Lecture 17 Sherali-Adams cont. Oct 29

ex) Max-SAT

Input: CNF formula $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ on n vars $x_1, \cdots, x_n$

Goal: Find assignment $x \in \{0,1\}^n$ maximizing the # of SAT clauses.

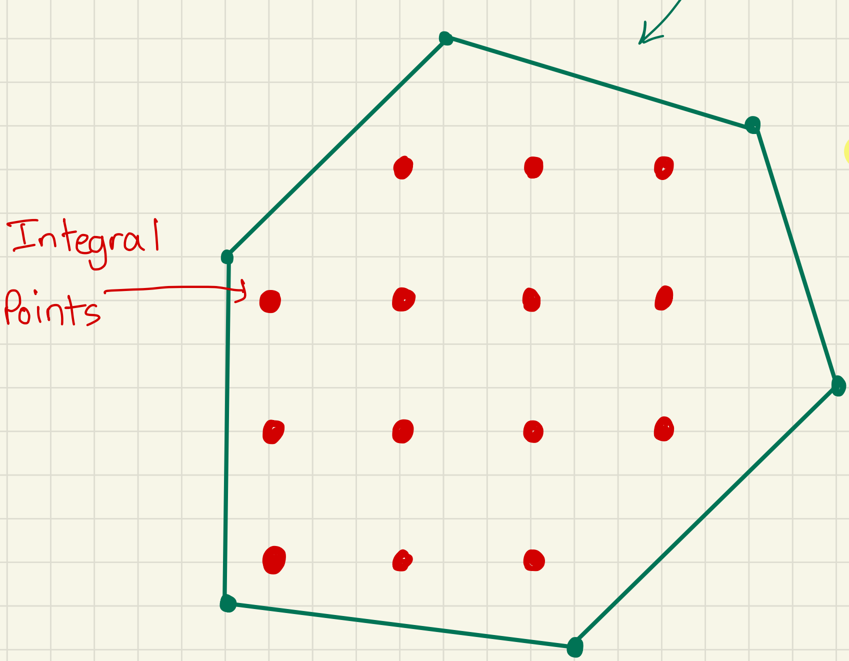
For clause $C = \bigvee_{i \in S} x_i \vee \bigvee_{j \in T} \overline{x_j}$ let $\tilde{C}_i = \sum_{i \in S} x_i + \sum_{j \in T} (1 - x_j)$

Integer Linear Program		LP Relaxation	
max $\sum_{i=1}^{m} c_i$		max $\sum_{i=1}^{m} c_i$	
s.t. $\tilde{C}_i \geq c_i$	$\forall i = 1 \cdots m$	s.t. $\tilde{C}_i \geq c_i$	$\forall i = 1 \cdots m$
$0 \leq x_j \leq 1$	$\forall j = 1 \cdots n$	$0 \leq x_j \leq 1$	$\forall j = 1 \cdots n$
$0 \leq c_i \leq 1$	$\forall i = 1 \cdots m$	$0 \leq c_i \leq 1$	$\forall i = 1 \cdots m$
$x_j, c_i \in \mathbb{Z}$		$\cancel{x_j, c_i \in \mathbb{Z}}$	

ILP:
- Represents Max-SAT exactly
- NP-Hard to solve

LP Relaxation:
- Doesn't represent Max-SAT exactly
- Can be solved in poly time!

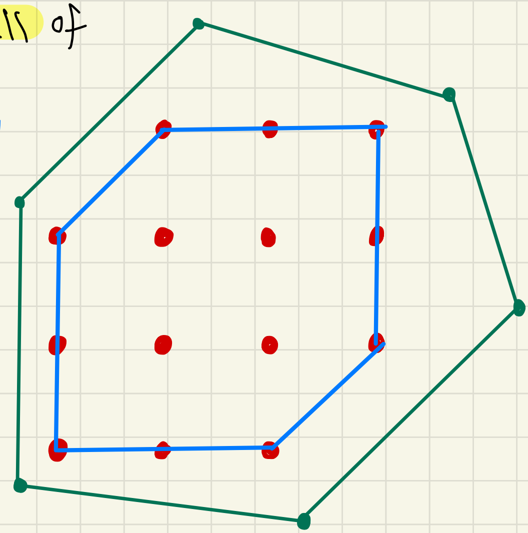
Geometric Picture



We want the optimal integral point, can only get the optimal fractional point.

Q. Is there a linear program that solves Max-SAT exactly?

A. **Yes!** Take the convex hull of integral points



Problems

(1) How do we get it?

A: Sherali-Adams

(2) There might be exponentially many inequalities in any description.

Today we focus on finding the integral hull using the SA hierarchy.

