Max-SAT

Input: CNF formula \( F = C_1 \land C_2 \land \cdots \land C_m \) on \( n \) vars \( x_1, \ldots, x_n \)

Goal: Find assignment \( x \in \{0,1\}^n \) maximizing the 

\# of SAT clauses.

For clause \( C = \bigvee_i x_i \lor \bigvee_j \overline{x_j} \) let \( \tilde{C}_i = \sum_i x_i + \sum_j (1-x_j) \)

**Integer Linear Program**

\[
\text{max } \sum_{i=1}^{m} \tilde{C}_i \\
\text{s.t. } \tilde{C}_i \geq c_i \quad \forall i = 1 \ldots m \\
0 \leq x_j \leq 1 \quad \forall j = 1 \ldots n \\
0 \leq c_i \leq 1 \quad \forall i = 1 \ldots m \\
x_i, c_i \in \mathbb{Z}
\]

\[\uparrow\]

- Represents **Max-SAT exactly**
- NP-Hard to solve

**LP Relaxation**

\[
\text{max } \sum_{i=1}^{m} \tilde{C}_i \\
\text{s.t. } \tilde{C}_i \geq c_i \quad \forall i = 1 \ldots m \\
0 \leq x_j \leq 1 \quad \forall j = 1 \ldots n \\
0 \leq c_i \leq 1 \quad \forall i = 1 \ldots m \\
\]

\[\uparrow\]

- Doesn’t represent **Max-SAT exactly**
- Can be solved in poly time!
Q. Is there a linear program that solves Max-SAT exactly?

A. Yes! Take the convex hull of integral points.

Problems

1. How do we get it?
   A: Sherali-Adams

2. There might be exponentially many inequalities in any description.
Today we focus on finding the \textit{integral hull} using the \textit{SA hierarchy}.

\textbf{Prelims}

- If $\vec{y}_1, \ldots, \vec{y}_m \in \mathbb{R}^n$ then a convex combination of the $y_i$s is any point of the form
  \[ \sum_{i=1}^{m} \alpha_i \vec{y}_i, \quad \alpha_i \geq 0, \quad \sum \alpha_i = 1 \]

- If $C \subseteq \mathbb{R}^n$ then $\text{conv}(C) \subseteq \mathbb{R}^n$ is all convex combos of points in $C$.

\[ C \ni c \mapsto \begin{cases} \text{red points} & \text{if } c \in \mathbb{R}^n \setminus C \\ \text{blue points} & \text{if } c \in \text{conv}(C) \end{cases} \]

- If $C \subseteq \mathbb{R}^n$ then $\text{hull}_Z(C) := \text{conv}(C \cap \mathbb{Z}^n)$
• We're given polytope \( P \subseteq [0,1]^n \) i.e. the constraints
\[ 0 \leq x_i \leq 1 \text{ are in } P. \]

• We want to describe \( \text{hull}_Z(P) \).

• Since \( P \subseteq [0,1]^n \), every point in \( \text{hull}_Z(P) \) can be written as
\[ \alpha \in \text{hull}_Z(P) \Rightarrow \alpha = \sum_{x \in [0,1]^n} \lambda_x x, \lambda_x \geq 0 \]

• Equivalently: \( \alpha \in \text{hull}_Z(P) \) represents a probability distribution of
\[ x \in P \cap [0,1]^n \]

• For each \( \alpha \) let \( \mu^{(\alpha)} : [0,1]^n \to \mathbb{R} \) s.t.
\[ \mu^{(\alpha)}(x) = \lambda_x \]
so \( \mu^{(\alpha)} \) is a prob. dist. over \( [0,1]^n \cap P \)

**Observation**: If we can test if \( \alpha \in \mathbb{R}^n \) represents a valid prob. dist. over \( P \cap [0,1]^n \), then \( \alpha \in \text{hull}_Z(P) \).

So: how do we test if \( \alpha \) represents a probability distribution?
\[ \mu : [0,1]^n \to \mathbb{R} \quad \mu(x) \geq 0 \text{ and } \sum \mu(x) = 1 \]
We modify this by adding more tests to verify that we are in $P$.

Sufficient to show $\mu$ is dist over $\mathbb{E}_0,\mathbb{I}^n$. Not for $\mathbb{E}_0,\mathbb{I}^n \cap P$

**Specific:** Test the **marginal distributions** of $\mu$.

If $S \subseteq [n]$ then

$$\mu_S : \{0,1\}^S \rightarrow \mathbb{R}$$

is defined by

$$\mu_S(\alpha) := \sum_{x \in \mathbb{E}_0,\mathbb{I}^n} \mu(x), \quad x |_S = \alpha$$

(If $\mu$ is a real prob. dist then)

$$\mu_S(\alpha) = \Pr_{x \sim \mu} \left[ \forall i \in S : x_i = \alpha_i \right]$$

**Lemma** $\mu : \{0,1\}^n \rightarrow \mathbb{R}$ is a prob. dist on $\mathbb{E}_0,\mathbb{I}^n$ iff

$$\forall S \subseteq T \subseteq [n], \forall \alpha \in \{0,1\}^S$$

1. $\mu_S(\alpha) = \sum_{\beta \in \{0,1\}^T} \mu_T(\beta)$  marginal $\alpha$ agree

2. $\mu_S(\alpha) \geq 0$  non-negativity

3. $\mu_\phi = 1$  normalizing  \( \text{explains} \ 1 \geq 0 \)
\[ \text{PF (\(\Rightarrow\)) Easy} \]
\[ (\Leftarrow) \text{Define } \Pr_{x \sim \mu}[x = y] := \mu(y) \]
\[ 1 = \mu \emptyset = \sum_{x \in \mathcal{X}_0,\mathcal{X}_1^n} \mu(x) \quad \square \]

