F := unsatisfiable CNF formula

\[ F = C_1 \land C_2 \land \ldots \land C_m \], each \( C_i \) is a clause (OR of lift)

Resolution Refutation of \( F \) is a list of clauses

\[ D_1, \ldots, D_m, D_m \rightarrow \ldots \rightarrow D_s = \bot \leftarrow \text{empty clause} \]

F All other clauses are obtained from earlier clauses by the resolution rule

\[
\frac{A \lor x \quad B \lor \overline{x}}{A \lor B}
\]

ex)f \( x_1 \land (x_1 \lor x_2) \land (x_2 \lor x_3) \land \overline{x_3} \)

\[
\frac{x_2 \quad \overline{x_2}}{1} \]

Defn A resolution refutation is tree-like if the graph underlying the refutation is a tree.

(i.e. every derived clause is used at most once.)

- Tree-like resolution can be exponentially weaker than general resolution, but, it is still complete.
**Complexity Measures of Refutations**

$F := \text{unsat CNF, on } n \text{ variables}$

$S_{\text{Res}}(F) := \min \# \text{ of clauses in any resolution ref. of } F$

$S_{\text{Tres}}(F) := \text{tree-like res. ref. of } F$

$D_{\text{Res}}(F) := \min \text{ depth of any resolution ref. of } F$

**Example DAG-like proof**

$F = y \land (\overline{y} \lor x) \land (x \lor z) \land (x \lor \overline{z})$

$w(F) := \frac{\text{width of the largest clause in } F}{\# \text{ of lits}}$

$w_{\text{Res}}(F) := \text{minimum width of any resolution refutation of } F.$

Assumes $F$ is minimally unsat, i.e. if we delete a clause it is SAT (safe assumption)

- $w(F) \leq w_{\text{Res}}(F) \leq D_{\text{Res}}(F) \leq n$
- $S_{\text{Res}}(F) \leq S_{\text{Tres}}(F) \leq 2^{O_{\text{Res}}(F)(\pm 1)}$
- $S_{\text{Res}}(F) \leq \frac{3}{4} \# \text{ of clauses of width } \leq w_{\text{Res}}(F)$
In particular, if $\text{wres}(F) = O(1)$ then $F$ has a polynomial-size proof.

Thm. [Haken 85] Any resolution refutation of $\text{PHP}_{n+1}^n$ requires length $2^{\omega(n)}$.

$\text{PHP}_{n+1}^n$ is defined on $(n+1)n$ variables

$X_{ij} \quad i \in [n+1] \ , j \in [n]$  

$X_{ij} = 1 \iff \text{Pigeon } i \text{ mapped to hole } j$

Clauses:

$\forall j \in [n] \quad \bigvee_{i=1}^{n} X_{ij}$ for all $i \in [n+1]$

$\bigvee_{i \neq k} X_{ij} \lor X_{kj}$ for all $i \neq k \in [n+1], j \in [n]$.

Proof. Two steps

(1) Any "proof" of $\text{PHP}_{n+1}^n$ requires a wide clause

(2) There is a partial restriction $\rho \in \{0,1,*\}^{(n+1)n}$ such that

$\text{PHP}_{n+1}^n \upharpoonright \rho = \text{PHP}_{m}^m$

$C \upharpoonright \rho := \text{plug } \rho \text{ into}$

- All wide clauses will be satisfied by $\rho$

in the proof.
Defn An assignment \( \alpha \in \{0,1\}^{(n+1)n} \) is \( i \)-critical if the only clause falsified in \( \text{PHP}_n^{n+1} \) is 
\[ \bigvee_{j=1}^{n} x_{ij}^\alpha. \]

2-critical

Defn If \( C \) is a clause over \( x_{ij} \) vars, let \( C^+ \) be the clause obtained by replacing every negative literal \( \overline{x_{ij}} \) with 
\[ \bigvee_{k \neq j} x_{ik}. \]

Let \( \Pi \) be a resolution proof of \( \text{PHP}_n^{n+1} \) let 
\[ \Pi^+ = \{ C^+ \mid C \in \Pi \}. \]

"relativization"

Claim \( \Pi^+ \) contains a clause \( C^+ \) with \( \mu(C^+) \geq \frac{n^2}{q} \).

Proof For any clause \( C \), define 
\[ \text{Crit}(C) := \{ i \in [n+1] : C(\alpha) = 0 \text{ for an } i\text{-crit assign } \alpha \} \]
\[ \mu(C) := |\text{Crit}(C)|. \]

If \( C \) is a clause of \( \text{PHP}_n^{n+1} \),
- \( \mu(C) = 0 \) if \( C \) is a "hole" clause
- \( \mu(C) = 1 \) if \( C \) is a "pigeon" clause
\[ \mu(\bot) = n+1. \]
If \( A = \text{Res}(B, C) \)

\( \mu(A) \leq \mu(B) + \mu(C) \)

\( A(x) = 0 \) for \( i \)-crit \( \alpha \), then \( \text{either } B(x) = 0 \text{ or } C(x) = 0 \)!

\[ \therefore \text{Let } C \text{ be any clause in the proof } \Pi \text{ with } \frac{n}{3} < \mu(C) \leq \frac{2n}{3} \text{ (uses subadditivity).} \]

Let \( i \in \text{Crit}(C) \), \( j \notin \text{Crit}(C) \).

