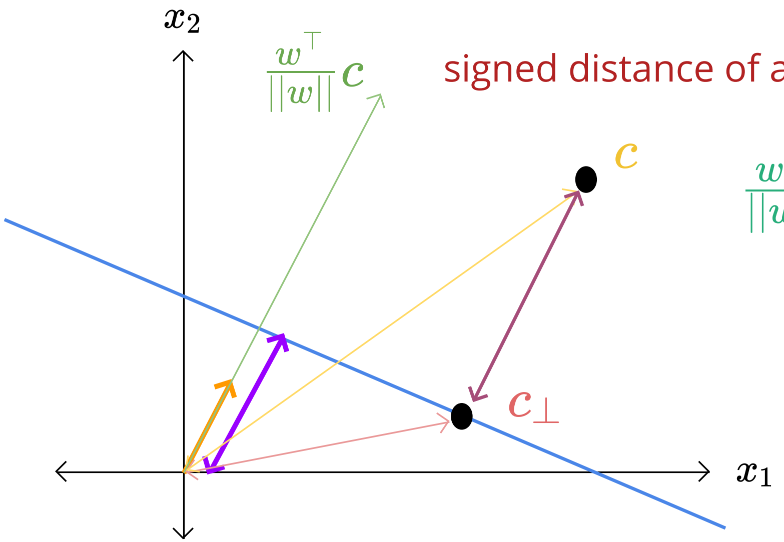
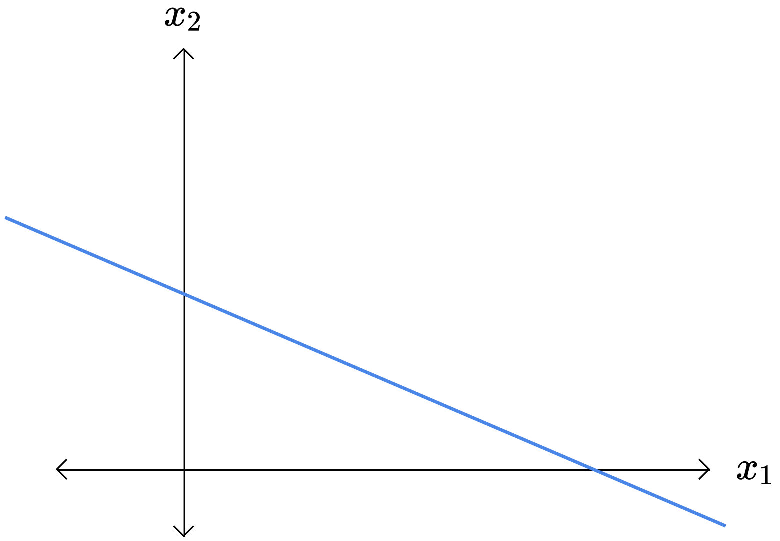


# Perceptron: objective

if $y^{(n)}\hat{y}^{(n)} < 0$ try to make it positive

label and prediction have different signs

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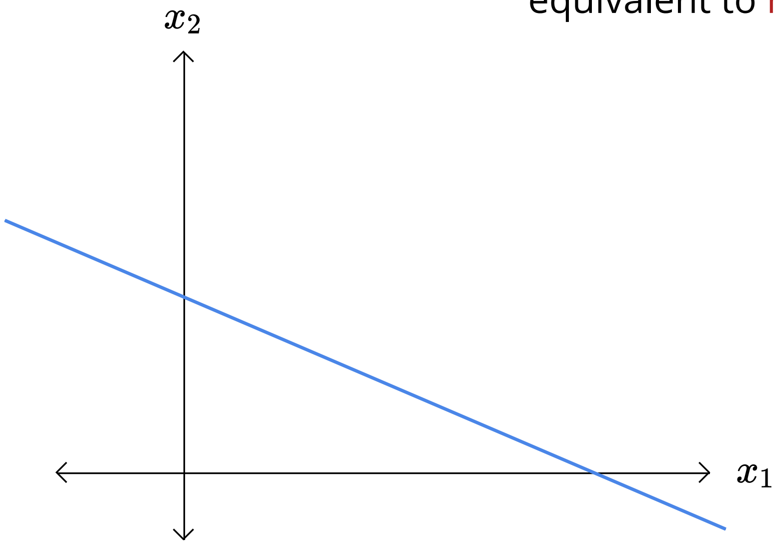

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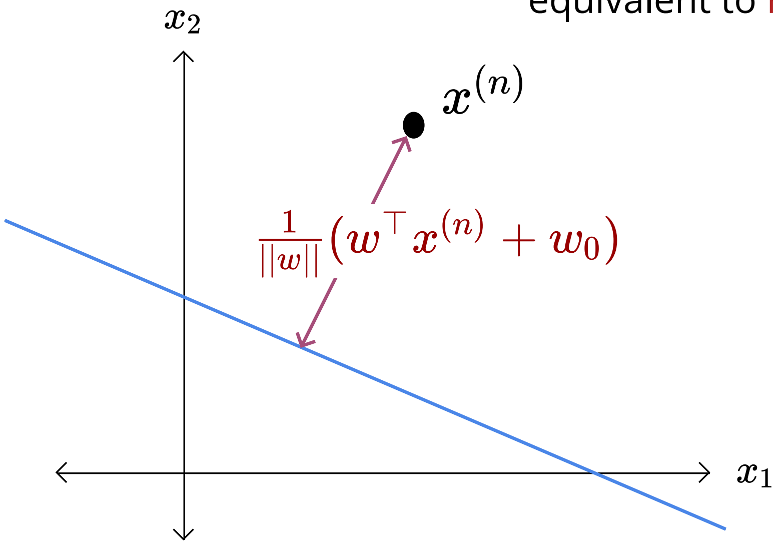

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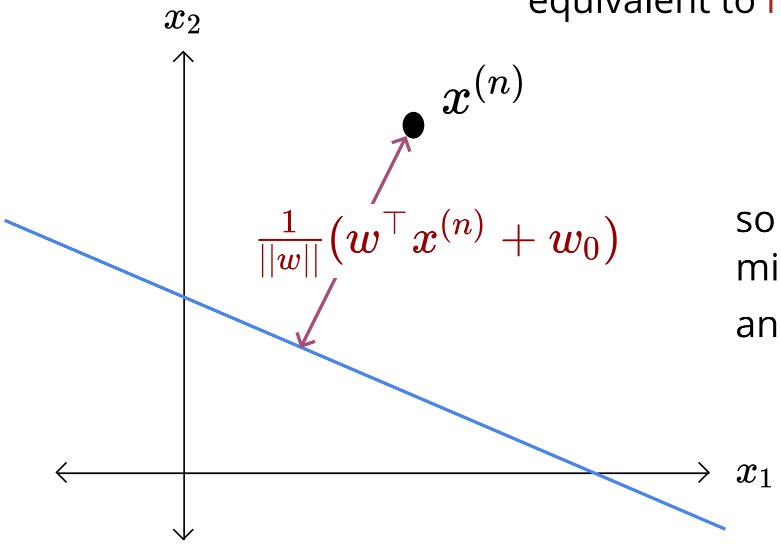


so perceptron tries to minimize the distance of misclassified points from the decision boundary and push them to the right side

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**Perceptron convergence theorem**

the algorithm is guaranteed to converge in finite steps if linearly separable

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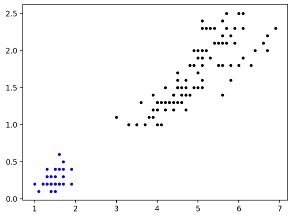
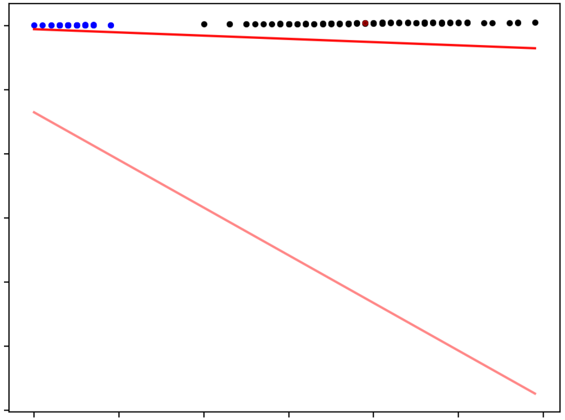



Iris dataset
(linearly separable case)

iteration 1



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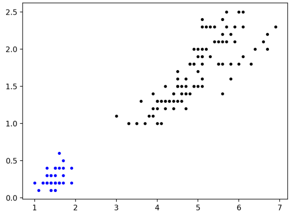

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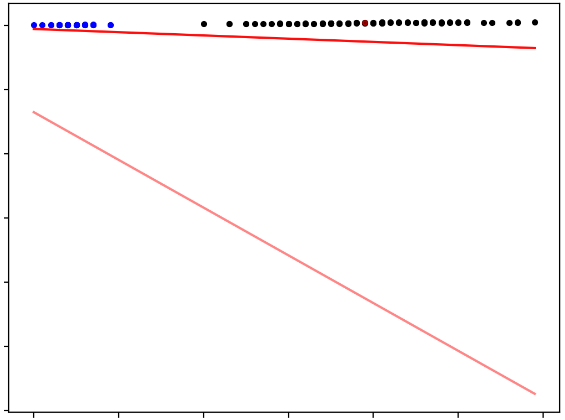

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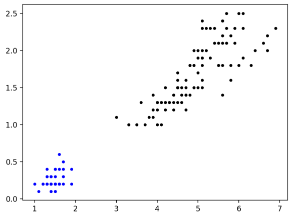


