Strongly Exponential Lower Bounds for Monotone Computation

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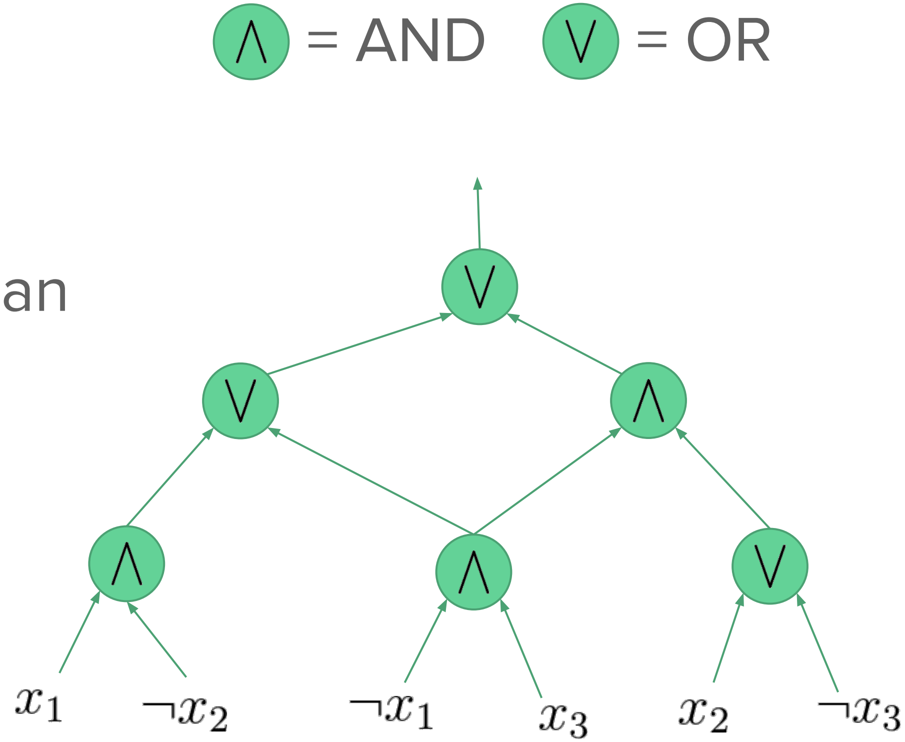
Boolean Circuits

Basic model for computing boolean functions $f : \{0,1\}^n \to \{0,1\}$

Assume fan-in 2, and a basis of AND, OR, NOT gates.



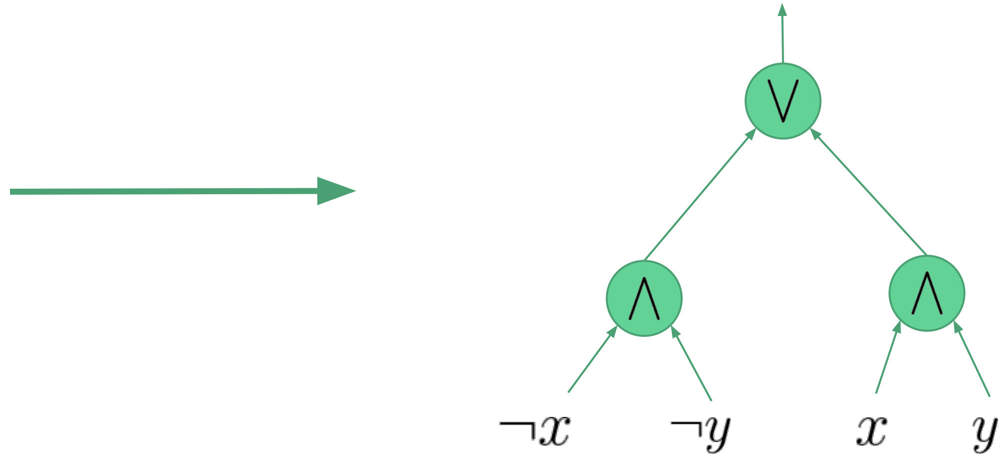
Central Question.