Let \( \alpha \) be \( i \)-crit, s.t. \( C(\alpha) = 0 \)

Go from \( \alpha \) to \( \alpha' \) which is \( j \)-critical by setting

\[ X_{ik} = 1 \quad X_{jk} = 0 \]

But \( C(\alpha') = 1 \) — so \( X_{ik} \) appears in \( C^+ \)!

Apply the same argument to all \( i \in \text{Crit}(C) \), \( j \notin \text{Crit}(C) \)

then

\[ w(C^+) \geq \mu(C) (n - \mu(C)) \geq n^2 \frac{1}{q}. \]

Aside: example of CNF formula \( F \) with small resolution proofs but large tree-like resolution proofs?

Answer: Q3 any Horn formula that is unsatisfiable has a polynomial-size Res. ref.
\[ C^+ - \text{obtained from } C \text{ by replacing } \]
\[ \overline{x_{ij}} \rightarrow \bigvee_{j \neq k} x_{ik} \]

**Fact** If \( \alpha \) is an \( i \)-critical assignment for some \( i \), then
\[ C(\alpha) = C^+(\alpha) \]

\( C(\alpha) = 1 \) why is \( C^+(\alpha) = 1 \)?

\( C^+(\alpha) = 1 \)
\[ x_{ik} = 1 \in C^+ \]
\[ \Rightarrow C \text{ contained } \overline{x_{ij}} \text{ for some } j \neq k \]

\( \alpha \) is \( i \)-critical \( \Rightarrow \) \( i \) not mapped (\( \overline{x_{ij}} = 1 \))

or

\( j \)-critical \( \Rightarrow \)
Horn := every clause has \leq 1 positive literal

\[ \forall x, \exists y \exists z \]

\[ x_1 \lor x_2 \lor x_3 \lor \overline{x_1} \lor \overline{x_4} \lor x_5 \]

\[ S_{\text{res}}(F \lor \neg x_1) \geq 2 \text{dres}(F) \]

Exercise: There is a Horn formula requiring large depth!

Defn If \( C \) is a clause over \( x_{ij} \) vars, let

\[ C^+ \text{ be the clause obtained by replacing every negative literal } \overline{x_{ij}} \text{ with } \]

\[ \bigvee_{k \neq j} x_{ik} \]

Let \( \Pi \) be a resolution proof of \( \text{PHP}^{n+1} \), let

\[ \Pi^+ = \{ C^+ \mid C \in \Pi \} \]

"relativization"

Claim \( \Pi^+ \) contains a clause \( C^+ \) with \( w(C^+) \geq \frac{n^2}{q} \).

Today: How do we kill all the wide clauses?

Notice all clauses \( C^+ \) are ORs of positive literals. So, restricting any variable in \( C^+ \) to 1 will kill the clause.

Say a clause \( C \) in the proof \( \Pi \) is wide if \( w(C^+) \geq \epsilon n^2 \) (choose \( \epsilon \) later).

Since every wide clause has an \( \epsilon \)-fraction of the variables, by averaging there is some literal \( x_{ij} \).
occurring in \( \geq eS \) of the wide clauses \( (\text{where } S \text{ is the } \# \text{ of wide clauses}) \).

- Pick \( x_{ij} \), set \( x_{ij} = 1 \), set \( x_{ik} = 0 \) for all \( k \neq i \).
- After this restriction we're left with \( \text{PHP}^n_{n-1} \) and the restricted proof is a refutation of the new instance.

How many times until all wide clauses are gone?

- After \( d \) restrictions, we have \( (1-e)^d S \) wide clauses remaining. To kill all wide clauses, we need

\[
(1-e)^d S \leq e^{-d \epsilon} S < 1.
\]

\[
\Rightarrow \ln S < d \epsilon \iff \frac{\ln S}{\epsilon} < d
\]

Choose \( d = \frac{\ln S}{\epsilon} \). After \( d \) restrictions, we have a proof of \( \text{PHP}^{n_1-d}_{n-d} \) with no wide clauses. By the Claim, there is a clause of width

\[
\frac{(n-d)^2}{q} \geq \frac{(n - \ln S / \epsilon)^2}{q}
\]