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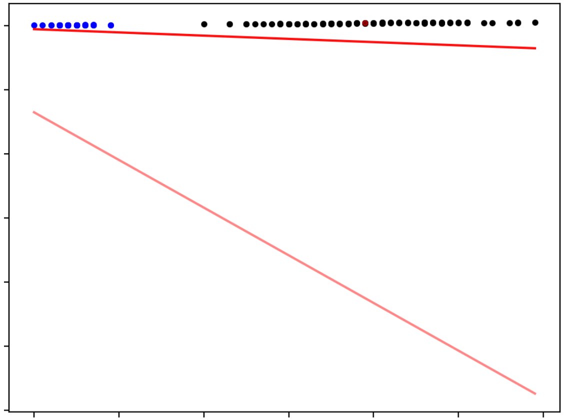

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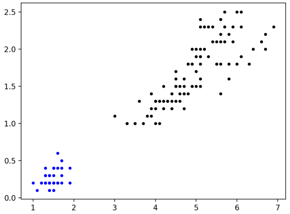


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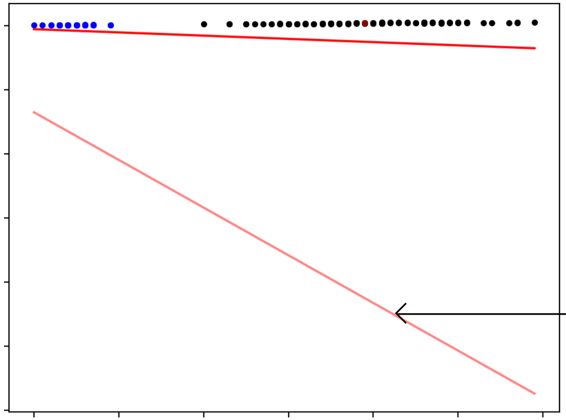

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initial decision boundary $w^\top x = 0$

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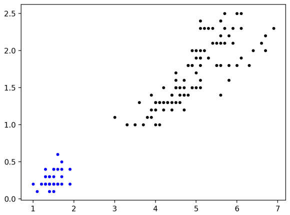


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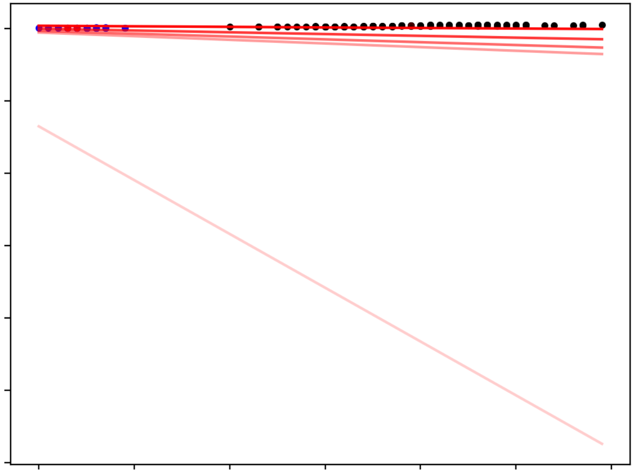

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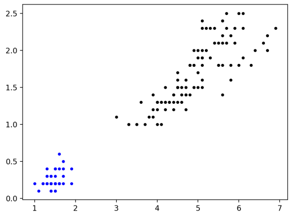


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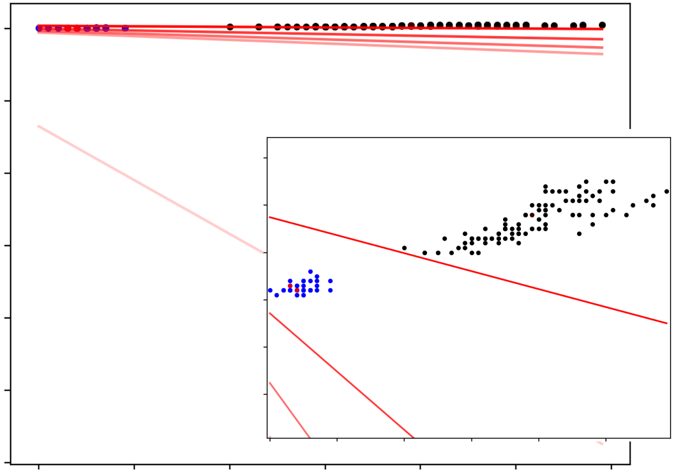

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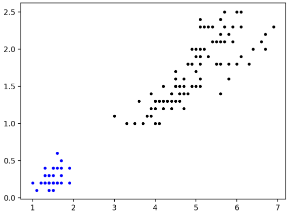

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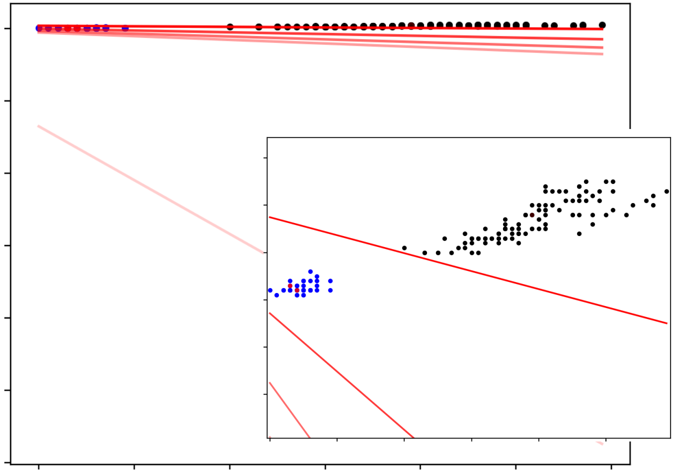

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## observations:

after finding a linear separator no further updates happen

the final boundary depends on the order of instances (different from all previous methods)

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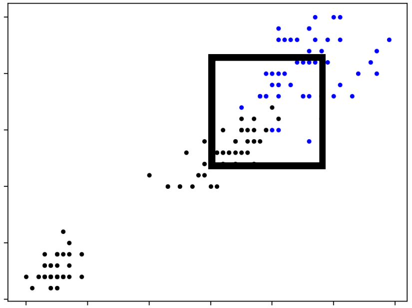

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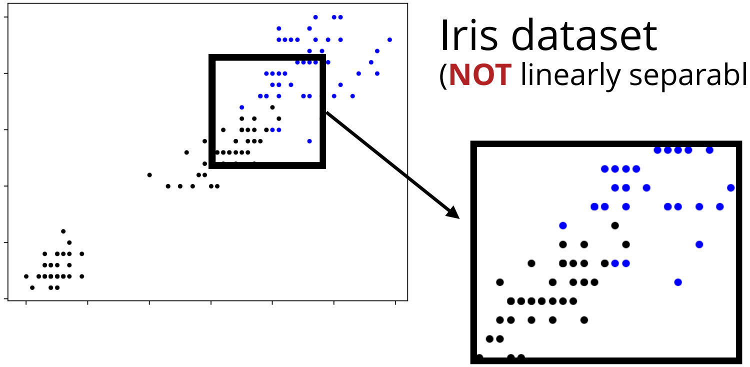

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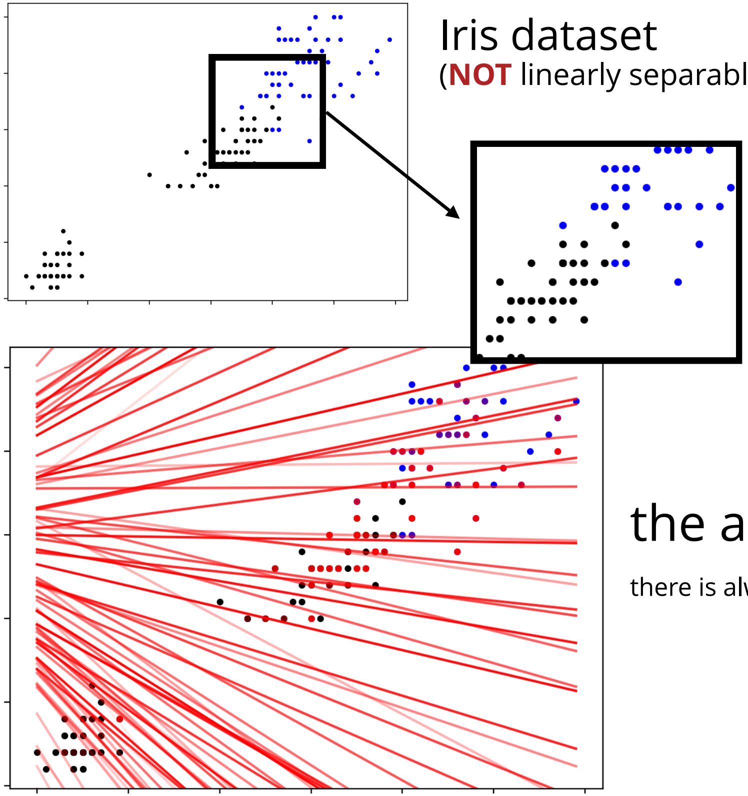