What boolean functions are hard to compute?

Boolean Circuits

Every $f:\{0,1\}^n \to \{0,1\}$ has a circuit of size $O(n2^n)$.

x	y	$f(x,y)$
0	0	1
0	1	0
1	0	0
1	1	1



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Theorem. [Shannon 1949] For every n, all but an exponentially small fraction of boolean functions on n bits require circuits with $\Omega\left(\frac{2^n}{n}\right)$ gates.

Proof. Simple counting argument (non-constructive).

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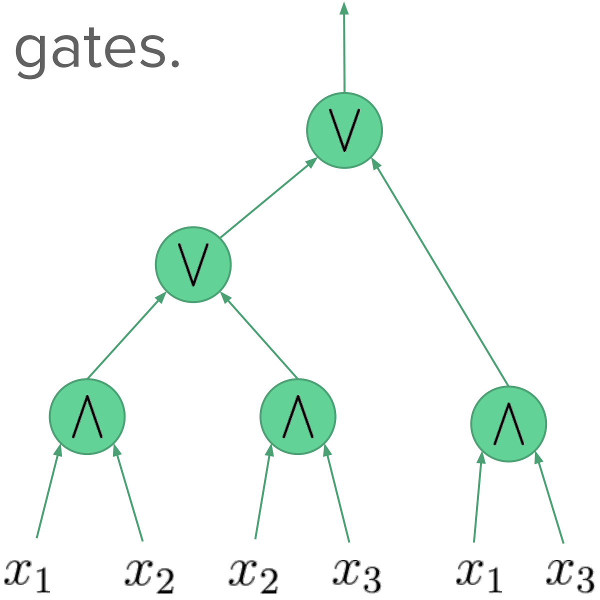
L	Switching Networks	$n^2/\log n$ [N. 66]	$2^n/n$ [S. 49]
Mod_p L	Span Programs	$n \log n$ [KW. 91]	**GF(2)** $\sqrt{2^{n+1}}$ [N.62]
CC	Comparator Circuits	$n \log n$ [KLMPSS. 95]	$2^n/n$ [S. 49]

Monotone Circuit Complexity

A circuit is *monotone* if it does not use NOT gates.



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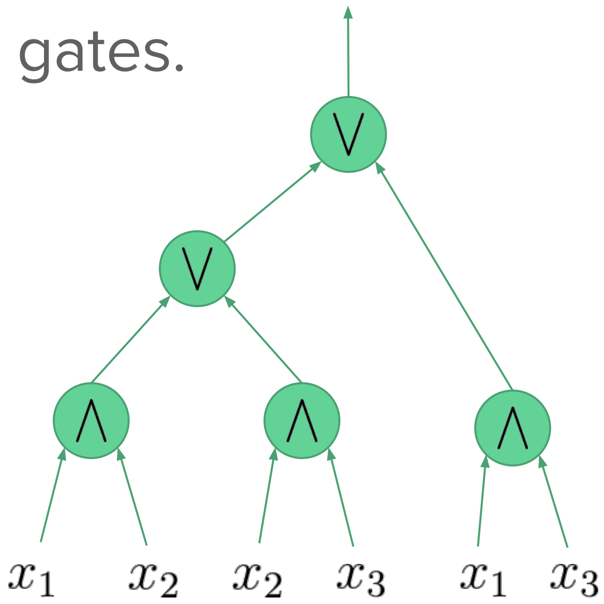