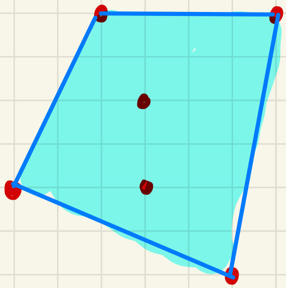
Prelims

- If $\vec{y}_1, \dots, \vec{y}_m \in \mathbb{R}^n$ then a convex combination of the y_is is any point of the form

$$\sum_{i=1}^{m} \alpha_i \vec{y}_i \qquad \alpha_i \geq 0, \; \sum \alpha_i = 1$$

- If $\mathcal{C} \subseteq \mathbb{R}^n$ then $\mathrm{conv}(\mathcal{C}) \subseteq \mathbb{R}^n$ is all convex combos of points in $\mathcal{C}$.

ex) $\mathcal{C}$ ↘



$\mathcal{C}$ red points

$\mathrm{conv}(\mathcal{C})$ blue points

- If $\mathcal{C} \subseteq \mathbb{R}^n$ then $\mathrm{hull}_{\mathbb{Z}}(\mathcal{C}) := \mathrm{conv}(\mathcal{C} \cap \mathbb{Z}^n)$

- We're given polytope $P \subseteq [0,1]^n$ i.e. the constraints

$$0 \le x_i \le 1 \quad \text{are in } P.$$

- We want to describe $\text{hull}_{\mathbb{Z}}(P)$.
- Since $P \subseteq [0,1]^n$, every point in $\text{hull}_{\mathbb{Z}}(P)$ can be written as

$$\alpha \in \text{hull}_{\mathbb{Z}}(P) \Rightarrow \alpha = \sum_{x \in \{0,1\}^n} \lambda_x x, \quad \lambda_x \ge 0, \quad \sum \lambda_x = 1$$

- Equivalently: $\alpha \in \text{hull}_{\mathbb{Z}}(P)$ represents a probability distribution of

$$x \in P \cap \{0,1\}^n$$

- For each α let $\mu^{(\alpha)} : \{0,1\}^n \to \mathbb{R}$ s.t.

$$\mu^{(\alpha)}(x) = \lambda_x$$

so $\mu^{(\alpha)}$ is a prob. dist. over $\{0,1\}^n \cap P$

Observation If **we can test** if $\alpha \in \mathbb{R}^n$ represents a valid prob. dist. over $P \cap \{0,1\}^n$ then $\alpha \in \text{hull}_{\mathbb{Z}}(P)$!

So: how do we test if α represents a probability distribution?

$$\mu : \{0,1\}^n \to \mathbb{R} \qquad \mu(x) \ge 0 \quad \text{and} \quad \sum \mu(x) = 1$$

We modify this by adding more tests to verify that we are in $\mathcal{P}$.

Sufficient to show μ is dist over $\{0,1\}^n$.

Not for $\{0,1\}^n \cap \mathcal{P}$

Specific: Test the marginal distributions of μ.

If $S \subseteq [n]$ then

$$\mu_S : \{0,1\}^S \longrightarrow \mathbb{R}$$

is defined by $\mu_S(\alpha) := \sum_{\substack{x \in \{0,1\}^n \\ x\restriction_S = \alpha}} \mu(x)$.

(if μ is a real prob. dist then

$$\mu_S(\alpha) = \Pr_{x \sim \mu}\left[\forall i \in S : x_i = \alpha_i\right])$$

Lemma $\mu : \{0,1\}^n \longrightarrow \mathbb{R}$ is a prob. dist on $\{0,1\}^n$

iff

$$\forall S \subseteq T \subseteq [n], \ \forall \alpha \in \{0,1\}^S$$

(1) $\mu_S(\alpha) = \sum_{\substack{\beta \in \{0,1\}^T \\ \beta\restriction_S = \alpha}} \mu_T(\beta)$ marginals agree

(2) $\mu_S(\alpha) \geq 0$ non-negativity

(3) $\mu_\emptyset = 1$ normalizing $\leftarrow$ explains $1 \geq 0$

Pf ($\Rightarrow$) Easy

($\Leftarrow$) Define $\Pr_{x\sim\mu}[x=y] := \mu(y)$

$$1 \overset{(3)}{=} \mu_\emptyset \overset{(1)}{=} \sum_{x\in\{0,1\}^n} \mu(x) \qquad \square$$

Answer: Where do non-negative juntas come from?

$y \in \{0,1\}^S$, let $S = T \cup U$ so that

$$y_i = \begin{cases} 1 & i\in T \\ 0 & i \in U \end{cases}$$

$$\mu_S(x) = \Pr_{x\sim\mu}[x\restriction_S = y] = \Pr_{x\sim\mu}\Big[\underbrace{\prod_{i\in T} x_i \prod_{j\in U}(1-x_j)}_{J_{T,U}} = 1\Big]$$

Non-negative juntas are random variables that describe the marginals of probability distributions over $\text{hull}_\mathbb{Z}(P)$.

This lemma is going to give a (somewhat crazy) LP for $[0,1]^n$.

- For each $S\subseteq[n]$ let $y_S \in \mathbb{R}$ be a variable for S.