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## the algorithm does not converge

there is always a wrong prediction and the weights will be updated

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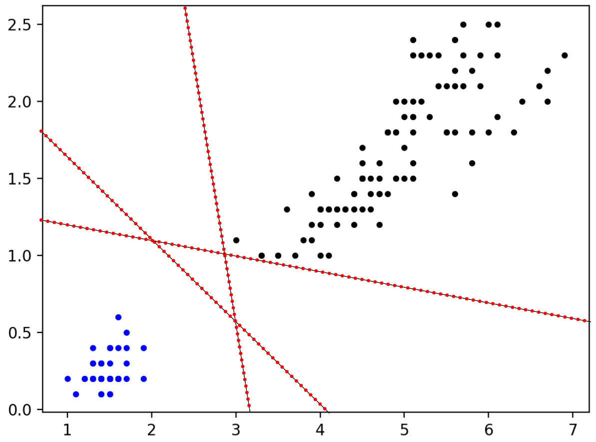

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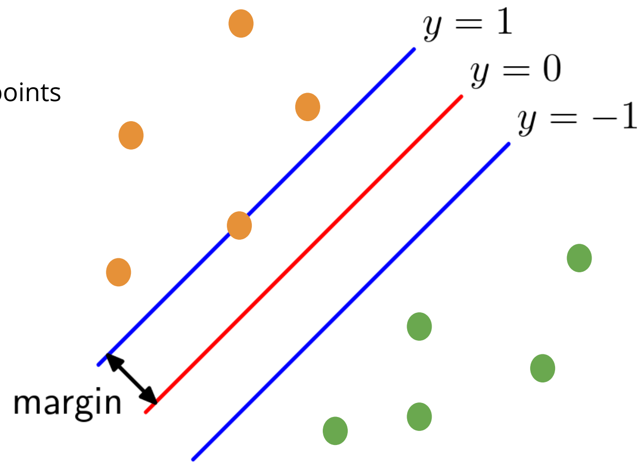
let's fix this problem first

assume linear separability

# Margin

the margin of a classifier (assuming correct classification)
is the distance of the closest point to the decision boundary

this is positive for correctly classified points



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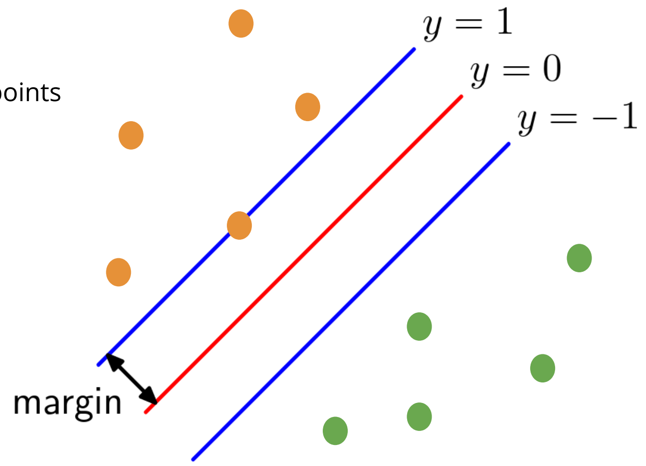

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correcting for sign (margin) $\frac{1}{||w||}y^{(n)}(w^\top x + w_0)$

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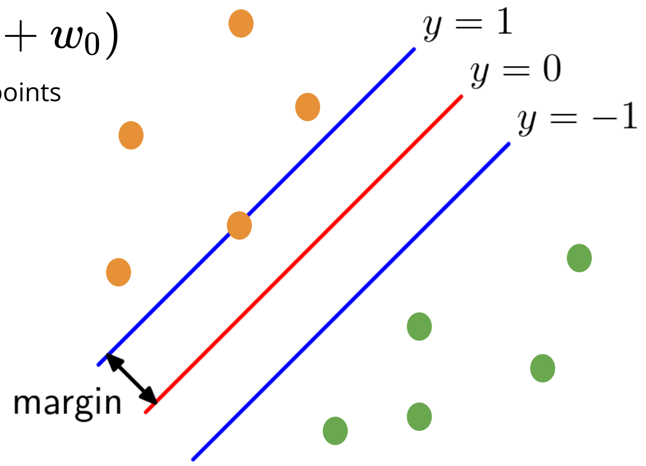
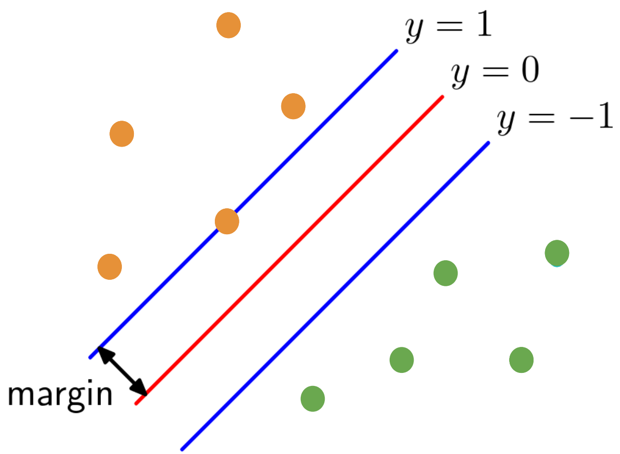


# Max margin classifier

find the decision boundary with maximum margin

margin is not maximal



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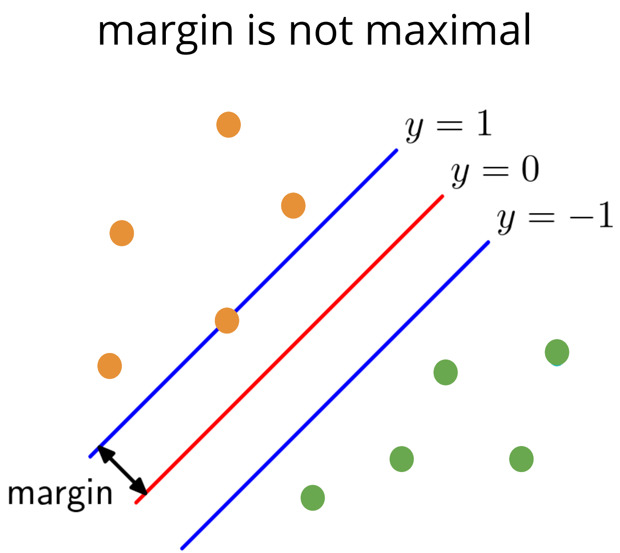


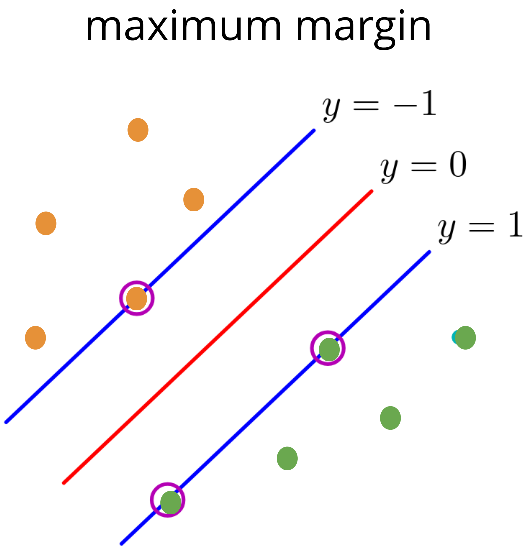

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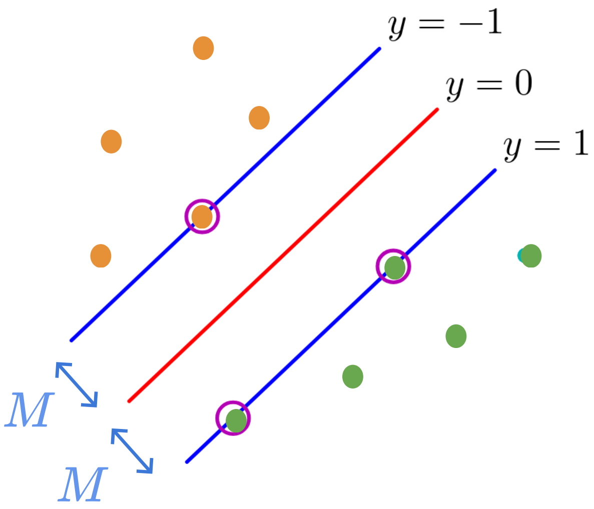


$$\begin{cases} \max_{w,w_0} M \\ M \leq \frac{1}{||w||_2} y^{(n)}(w^\top x^{(n)} + w_0) \quad \forall n \end{cases}$$

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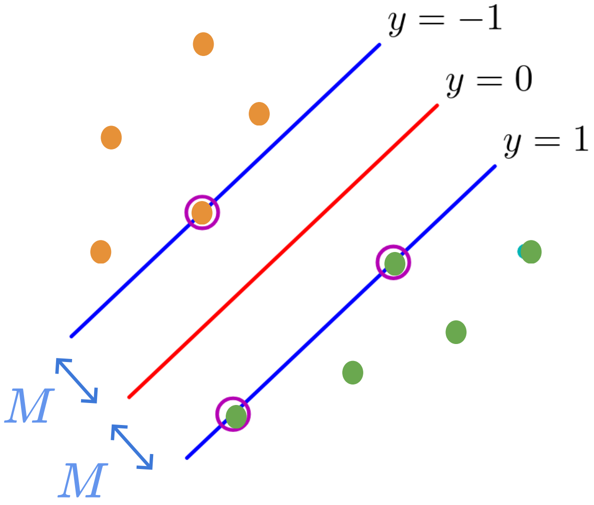


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$M = \frac{1}{||w||_2} y^{(n)}(w^\top x^{(n)} + w_0)$ matter in finding the boundary