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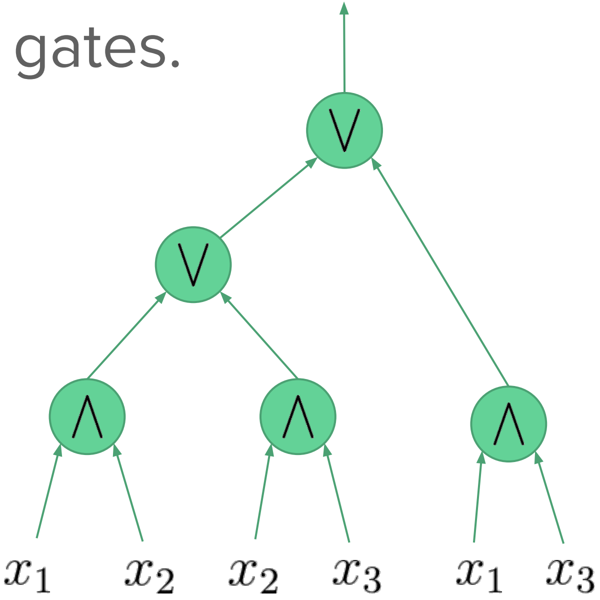
Monotone circuits have a number of applications in cryptography, proof complexity, communication theory ....



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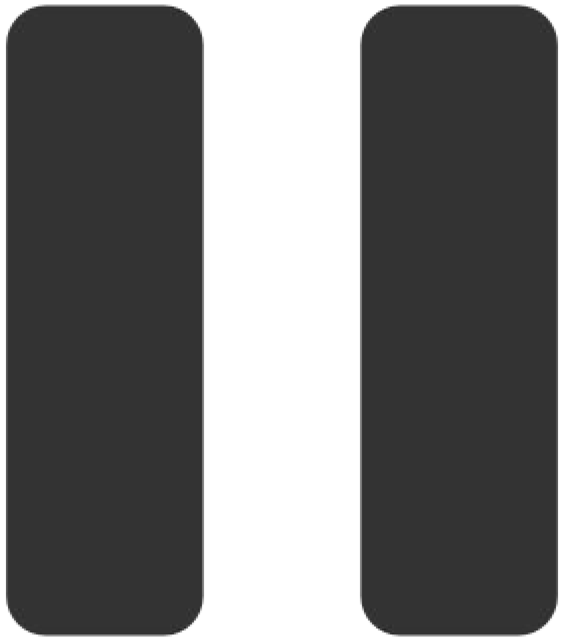
Result

Main Theorem. There is a monotone boolean function f computable in **NP** (CSP-SAT) such that every monotone

1. formula,
2. switching network,
3. real span program, or
4. comparator circuit

computing f requires size $2^{\alpha n}$ for some universal constant $\alpha > 0$.



The Proof (A Flavor)

Columns labelled with $y \in f^{-1}(0)$

Rows labelled with $x \in f^{-1}(1)$

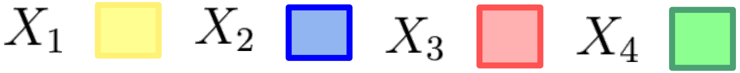
$f^{-1}(1) \times f^{-1}(0)$

Let $f : \{0,1\}^N \to \{0,1\}$ be a monotone boolean function.

$\text{KW-Search}^+(f) \subseteq f^{-1}(1) \times f^{-1}(0) \times [N]$

Input: $(x, y) \in f^{-1}(1) \times f^{-1}(0)$

Output: $i \in [N] \quad x_i = 1, y_i = 0$



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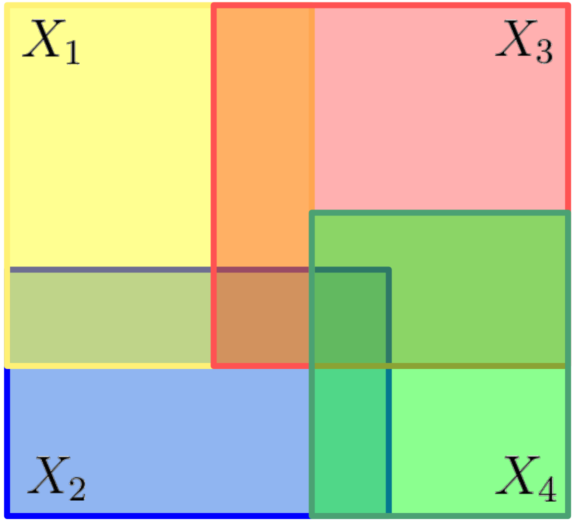