Answer: Where do non-negative juntas come from?

\[ y \in \mathcal{X}_0,\mathcal{X}_1^n, \text{ let } S = TuU \text{ so that} \]
\[ \begin{align*} 
\mu_S(y) & = \sum_{i \in T} 1 \\
Y_i & = \begin{cases} 
1 & \text{if } i \in U \\
0 & \text{if } i \notin U 
\end{cases} \\
\Pr_{x \sim \mu}[x|_S = y] & = \Pr_{x \sim \mu}[\Pi_{i \in T} x_i \cdot \Pi_{j \in U} (1 - x_j) = 1] 
\end{align*} \]

Non-negative juntas are random variables that describe the marginals of probability distributions over \(\mathcal{X}_0,\mathcal{X}_1^n\).

This lemma is going to give a (somewhat crazy) LP for \([0, 1]^n\).

- For each \(S \subseteq [n]\) let \(Y_S \in \mathbb{R}\) be a variable for \(S\).

Intuitively:
\[ Y_S = \mu_S(\hat{1}) = \Pr_{x \sim \mu}[x_i = 1 \ \forall i \in S] \]
\[ = E_{\mu} \left[ \prod_{i \in S} x_i \right] \leq \text{moment for } S \]
Define an LP on $\{y_s \mid S \subseteq [n]^2 \}$ with constraints

(a) $\gamma \phi = 1$

(b) $\forall S \subseteq [n], \forall TuU = S, TuU = \emptyset$

$$\sum (-1)^{u'} \gamma_{u'uT} \geq 0 \quad (1 \geq 0)$$

Claim $(1), (2) \Rightarrow (b)$

$S \subseteq T$

(1) $\mu_S(\beta) \geq 0$

(2) $\mu_S(\beta) = \sum_{\beta \in \phi, 1 \subseteq T} \mu_T(\beta)$

$0 \leq \mu_S(\beta) = \Pr_{x \sim \mu} [x_{\cap S} = \beta]$

(1) $0 \leq \mu_S(\beta) = \Pr [\prod_{i \in T} x_i \prod_{j \in U} (1-x_j) = 1]$

$= \mathbb{E} [\prod_{i \in T} x_i \prod_{j \in U} (1-x_j)]$

$= \sum \mathbb{E} [(-1)^{u'} \prod_{i \in u' \cap T} x_i]$

$u' \subseteq u$

$= \sum (-1)^{u'} \gamma_{u'uT}$
How to include constraints from $P$?

Write $P$ as

$$Ax \leq b = \begin{cases} a_1 \cdot x \leq b_1 \\ a_2 \cdot x \leq b_2 \\ \vdots \\ a_m \cdot x \leq b_m \end{cases} \quad 0 \leq x \leq 1$$

Let $Q = \left\{ b_1 - a_1 x \geq 0, \ldots, b_m - a_m x \geq 0 \right\}$

**Defn** The degree-$d$ Sherali-Adams tightening of $Q$ is obtained by the following two steps

1. For each inequality $q_i \geq 0$ in $Q$ add
   $$\sum_{s,T} q_i \geq 0$$
   to $Q$ where $|S \cup T| \leq d$, $S \cap T = \emptyset$

2. For every inequality $p_i \geq 0$ created in the last step, linearize $p_i$ by
   - Replace each $x_i$ term with $x_i$
   - For every monomial $\prod_{i \in S} x_i$, replace it with $y_S$.

**Lemma** (Next class) The degree-$n$ Sherali-Adams tightening is exactly $\text{hull}_Z(P)$. 

1. \hspace{1cm} \text{hull}_Z(P)$.
Max-SAT

Let \( F = (x_1 \lor x_2) \land (x_1 \lor \bar{x}_2) \land (\bar{x}_1) \) \n
LP Relaxation

Constraints:

\[
\begin{align*}
    x_1 + x_2 & \geq x_3 \\
    x_1 + (1-x_2) & \geq x_4 \\
    (1-x_1) & \geq x_5 
\end{align*}
\]

The degree-2 SAT tightening would add

\[
\begin{align*}
    y_{3i}z & \geq 0 \quad (1-y_{3i}z) \geq 0 \quad \forall i, j, i \neq j \\
    y_i (1-y_j) & \geq 0 \quad y_i y_j & \geq 0 \\
    y^0_i - y_{3i}y_j & \geq 0 \quad y_{3i}y_j & \geq 0 
\end{align*}
\]

Then, we multiply each constraint in the LP by a non-neg junta and linearize

\[
\begin{align*}
    x_1 + x_2 & \geq x_4 \implies x_i^o (x_1 + x_2 - x_4) & \geq 0 \\
        & \quad (1-x_i^o) (x_1 + x_2 - x_4) & \geq 0 \\
        & \quad \forall i, j, i \neq j \\
        & \quad x_i^o x_j^o (x_1 + x_2 - x_4) & \geq 0 \\
        & \quad \forall i, j, i \neq j \\
        & \quad (1-x_i^o) (1-x_j^o) (x_1 + x_2 - x_4) & \geq 0 
\end{align*}
\]
\[ x_1^0 (x_1 + x_2 - x_4) \geq 0 \]
\[ \delta \]
\[ \chi_{i,1}^2 + \gamma_{3i,2} - \gamma_{3i,4} \geq 0 \]
\[ n \in \mathbb{N} \cap d \rightleftharpoons m \cap n \]