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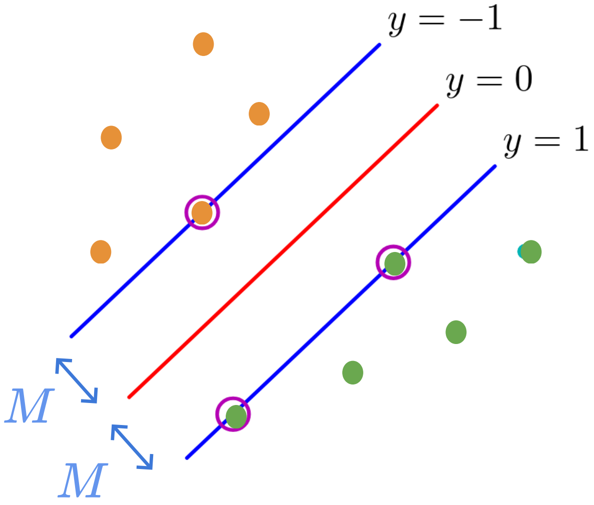


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these are called **support vectors**

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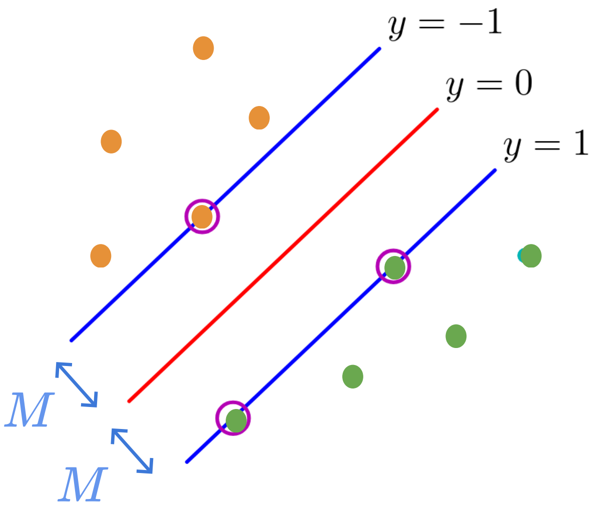


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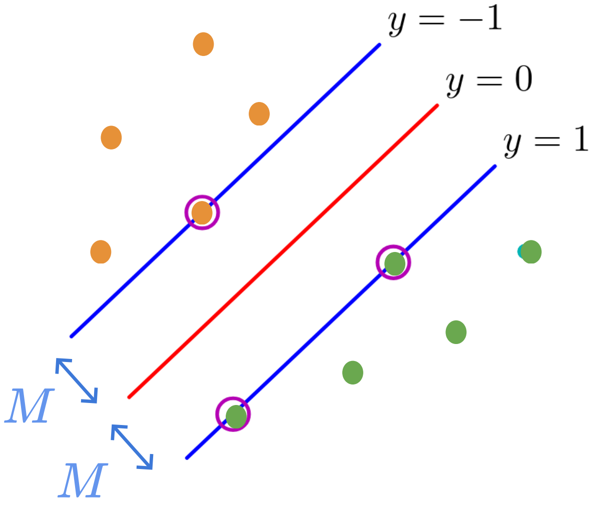
max-margin classifier is called **support vector machine** (SVM)

# Support Vector Machine

find the decision boundary with maximum margin

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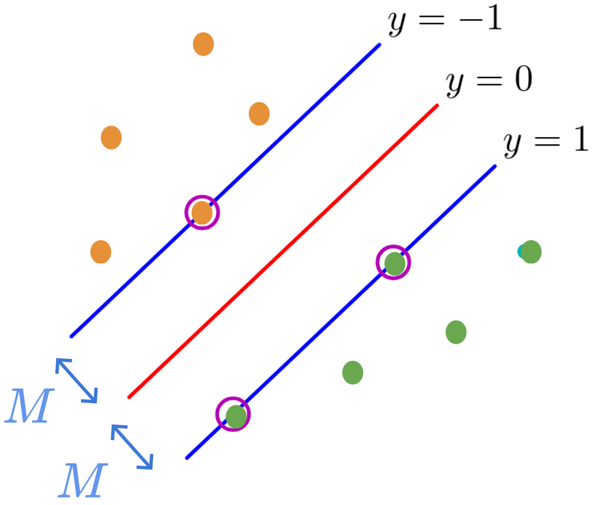


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if $w^*, w_0^*$ is an optimal solution then

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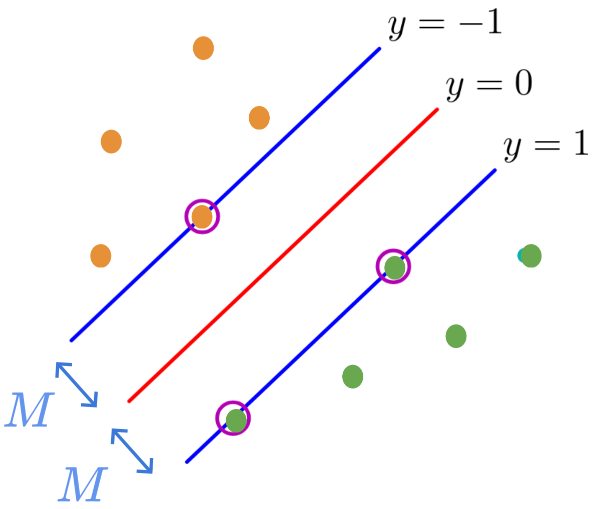
find the decision boundary with maximum margin

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$cw^*, cw_0^*$ is also optimal (same margin)

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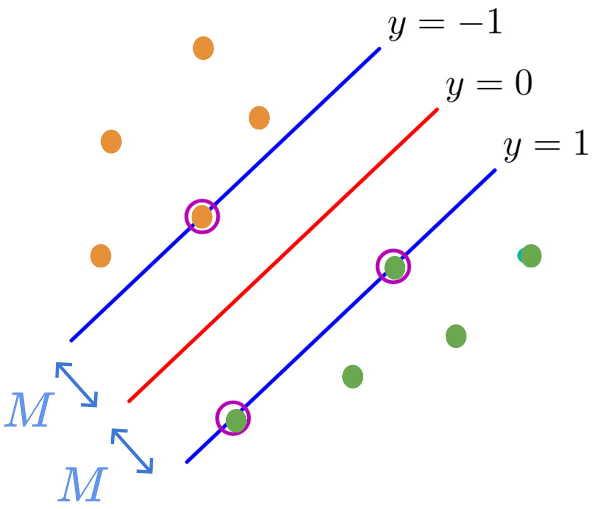


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fix the norm of w to avoid this $||w||_2 = \frac{1}{M}$

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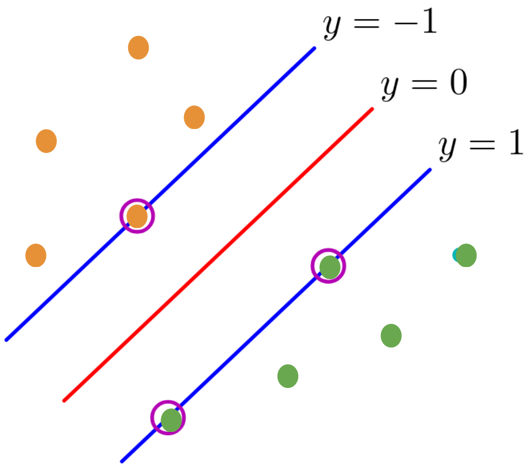


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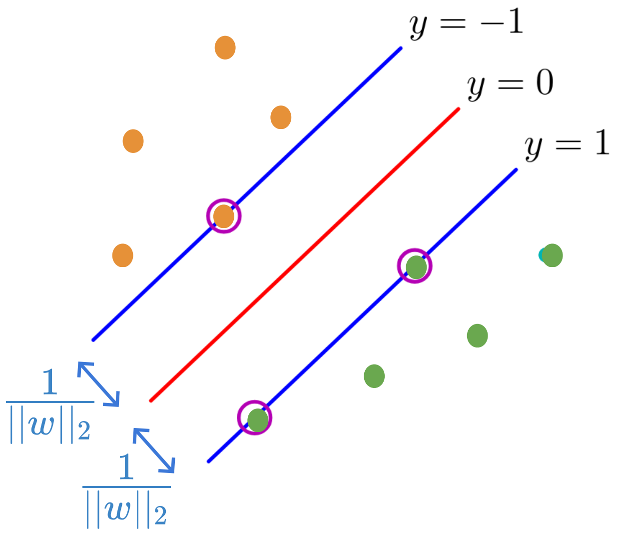


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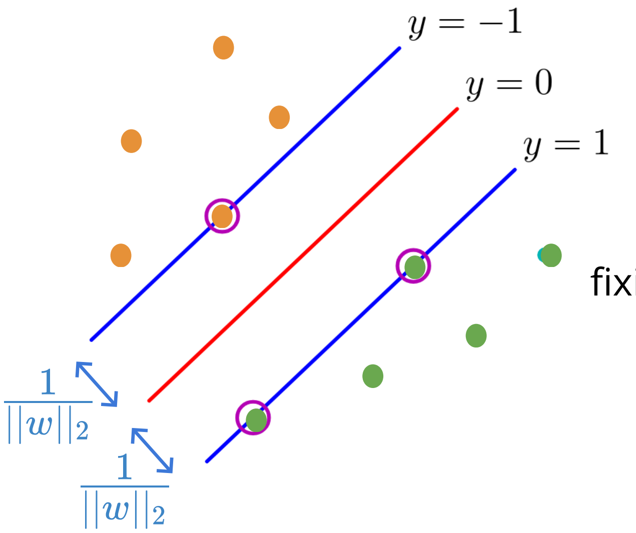


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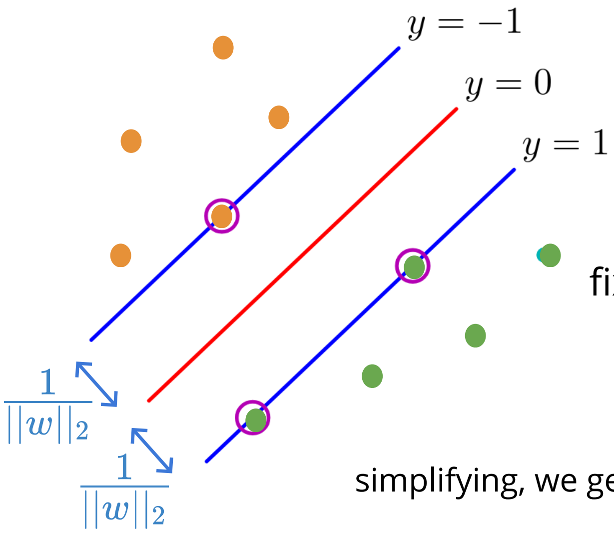


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simplifying, we get hard margin SVM objective

$$\begin{cases} \min_{w,w_0} ||w||_2^2 \\ y^{(n)}(w^\top x^{(n)} + w_0) \geq 1 \quad \forall n \end{cases}$$

# Perceptron: issues

cyclic updates if the data is not linearly separable?