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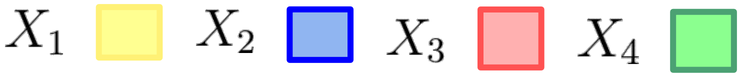

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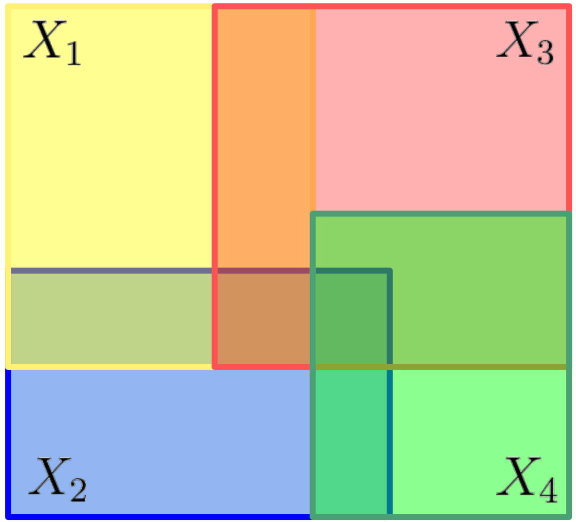

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Theme: Complexity of $\text{KW-Search}(f) \approx$ Circuit Complexity of f

Example: Formulas

$\wedge$ = AND $\vee$ = OR

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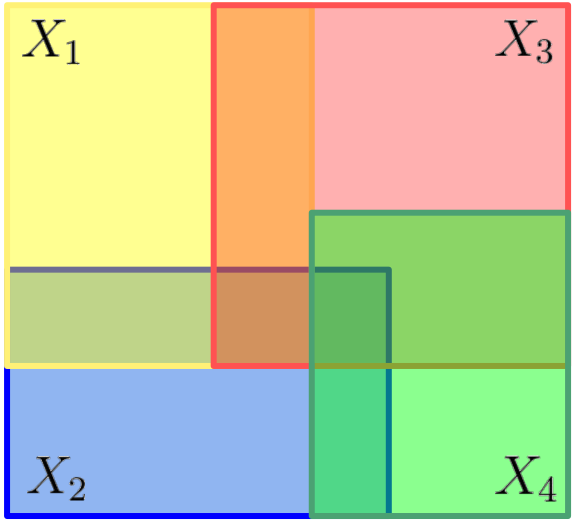