So, if \( (n-d)^2 / q \geq \epsilon n^2 \) we have a contradiction. Towards this, assume \( S \leq e^{\frac{\epsilon n}{4}} \), then

\[
\frac{(n-d)^2}{q} \geq \frac{(n - \frac{\epsilon n}{4})^2}{q} = \frac{n^2}{16}
\]

Then, if \( \epsilon < \frac{1}{16} \) we have a contradiction. \( \therefore S \geq e^{\frac{\epsilon n}{4}} \). \( \square \)
Thm For any unsat CNF $F$, we have

1. $S_{\text{Res}}(F) \geq 2^{\frac{W_{\text{Res}}(F) - w(F)}{16n}}$

2. $S_{\text{Res}}(F) \geq 2^{\frac{W_{\text{Res}}(F) - w(F)}{16n}}$

"width gap is $w(\ln n)$ then lower bds"

(1) cannot naively be applied to get $\mathsf{P} \neq \mathsf{NP}$ lower bds

Pf (1)

Prove by induction on $b$ (parameter) and $n$ ($\# \text{vars}$) that if

$S_{\text{Res}}(F) \leq b$

then $W_{\text{Res}}(F) \leq b + w(F)$.

Notation

$x^0 := \overline{x}$, $x^1 := x$

$F \upharpoonright x=a :=$ New CNF formula from $F$ by substituting $a \in \{0,1\}$ for $x$.

So, if $C \subseteq F$ contains $x^a$ we remove $C$, if $x^{1-a} \in C$ we delete $x^{1-a}$ from the clause.

Claim If $W_{\text{Res}}(F \upharpoonright x=a) \leq K$ and

$W_{\text{Res}}(F \upharpoonright x=1-a) \leq K-1$ then

$W_{\text{Res}}(F) \leq \max \{ K, w(F) \}$. 
For simplicity, we assume resolution has the weakening rule

$$\frac{C}{C \lor x}$$

for any $x \in C$.

(Note: This just makes the proof cleaner and can be removed.)

How to combine $\Delta$ and $\Delta'$ into a refutation of $F$?

- Let $\Pi_1$ be the refutation of $F \land x = a$, in width $k-1$.
- Let $\Pi_1'$ be obtained by adding $x^{1-a}$ to every clause in $\Pi_1$.

- Every clause at start of the proof of $\Pi_1'$ is either a clause in $F$ or a weakening of a clause in $F$.

$$x^{1-a} \lor A \lor x \quad x^{1-a} \lor B \lor \overline{x}$$

- Observe adding $x^{1-a}$ doesn't affect the correctness of any resolution step.
Let \( \{1\} \leq b \)

\text{(Almost) done!}

If \( b = 0 \) then \( |\{1\}| = 1 \), \( \mu \) means \( LEF \).

Now substitute the new proof to derive \( \mu x = 1 \).

\( \mu \)

\( \mu - a \)

\( \mu \) width \( \mu \)

\( \mu \) width \( \mu \)

\( \mu \) width \( \mu \)

\( \mu \) width \( \mu \)

\( \mu \) width \( \mu \)

\( \mu \) width \( \mu \)
Otherwise: the last step of \( \Pi \) resolved two literals \( x \) and \( \overline{x} \).

\[
\begin{array}{c}
\Pi \\
\setminus
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\setminus \\
x
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\setminus \\
\overline{x}
\end{array}
\end{array}
\]

Size \( \leq 2^b \)

Assume w.l.o.g. \( |\Pi_L| \leq |\Pi_R| \), so \( |\Pi_L| \leq \frac{|\Pi|}{2} \leq 2^{b-1} \).

- Induction on \( b \) for \( \Pi_L \), we get a width \( b-1 \) proof of \( F \wedge x = 0 \).

- Induction on \( n \) for \( \Pi_R \) we get a width \( b \) proof of \( F \wedge x = 1 \).

Apply the claim and we are done!

This will massively restructure the proof — low width at cost of doubly-exponential blow-up in size!
Q. Do you have to pay this cost?
(i.e. can we optimize width and size at the same time?)

No!

[Razborov 2016] Doubly-exponential blow-up is necessary for some formulas!

Next time:

$$S_{\text{Res}}(F) \geq 2 \frac{(\text{w}_{\text{Res}}(F) - \text{w}(F))^2}{16n}.$$  

$$\begin{align*}
\forall \sqrt{\gamma} & \quad z_1 \land z_2 = (\overline{z_1} \lor \overline{z_2}) \land (z_1 \lor z_2) \\
\overline{z_1 \land z_2} & = (\overline{z_1} \lor \overline{z_2}) \land (\overline{z_1} \lor z_2) \\
((z_1 \lor z_2) \land (\overline{z_1} \lor z_2)) & \lor ((\overline{z_3} \lor z_4) \land (\overline{z_3} \lor z_4)) \\
\end{align*}$$

(rewrite in CNF.)