- try make the data separable using additional features?
- data may be inherently noisy

even if linearly separable

convergence could take many iterations

the decision boundary may be suboptimal

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# Perceptron: issues

cyclic updates if the data is not linearly separable?

- try make the data separable using additional features?
- data may be inherently noisy

now lets fix this problem
maximize a **soft** margin

even if linearly separable
convergence could take many iterations

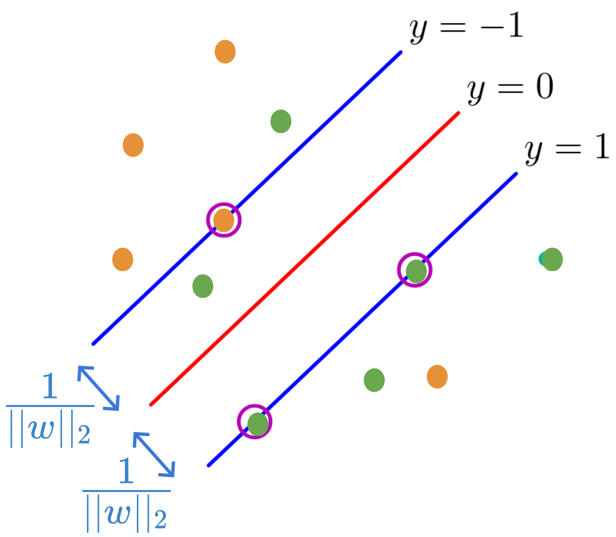
the decision boundary may be suboptimal

maximize the **hard** margin

# Soft margin constraints

allow points inside the margin and on the wrong side

but penalize them

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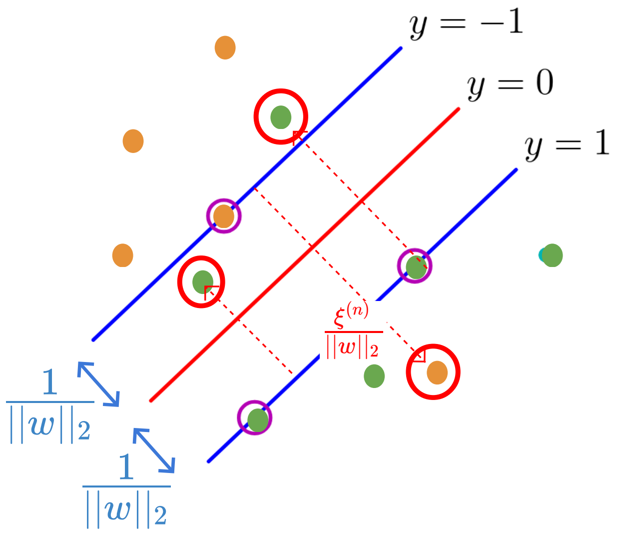


instead of hard constraint $\quad y^{(n)}(w^\top x^{(n)} + w_0) \geq 1 \quad \forall n$

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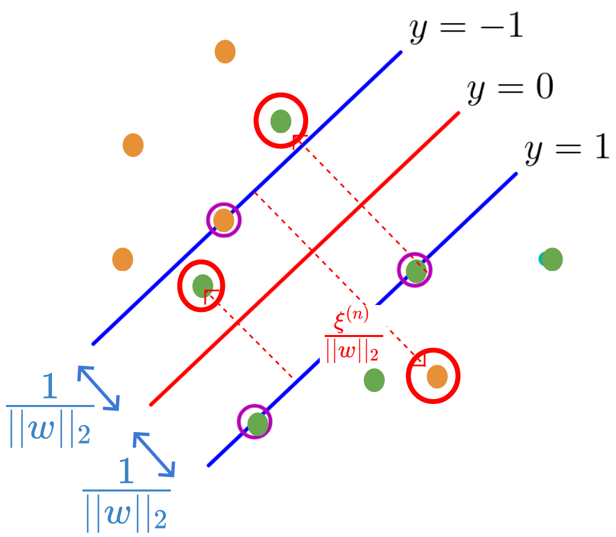


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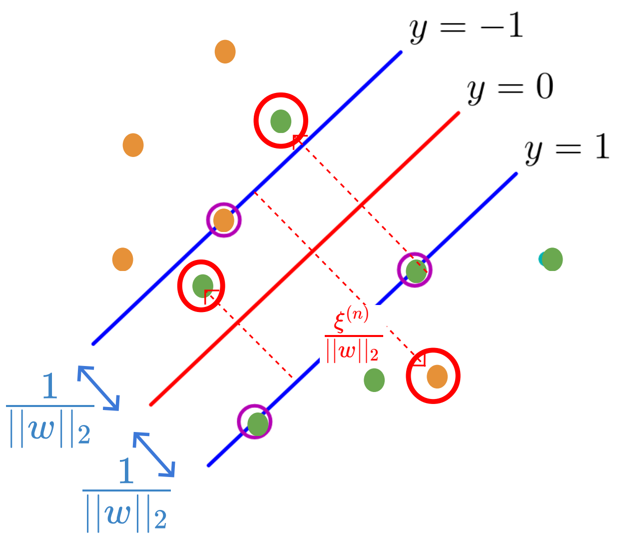


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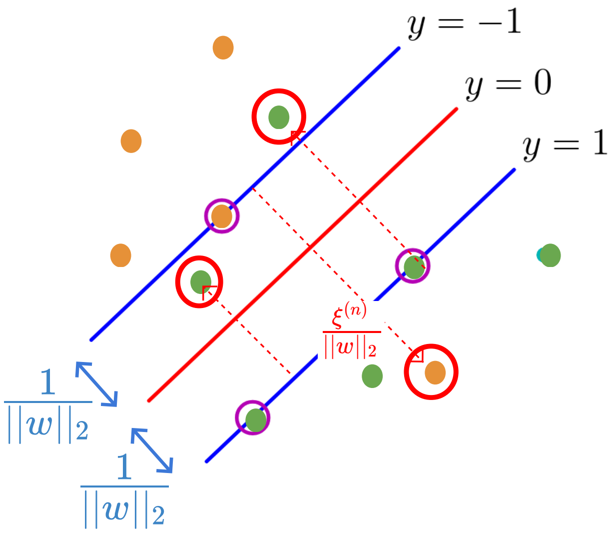


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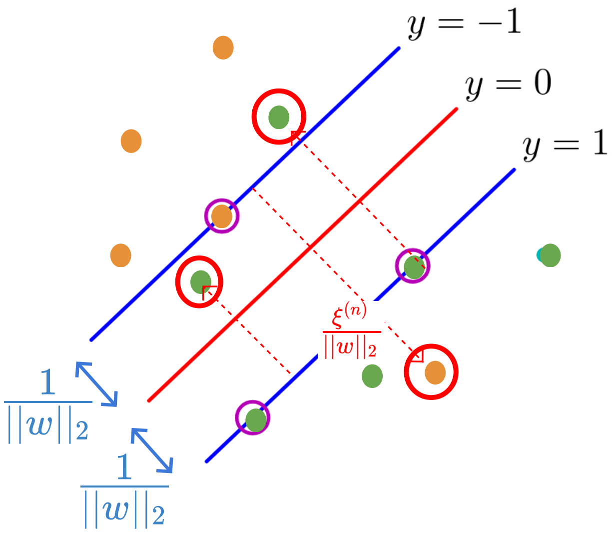


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$0 < \xi^{(n)} < 1$ if correctly classified but inside the margin