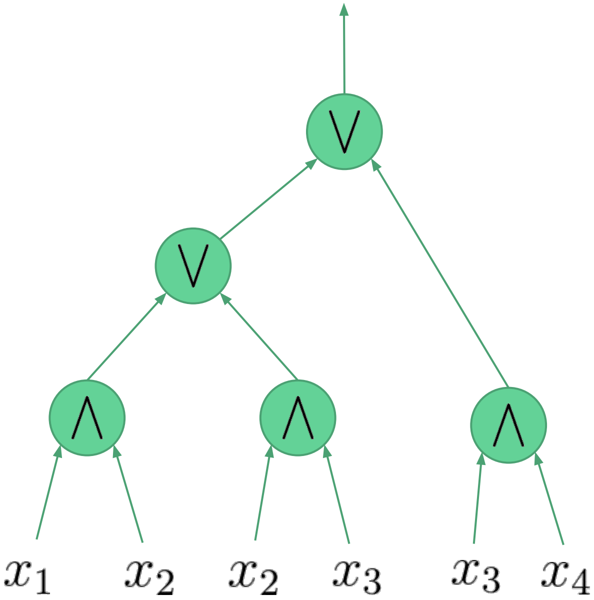

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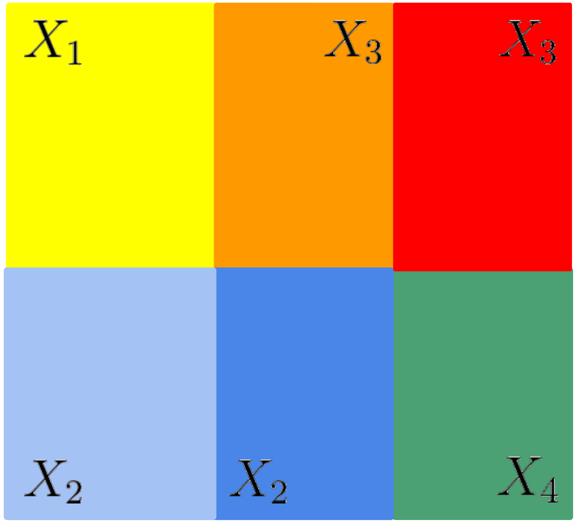

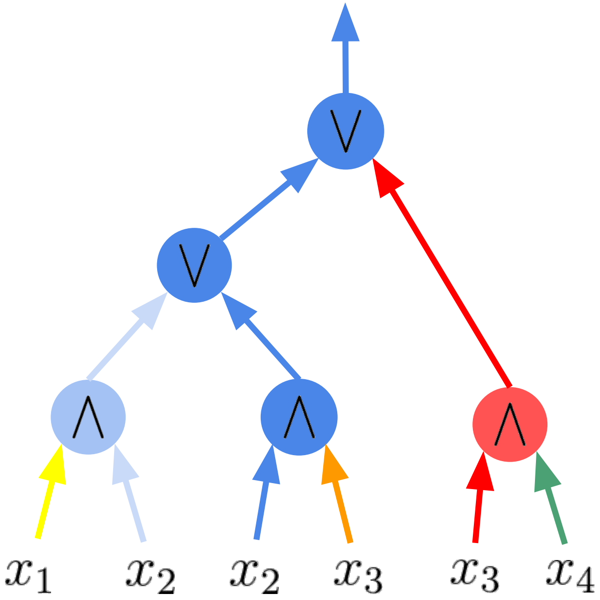

Lemma. [Khrapchenko 71] Formula for $f : \{0,1\}^n \to \{0,1\}$ with **s** leaves yields a partition of $f^{-1}(1) \times f^{-1}(0)$ into **s** mono. rectangles.

Let $\chi(f)$ denote the minimum number of rectangles in any monochromatic partition of $f^{-1}(1) \times f^{-1}(0)$

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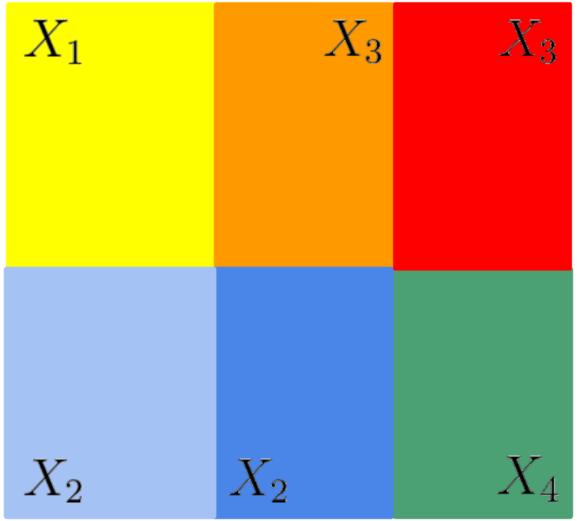

Idea [Razb. 90]: Use rank to lower bound $\chi(f)$!

