1. To this end, consider the following slight modification to Polynomial Calculus: replace the multiplication rule with the following rule

\[
\frac{p}{qp}
\]

where \( p \) is an arbitrary polynomial and \( q \) is any degree \( \leq 1 \) polynomial. Let \( \text{mPC} \) denote this modified polynomial calculus.

Prove that if there is a tree-like \( \text{mPC} \) refutation of a set of polynomials \( F \) with depth \( h \), then there is a Nullstellensatz refutation of \( F \) in degree \( h + \max_{f \in F} \deg(f) \).

**Solution.** We prove, by induction on \( h \), that if there is an \( \text{mPC} \) derivation of a polynomial \( p \) from \( F \) in depth \( h \), then there is a Nullstellensatz derivation of \( p \) from \( F \) in the desired degree; this gives the solution by setting \( p = 1 \). First, if \( h = 0 \) then the polynomial \( p \) must be in \( F \), and so the claim is clearly true. By induction, consider a depth-\( h \) \( \text{mPC} \) proof of \( p \). Let \( d = h - 1 + \max_{f \in F} \deg(f) \). We have two cases.

**Case 1.** \( p = qr \) where \( q \) has degree at most 1 and \( r \) is provable from \( F \) in depth \( h - 1 \). By induction, \( q \) has a degree \( d \) Nullstellensatz proof from \( F \), which we write as

\[
\sum_{f \in F} g_f f = q.
\]

Then

\[
\sum_{f \in F} (qg_f) f = q \sum_{f \in F} g_f f = qr = p
\]

is a degree \( d + 1 \) proof of \( p \).

**Case 2.** \( p = q + r \) for two polynomials \( q, r \) provable in depth \( h - 1 \) from \( F \). This case is analogous to the previous case: by induction \( q \) and \( r \) have degree \( d \) Nullstellensatz proofs, which we write as

\[
\sum_{f \in F} g_f f = q, \quad \sum_{f \in F} h_f f = r.
\]

Then

\[
\sum_{f \in F} (g_f + h_f) f = q + r
\]

is a degree at most \( d \leq d + 1 \) Nullstellensatz proof of \( q + r \).
2. In this problem we show that Sherali-Adams (unlike Nullstellensatz!) can efficiently simulate Resolution in both degree and size. Here we crucially need to use twin variables. Let \( C \) be a clause over \( x_1, x_2, \ldots, x_n \) and we consider Sherali-Adams proofs with twin variables \( x_1, \ldots, x_n, x'_1, \ldots, x'_n \) and boolean constraints \( x_i^2 - x_i = 0, x_i + x'_i = 1 \).

(a) First show that Sherali-Adams can derive multiplicative encodings of clauses from the linear encodings of clauses.

For any sets \( A, B \subseteq [n] \) with \( A \cap B = \emptyset \) let \( J_{A,B} = \prod_{i \in A} x_i \prod_{j \in B} x'_j \) be the non-negative junta corresponding to \( A, B \). Consider the clause \( C = \bigvee_{i \in S} x_i \vee \bigvee_{j \in T} \overline{x_i} \), and its translation into a linear inequality

\[
\sum_{i \in S} x_i + \sum_{j \in T} x'_j \geq 1
\]

(1)

Prove that Sherali-Adams can derive the polynomial \(-J_{T,S}\) from Equation (1) and the boolean constraints in degree \(|C|+1\) and size \(O(|C|)\).

**Solution.** First observe that from the boolean constraints \( x_i^2 - x_i \) and \( x_i + x'_i - 1 \) we can derive the polynomial \( px_i x'_i \) for any \( p \) in degree \( \deg(p) + 2 \) by

\[
px_i(x_i + x'_i - 1) + p(x_i^2 - x_i) = px_i x'_i.
\]

Multiplying the translation of the clause inequality by the junta \( J_{T,S} \) yields

\[
J_{T,S} \left( \sum_{i \in S} x_i + \sum_{j \in T} x'_j - 1 \right) = \sum_{i \in S} x_i x'_i J_{T,S \setminus i} + \sum_{j \in T} x_j x'_j J_{T \setminus j, S} - J_{T,S}.
\]

Now, by deriving \(-J_{T,S \setminus i} x_i x'_i\) and \(-J_{T \setminus j, S} x_j x'_j\) for each \( i, j \) and adding them to the previous polynomial we have derived \(-J_{T,S}\) in the desired degree and size.

(b) Now show Sherali-Adams can derive the following “weakening” and “resolution” rules. Let \( S, T, S', T' \subseteq [n] \) be sets with \( S \cap T = \emptyset \), \( S' \cap T' = \emptyset \), and let \( i \not\in S \cup T \).