$\xi^{(n)} > 1$ incorrectly classified

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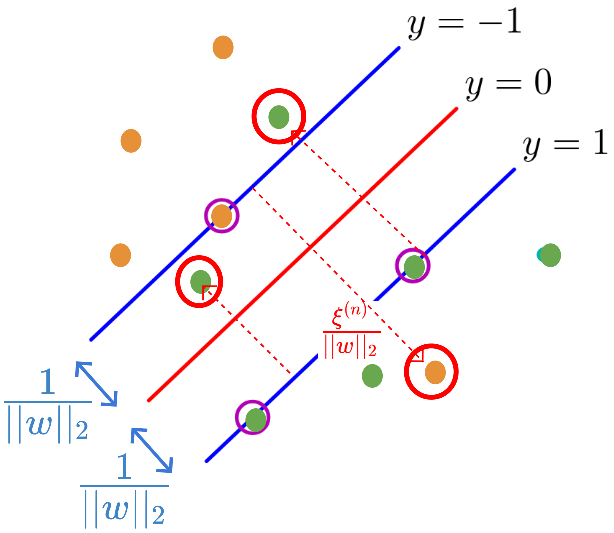


soft-margin objective

$$\min_{w,w_0} \frac{1}{2}||w||_2^2 + \gamma \sum_n \xi^{(n)}$$

$$y^{(n)}(w^\top x^{(n)} + w_0) \geq 1 - \xi^{(n)} \quad \forall n$$

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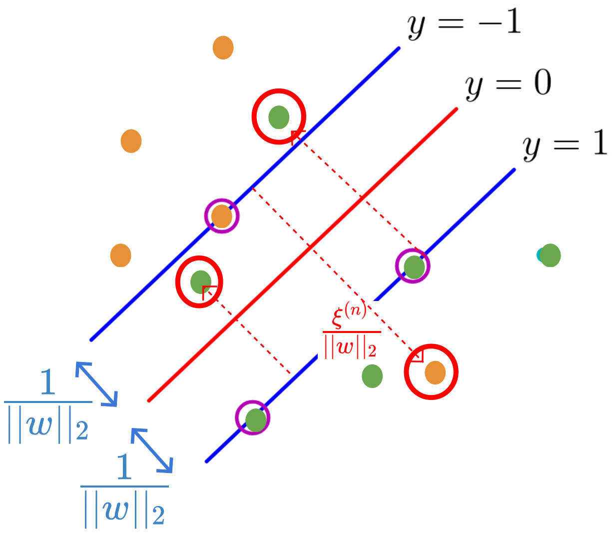


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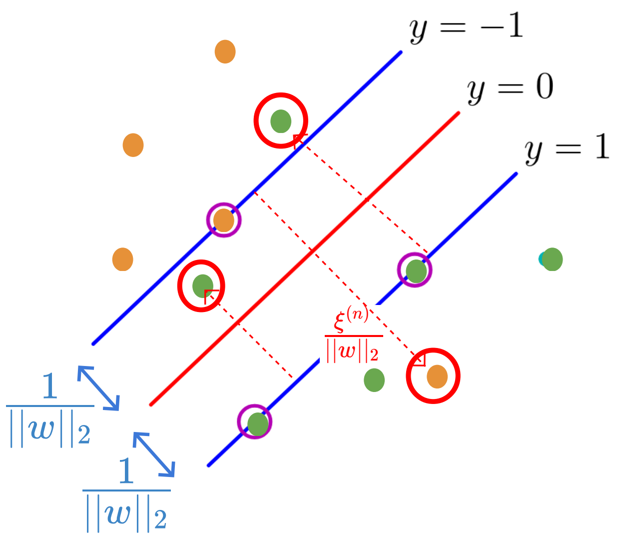
$\gamma$ is a hyper-parameter that defines the importance of constraints
for very large $\gamma$ this becomes similar to hard margin svm

# Hinge loss

would be nice to turn this into an unconstrained optimization

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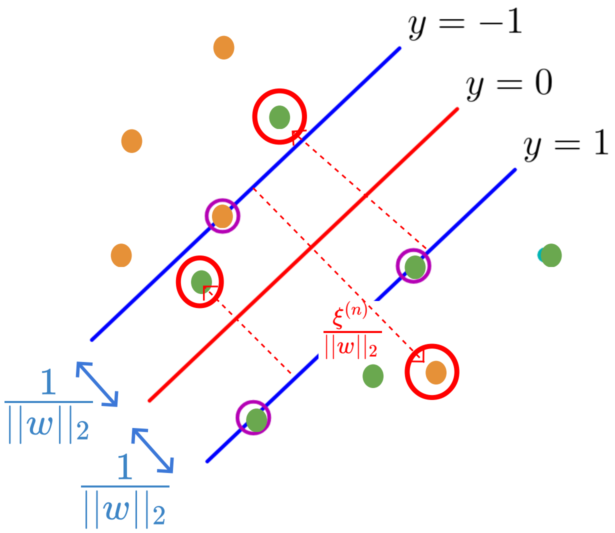


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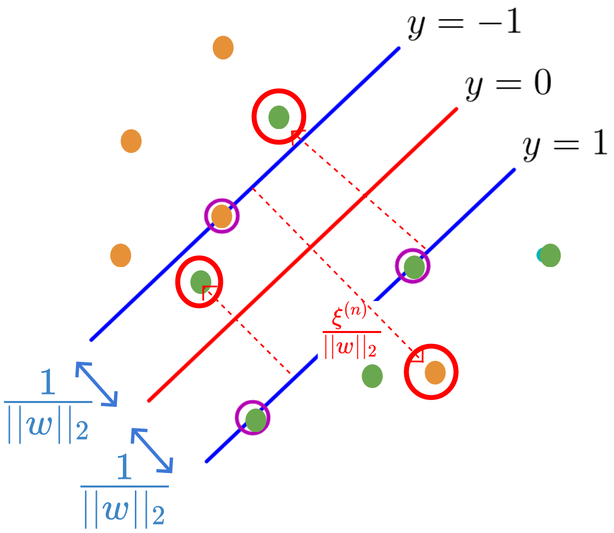


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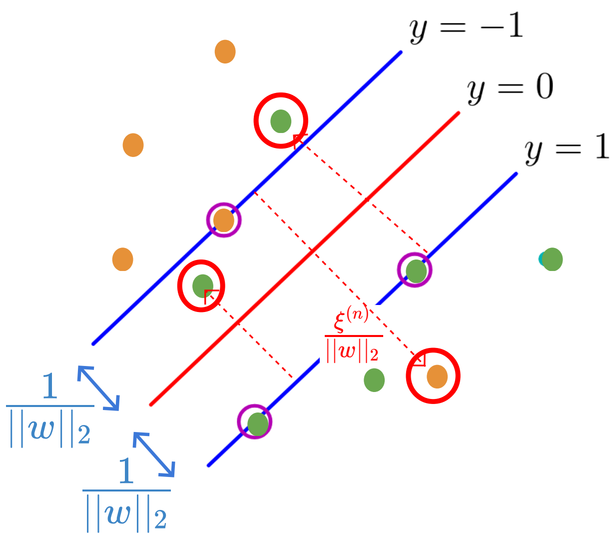


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so the optimal slack satisfying both cases

$$\xi^{(n)} = \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$$

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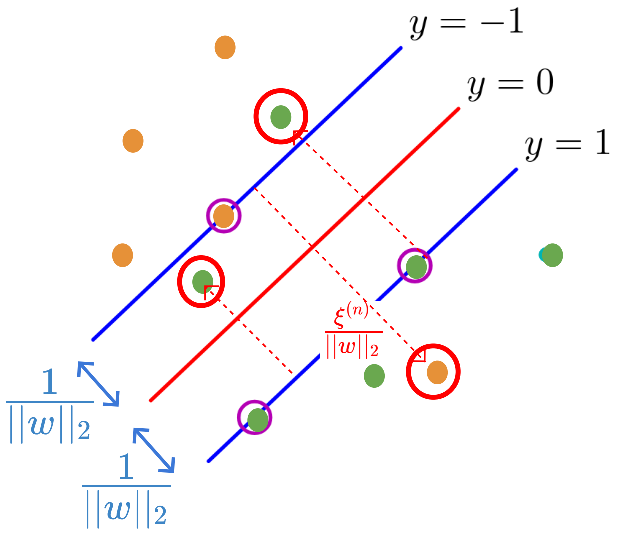


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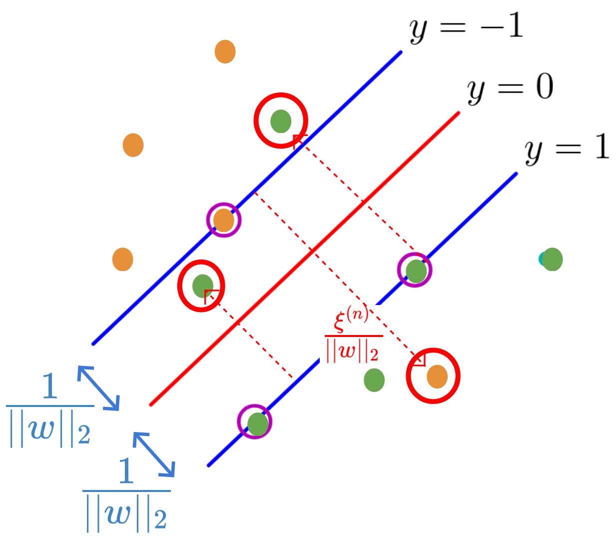


replace $\xi^{(n)} = \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$

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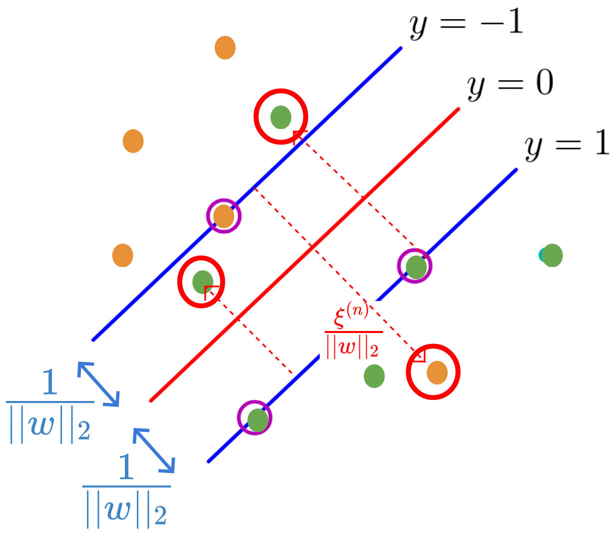