Let $\chi(f)$ denote the minimum number of rectangles in any monochromatic partition of $f^{-1}(1) \times f^{-1}(0)$

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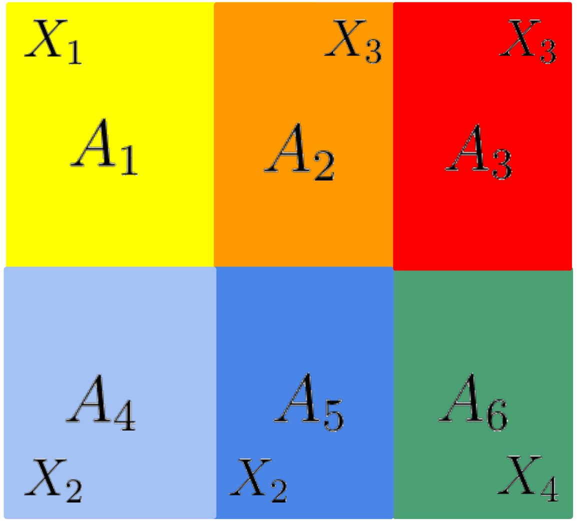
A

Rows labelled with $x \in f^{-1}(1)$



Idea [Razb. 90]: Use rank to lower bound $\chi(f)$!

Let A be any $|f^{-1}(1)| \times |f^{-1}(0)|$ matrix over a field **F**.

$$A = \sum_{i=1}^{\chi(f)} A_i$$

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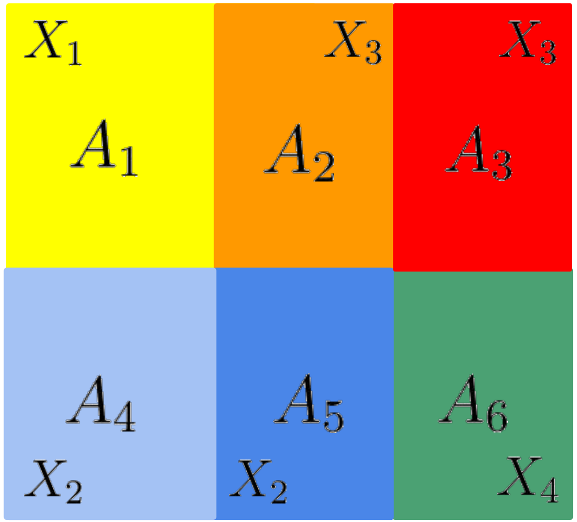

$$\operatorname{rank}(A) \leq \chi(f) \max_i \operatorname{rank}(A_i)$$

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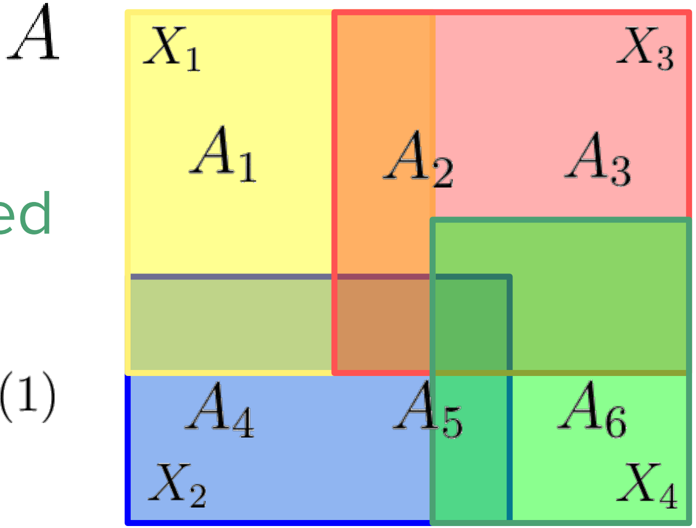

$$\mathrm{rank}(A) \le \chi(f) \max_i \mathrm{rank}(A_i)$$
$$\le \chi(f) \max_{i \in [n]} \mathrm{rank}(A \upharpoonright X_i)$$

Rearranging,

$$\chi(f) \ge \frac{\mathrm{rank}(A)}{\max_{i \in [n]} \mathrm{rank}(A \upharpoonright X_i)}$$

Rank Measure

Theorem [Razb. 90]. For any monotone boolean function f and any $f^{-1}(1) \times f^{-1}(0)$ matrix A over any field, the quantity

$$\mu_A(f) = \frac{\operatorname{rank}(A)}{\max_{i \in [n]} \operatorname{rank}(A \restriction X_i)}$$

is a lower bound on $\chi(f)$ (and the monotone formula size of f).