Intuitively: $y_S = \mu_S(\vec{1}) = \Pr_{x\sim\mu}[x_i = 1 \ \forall i\in S]$

$$= \mathbb{E}_\mu\Big[\prod_{i\in S} x_i\Big] \quad \leftarrow \text{moment for } S$$

Define an LP on $\{y_S \mid S \subseteq [n]\}$, with constraints

(a) $y_\emptyset = 1$

$$y_{\{i\}} = x_i$$

(b) $\forall S \subseteq [n]$, $\forall T \cup \mathcal{U} = S$, $T \cap \mathcal{U} = \emptyset$

$$\sum_{\mathcal{U}' \subseteq \mathcal{U}} (-1)^{|\mathcal{U}'|} y_{\mathcal{U}' \cup T} \geq 0 \qquad (1 \geq 0)$$

<u>Claim</u> (1), (2) $\Rightarrow$ (b)

$S \subseteq T$

(1) $\mu_S(\alpha) \geq 0$ (2) $\mu_S(\alpha) = \sum_{\substack{\beta \in \{0,1\}^T \\ \beta \upharpoonright_S = \alpha}} \mu_T(\beta)$

$$0 \overset{(1)}{\leq} \mu_S(\alpha) \overset{(2)}{=} \Pr_{x \sim \mu}[x\upharpoonright_S = \alpha] \qquad \text{let } \alpha_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{if } i \in \mathcal{U} \end{cases}$$

$$= \Pr\left[\prod_{i \in T} x_i \prod_{j \in \mathcal{U}} (1 - x_i) = 1\right]$$

$$= \mathbb{E}\left[\prod_{i \in T} x_i \prod_{j \in \mathcal{U}} (1 - x_j)\right]$$

$$= \sum_{\mathcal{U}' \subseteq \mathcal{U}} \mathbb{E}_{\mu}\left[(-1)^{|\mathcal{U}'|} \prod_{i \in \mathcal{U}' \cup T} x_i\right]$$

$$= \sum_{\mathcal{U}' \subseteq \mathcal{U}} (-1)^{|\mathcal{U}'|} y_{\mathcal{U}' \cup T}$$

How to include constraints from P?

Write P as

$$Ax \le b \quad = \begin{cases} a_1 \cdot x \le b_1 \\ a_2 \cdot x \le b_2 \\ \vdots \\ a_m \cdot x \le b_m \end{cases}$$

$$0 \le x \le 1$$

Let $Q = \{b_1 - a_1 x \geq 0, \cdots, b_m - a_m x \geq 0\} \cup \{1 \geq 0\}$

Defn The degree-d Sherali-Adams tightening of Q is obtained by the following two steps

(1) For each inequality $q_i \geq 0$ in Q add

$$J_{S,T}\, q_i \geq 0 \qquad \leftarrow J_{S,T} = \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j)$$

to Q where $|S \cup T| \le d$, $S \cap T = 0$

(2) For every inequality $p_i \geq 0$ created in the last step, linearize p_i by

- Replace each x_i^c term with x_i
- For every monomial $\prod_{i \in S} x_i$, replace it with y_S.

Lemma (Next class) The degree-n Sherali-Adams tightening is exactly

$$\mathrm{hull}_{\mathbb{Z}}(P).$$

ex) Max-SAT

Let $F = (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_2}) \wedge (\overline{x_1})$

LP Relaxation

constraints: $x_1 + x_2 \geq x_3$

$x_1 + (1 - x_2) \geq x_4$

$(1 - x_1) \geq x_5$

The degree-2 SA tightening would add

$y_{\{i\}} \geq 0 \quad (1 - y_{\{i\}}) \geq 0 \quad \forall i, j \quad i \neq j$

$y_i (1 - y_j) \geq 0 \quad y_i y_j \geq 0$

$\downarrow \qquad \downarrow$

$y_i - y_{\{i,j\}} \geq 0 \quad y_{\{i,j\}} \geq 0$

Then, we multiply each constraint in the LP by a non-neg junta and linearize

ex) $x_1 + x_2 \geq x_4 \rightsquigarrow$

$$x_i (x_1 + x_2 - x_4) \geq 0$$

$$(1 - x_i)(x_1 + x_2 - x_4) \geq 0$$

$\forall i, j$

$i \neq j$

$$x_i x_j (x_1 + x_2 - x_4) \geq 0$$

$$x_i (1 - x_j)(x_1 + x_2 - x_4) \geq 0$$

$$(1 - x_i)(1 - x_j)(x_1 + x_2 - x_4) \geq 0$$

$$x_i \cdot (x_1 + x_2 - x_4) \geq 0$$

$$\Updownarrow$$

$$y_{\{i,1\}} + y_{\{i,2\}} - y_{\{i,4\}} \geq 0$$

$$\sim m \binom{n}{\leq d} = m\, n^{O(d)}$$