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replace $\xi^{(n)} = \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$

we get $\min_{w,w_0} \frac{1}{2}||w||_2^2 + \gamma \sum_n \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$

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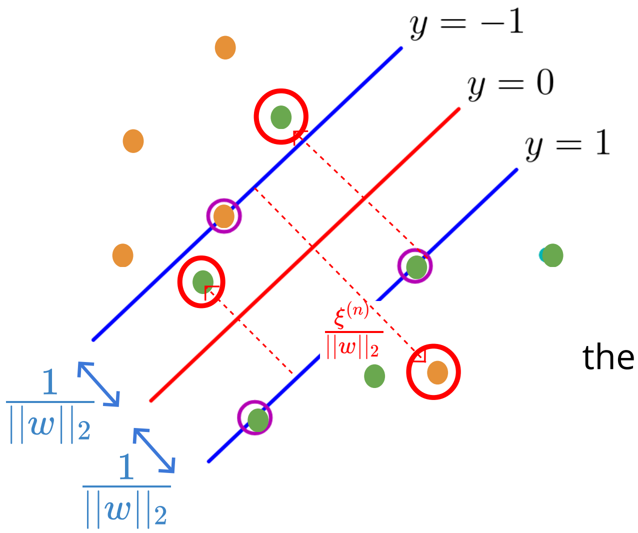


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the same as $\min_{w,w_0} \sum_n \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0)) + \frac{1}{2\gamma}||w||_2^2$

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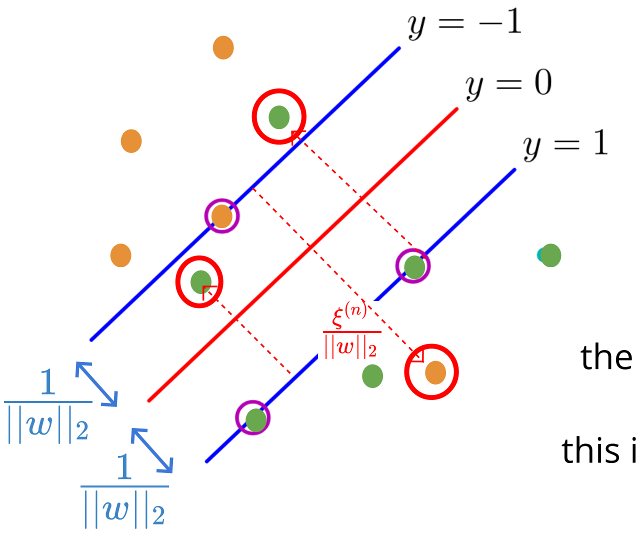


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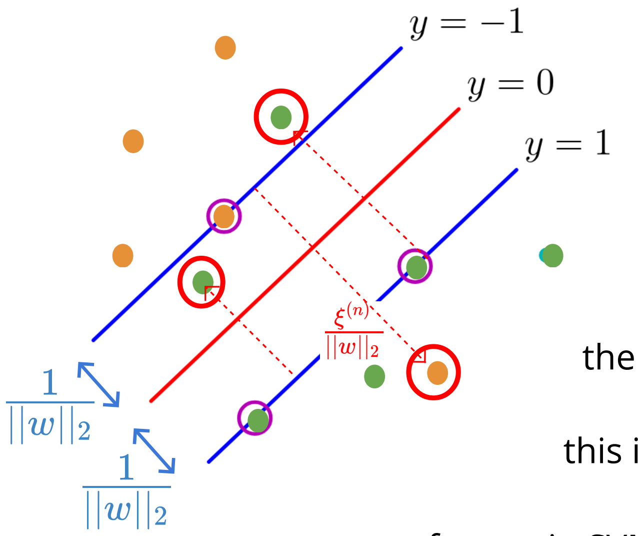
the same as $\min_{w,w_0} \sum_n \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0)) + \frac{1}{2\gamma}||w||_2^2$

this is called the hinge loss $L_{hinge}(y, \hat{y}) = \max(0, 1 - y\hat{y})$

# Hinge loss

would be nice to turn this into an unconstrained optimization




$$\min_{w,w_0} \frac{1}{2}||w||_2^2 + \gamma \sum_n \xi^{(n)}$$

$$y^{(n)}(w^\top x^{(n)} + w_0) \geq 1 - \xi^{(n)}$$

$$\xi^{(n)} \geq 0 \quad \forall n$$

replace $\xi^{(n)} = \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$

we get $\min_{w,w_0} \frac{1}{2}||w||_2^2 + \gamma \sum_n \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0))$

the same as $\min_{w,w_0} \sum_n \max(0, 1 - y^{(n)}(w^\top x^{(n)} + w_0)) + \frac{1}{2\gamma}||w||_2^2$

this is called the hinge loss $L_{hinge}(y, \hat{y}) = \max(0, 1 - y\hat{y})$

soft-margin SVM is doing L2 regularized hinge loss minimization

# Perceptron vs. SVM

### Perceptron

if correctly classified evaluates to zero
otherwise it is $\min_{w,w_0} -y^{(n)}(w^\top x^{(n)} + w_0))$

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stochastic gradient descent with fixed learning rate

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depending on the formulation we have many choices

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cost $J(w) = \sum_n \max(0, 1 - y^{(n)} w^\top x^{(n)}) + \frac{\lambda}{2}||w||_2^2$

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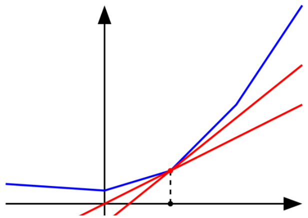

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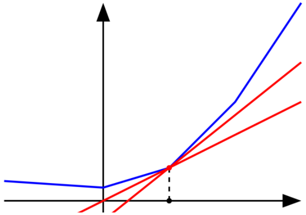

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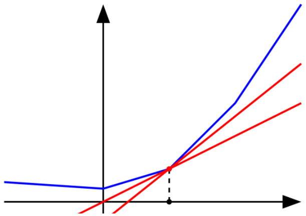

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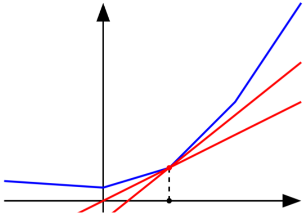

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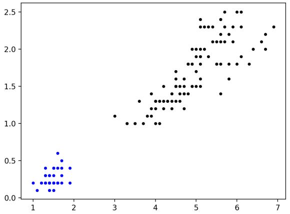


Iris dataset (D=2)

(linearly separable case)

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def SubGradientDescent(X,y,lr=1,eps=1e-18, max_iters=1000, lamb=1e-8):
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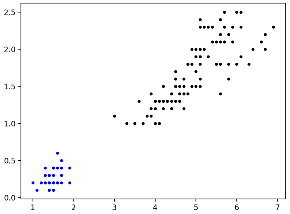


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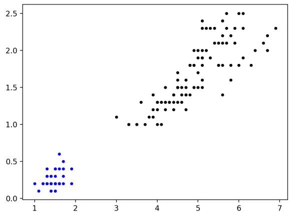

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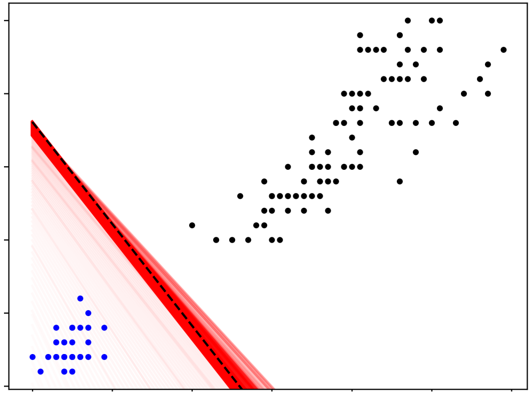
max-margin boundary (using small lambda $\lambda = 10^{-8}$ )



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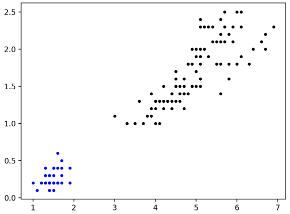


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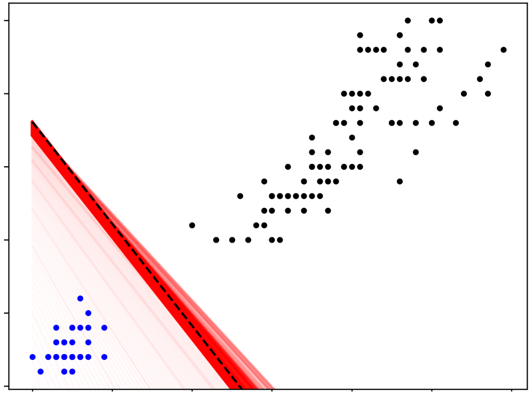
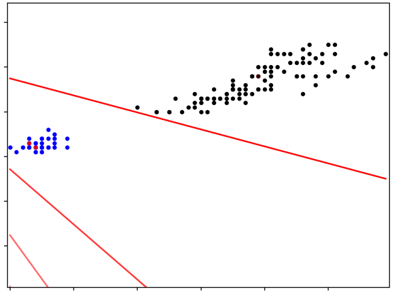


compare to Perceptron's decision boundary



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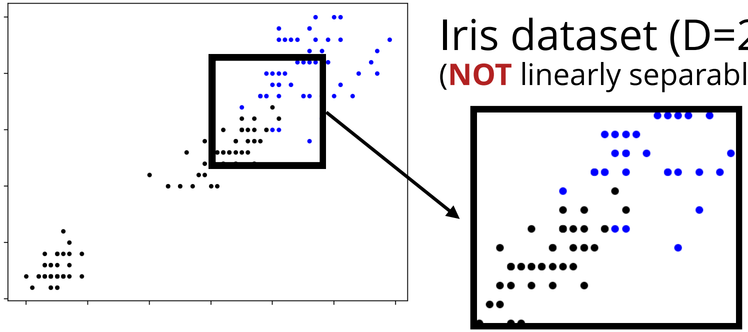