Prove that Sherali-Adams can derive, from the inequalities \( 1 \geq 0 \) and the boolean inequalities, the following equalities:

- \( J_{S,T} - J_{S \cup \{i\}, T} \geq 0 \)

**Solution.** We give the derivation below:

\[
-J_{S,T}(x_i + x'_i - 1) + J_{S,T \cup i} \cdot 1 = (1 - x_i) J_{S,T} - x'_i J_{S,T} + x'_i J_{S,T} = (1 - x_i) J_{S,T} = J_{S,T} - J_{S \cup i, T}.
\]

- \( J_{S,T} - J_{S \cup \{i\}, T} \geq 0 \)

**Solution.** The derivation is symmetric to the previous case so we omit it.

- \( J_{S \cup \{i\}, T} + J_{S', T' \cup \{i\}} - J_{S \cup S', T \cup T'} \geq 0 \)

**Solution.** Let \( A = S \cup S' \) and \( B = T \cup T' \); then

\[
J_{A,B}(x_i + x'_i - 1) = J_{A \cup \{i\}, B} + J_{A, B \cup \{i\}} - J_{A,B}.
\]

(2)
which is the special case of the above inequality when $S = S'$, $T = T'$. Now, using the previous two steps derive the two polynomials
\[ J_{S \cup \{i\}, T} - J_{S \cup S' \cup \{i\}, T \cup T'}, \quad J_{S, T \cup \{i\}} - J_{S \cup S', T \cup T' \cup \{i\}}, \]
and add them to (2) to obtain the desired inequality.

(c) Finally, prove that if there is a Resolution refutation of a CNF formula $F$ in width $w$ and size $s$, then there is a Sherali-Adams refutation in degree $w + 1$ and size polynomial in $w$ and $s$.

**Solution.** This is a simple combination of the other parts of this question. Given any clause $C = \bigvee_{i \in S} x_i \lor \bigvee_{j \in T} \neg x_j$ let $\tilde{C} = \sum_{i \in S} x_i + \sum_{j \in T} \neg x_j - 1$ denote the translation of $C$ to a linear inequality, and let $J_C := J_{T, S}$ denote the non-negative junta that is 1 when $C$ is false and 0 otherwise. Let $\Pi = (C_1, C_2, \ldots, C_s)$ be a resolution refutation of $F$, where each clause $C_i$ is either in $F$ or is derived from earlier clauses $C_j, C_k$ by applying the resolution rule. We prove by induction that we can derive the polynomial $-J_{C_i}$ from $F$ in Sherali-Adams. Since $C_s$ is empty, $-J_{C_s} = -1$, and thus we have refuted $F$.

First, $C_1$ must be in $F$, and so we can apply part (a) to derive $-J_{C_1}$. By induction, consider the clause $C_i$. If $C_i$ in $F$ then we can apply part (a) again to derive $-J_{C_i}$. So, suppose that $C_i$ is derived from $C_j = A \lor x_\ell$ and $C_k \lor \neg x_\ell$ by resolving the variable $x_\ell$. By induction we can derive $-J_{C_j}, -J_{C_k}$ from $F$. Using part (c) we derive $J_{C_j} + J_{C_k} - J_{C_i}$ from the boolean axioms and $1 \geq 0$, and adding $-J_{C_j} - J_{C_k}$ gives a derivation of $-J_{C_i}$.

Finally, observe that the degree of the derivation is at most $w + 1$ (obtained at part (a)), and the size of the refutation is at most polynomial in $w, s$, as we must pay an $O(|C_i|)$ size cost to derive each monomial $-J_{C_i}$, and we need a copy of $-J_{C_i}$ for each time the clause $C_i$ is resolved upon in $\Pi$.

3. Prove that any Sherali-Adams proof of the inequality $\sum_{i=1}^n x_i \leq 1$ from the set of inequalities
\[
\{x_i + x_j \leq 1 \mid \forall i, j \in [n] : i \neq j\} \cup \{x_i^2 - x_i = 0\}_{i=1}^n
\]
requires degree $\Omega(n)$.

**Solution.** Suppose by contradiction that there is a proof of $1 - \sum_{i=1}^n x_i$ in degree $d = o(n)$. We can use this to refute the pigeonhole principle $\text{PHP}_{n+1}^n$ in degree $o(n)$, contradicting the known $\Omega(n)$ degree lower bound for refuting $\text{PHP}_{n+1}^n$ in Sherali-Adams.

Observe that $\text{PHP}_{n+1}^n$ has the inequalities $\sum_{j=1}^n x_{ij} - 1$ for each $i \in [n + 1]$ among its axioms. Summing these over all $i$ yields the polynomial $-(n + 1) + \sum_{i=1}^{n+1} \sum_{j=1}^n x_{ij}$. Using the assumption, we can derive $1 - \sum_{i=1}^{n+1} x_{ij}$ for each $j \in [n]$ in degree $o(n)$, and so by summing all of these polynomials up we can derive $n - \sum_{i=1}^{n+1} \sum_{j=1}^n x_{ij}$. Adding these two polynomials yields $-1$, and thus we have a refutation of $\text{PHP}_{n+1}^n$ in degree $o(n)$. 

3