Theorem [G. 01, RPRC. 16]. $\mu_A(f)$ is also a lower bound on monotone switching networks, monotone span programs, and monotone comparator circuits computing f.

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Main Theorem (Restated). There is an explicit function f computable in NP and a matrix A such that $\mu_A(f) \geq 2^{\alpha n}$.

Proving Lower Bounds on $\mu_A(f)$

Theorem [Razb. 90] There is a monotone boolean function $f : \{0,1\}^n \to \{0,1\}$ in NP and a 0/1 matrix A satisfying

$$\mu_A(f) \geq n^{\Omega(\log n)}$$

[RPRC 16, PR 17] "Lifting theorem" to prove lower bounds against $\mu_A(f)$

1. Reduce lower bounds on $\mu_A(f)$ to **query complexity** lower bounds for a search problem $\mathrm{Search}(\mathcal{C})$ related to $\text{KW-Search}^+(f)$
2. Prove strong query complexity lower bounds for $\mathrm{Search}(\mathcal{C})$

Search Problems and Algebraic Gaps

$\mathcal{C} = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ is an unsatisfiable **k**-CNF with variables **z**.

$\mathrm{Search}(\mathcal{C})$:= given assignment to **z**, output index of falsified clause.

Algebraic Gap Complexity

Ex. $$\mathcal{C} = \overline{x}_1 \wedge \overline{x}_2 \wedge \cdots \wedge \overline{x}_n \wedge \left(\bigvee_{i=1}^{n} x_i \right)$$

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Certificate = minimal partial restriction falsifying a clause

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$$\mathrm{Cert}(\mathcal{C}) \quad x_1 = 1 \;\; x_2 = 1 \quad\quad x_n = 1 \quad x_1 = 0, x_2 = 0, \cdots, x_n = 0$$

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$\mathrm{Cert}(\mathcal{C})$ $\quad x_1 = 1 \quad x_2 = 1 \qquad x_n = 1 \qquad x_1 = 0, x_2 = 0, \cdots, x_n = 0$

Algebraic Gap Complexity. Find a polynomial $p : \{0,1\}^n \to \mathbb{R}$

so that $\mathrm{gap}_p(\mathcal{C}) = \deg(p) - \max_{\pi \in \mathrm{Cert}(\mathcal{C})} \deg(p \restriction \pi)$ is maximized.

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$p = OR_n \implies \deg(OR_n) = n$ and $\max_{\pi \in \mathrm{Cert}(\mathcal{C})} \deg(OR_n) = 0$

Algebraic Gap Complexity vs. Rank Measure

Algebraic Gap Complexity. Given $\mathrm{Search}(\mathcal{C})$, find polynomial $p : \{0,1\}^n \to \mathbb{R}$ so that $\mathrm{gap}_p(\mathcal{C}) = \deg(p) - \max_{\pi \in \mathrm{Cert}(\mathcal{C})} \deg(p \restriction \pi)$ is maximized.

Rank Measure $\mu_A(f)$. Given $f : \{0,1\}^N \to \{0,1\}$, find matrix A such that

$$\mu_A(f) = \frac{\mathrm{rank}(A)}{\max_{i \in [n]} \mathrm{rank}(A \restriction X_i)}$$

is maximized.

Rank Measure Lifting

Theorem [RPRC 16].