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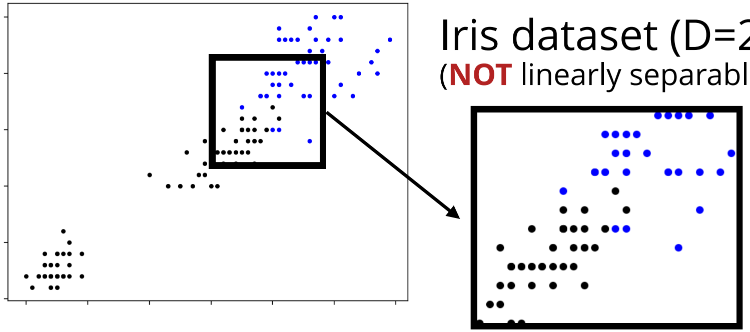

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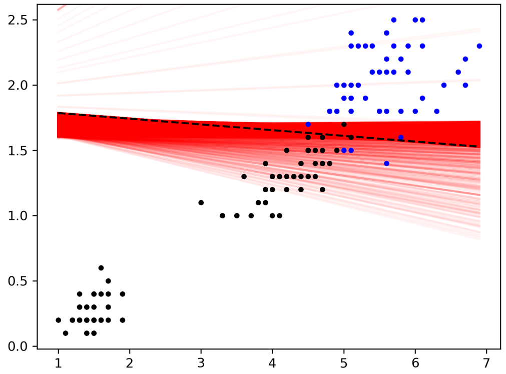

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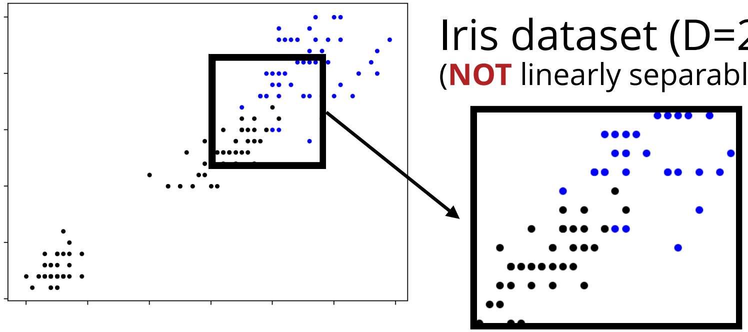

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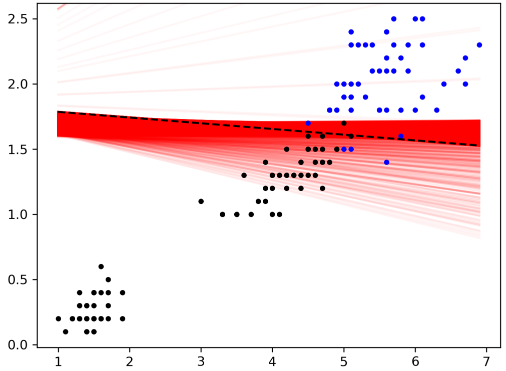
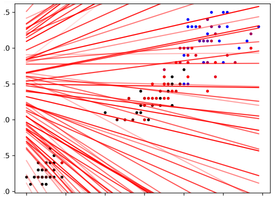



Perceptron does not converge



# SVM vs. logistic regression

recall: **logistic regression** simplified cost for $y \in \{0,1\}$

$$J(w) = \sum_{n=1}^{N} y^{(n)} \log\left(1 + e^{-z^{(n)}}\right) + (1 - y^{(n)}) \log\left(1 + e^{z^{(n)}}\right) \quad \text{where} \quad z^{(n)} = \boldsymbol{w}^\top \boldsymbol{x}^{(n)}$$

includes the bias

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for $y \in \{-1, +1\}$ we can write this as

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*also added L2 regularization*

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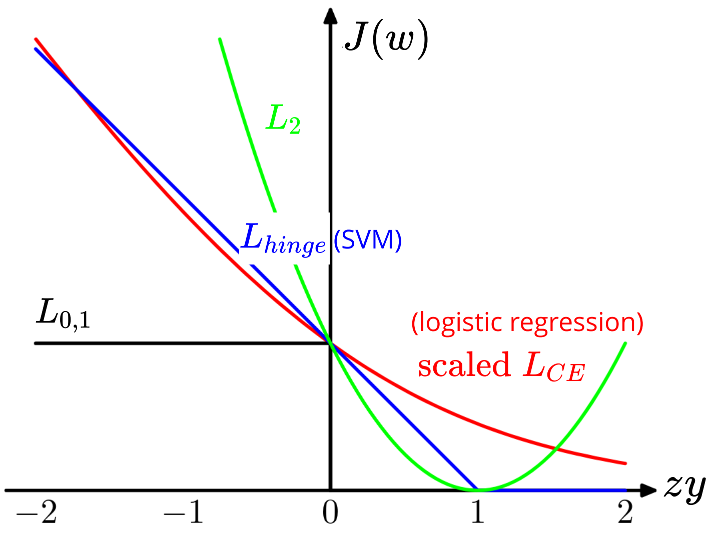
they both try to approximate 0-1 loss (accuracy)



# Multiclass classification

can we use multiple binary classifiders?

**one versus the rest**

image credit: Andrew Zisserman

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**one versus the rest**

**training:**

train C different 1-vs-(C-1) classifiers $z_c(x) = w_{[c]}^\top x$

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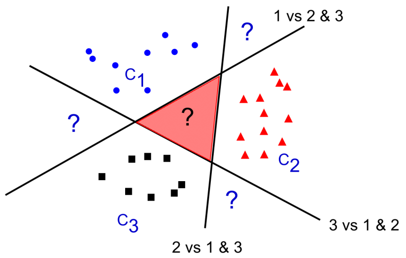


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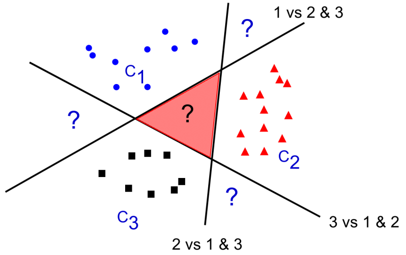
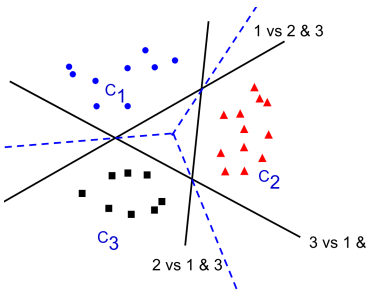



**test time:**

choose the class with the highest score

$$z^* = \arg\max_c z_c(x)$$

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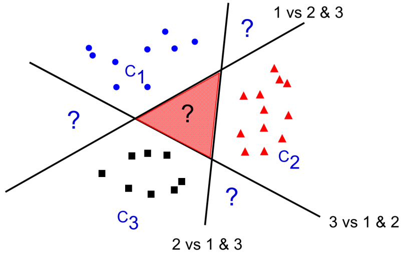


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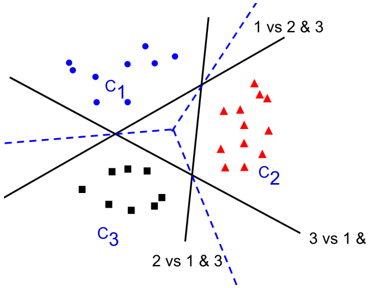


**problems:**

class imbalance

not clear what it means to compare $z_c(x)$ values

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**one versus one**

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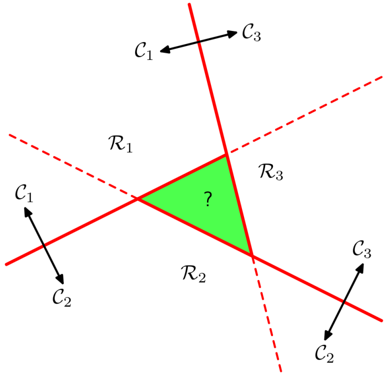

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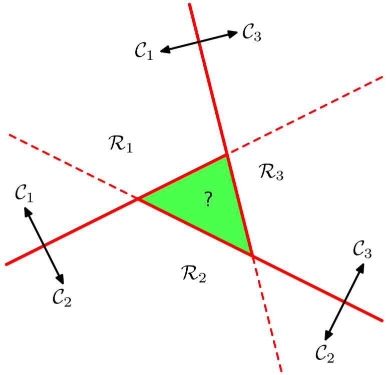


**problems:**

computationally more demanding for large C

ambiguities in the final classification

# Summary

- geometry of linear classification
- Perceptron algorithm
- distance to the decision boundary (margin)
- max-margin classification
- support vectors
- hard vs soft SVM
- relation to perceptron
- hinge loss and its relation to logistic regression
- some ideas for max-margin multi-class classification