For any unsatisfiable **k**-CNF $\mathcal{C}$ with **m** clauses there is a function $f_{\mathcal{C}}$ computable in NP with $N \leq m^{2k+1}$ variables and a real matrix A such that

$$\mu_A(f_{\mathcal{C}}) \geq \Omega(m^{\mathrm{gap}(\mathcal{C})}) \geq \Omega(N^{\mathrm{gap}(\mathcal{C})/2k+1})$$

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[RPRC 16]. $\mathcal{C}$ = “pebbling contradiction”, then $\mathrm{gap}(\mathcal{C}) \geq m/\log m \geq \tilde{\Omega}(N^{1/2k+1})$

Yields $2^{\Omega(N^{\varepsilon})}$ lower bounds!

Problem is the number of variables!

Gadget Size Blues

Query Complexity		$\leq$	**Circuit Complexity**	
Decision Tree Depth	[RM 99]		Monotone Circuit Depth	[RM 99]
Critical Block Sensitivity	[HN 12, GP 16]		Avg. Case Monotone Depth	[HN 12, GP 16]
Algebraic Gap Complexity	[RPRC 16]		**(Logarithm of) Rank Measure**	[RPRC 16]

For decision trees vs. depth, current constructions yield $N = \omega(m)$ variables.

For critical block sensitivity, we can take $N = O(m)$ variables, but best query lower bounds are $\Omega(m/\log m)$.

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Rank Measure Lifting (Refined)

Theorem [PR 17].

For any unsatisfiable **O(1)**-CNF $\mathcal{C}$ with **m** clauses **satisfying** $\mathrm{gap}(\mathcal{C}) = \Omega(m)$ there is a function $f_{\mathcal{C}}$ computable in NP with $N = O(m)$ variables and a real matrix A such that

$$\mu_A(f_{\mathcal{C}}) \geq 2^{\Omega(m)} \geq 2^{\Omega(N)}$$

Rank Measure Lifting (Refined)

Theorem [PR 17].

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$$\mu_A(f_{\mathcal{C}}) \geq 2^{\Omega(m)} \geq 2^{\Omega(N)}$$

Proof. [RPRC 16] $\text{KW-Search}^{+}(f_{\mathcal{C}}) \equiv \text{Search}(\mathcal{C} \circ g^n(x, y))$

Rank of **pattern matrix** $A = [p(g^n(x, y))]_{x,y \in \mathcal{X}^n \times \mathcal{Y}^n} \approx \exp(\deg(p))$

Proving Large Algebraic Gaps

Algebraic Gap Complexity. Given $\mathrm{Search}(\mathcal{C})$, find polynomial $p : \{0,1\}^n \to \mathbb{R}$ so that $\mathrm{gap}_p(\mathcal{C}) = \deg(p) - \max_{\pi \in \mathrm{Cert}(\mathcal{C})} \deg(p \restriction \pi)$ is maximized.

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Theorem. $\mathrm{gap}(\mathrm{Tseitin}_G) \geq \mathrm{Expansion}(G) \cdot m/3d$

Proof. Reduction to resolution width of $\mathrm{Tseitin}_G$

Rank Measure Lifting

Theorem [PR 17]. For any unsatisfiable **O(1)**-CNF $\mathcal{C}$ with **m** clauses **satisfying** $\text{gap}(\mathcal{C}) = \Omega(m)$ there is a function $f_\mathcal{C}$ computable in NP with $N = O(m)$ variables and a real matrix A such that

$$\mu_A(f_\mathcal{C}) \geq 2^{\Omega(m)} \geq 2^{\Omega(N)}$$

Theorem. $\text{gap}(\text{Tseitin}_G) \geq \text{Expansion}(G) \cdot m/3d$

Choose G to be a strong constant-degree expander and the main theorem is proved!

Conclusion

Prove the first strongly exponential lower bounds for any explicit function, asymptotically matching non-explicit lower bounds from counting in the monotone setting.

Can we sharpen it further?

Further applications of the framework? (In particular, a deeper understanding of the **algebraic gap complexity** and other exotic query complexity measures for search problems.)

Thanks!