

Minimization via Duality

Prakash Panangaden
School of Computer Science
McGill University

Verification, Model Checking and Abstract Interpretation
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Relevant papers

- Canonical regular expressions and minimal state graphs for definite events, by Jan Brzozowski in *Mathematical Theory of Automata*, 1962.
- Brzozowski's algorithm co-algebraically, by Bonchi, Bonsangue, Rutten and Silva, in Kozen Festschrift, Lecture Notes in Computer Science 7230, 2012.
- Minimization via duality: Bezahanishvili, Kupke and P.; proceedings of WoLLIC 2012, Lecture Notes in Computer Science 7456.
- Algebra-coalgebra duality in Brzozowski's minimization algorithm: Bonchi, Bonsangue, Hansen, P., Rutten and Silva, ACM Transactions on Computational Logic 2013.
- Longer paper with above authors plus Bezhanishvili, Kozen and Kupke in preparation.

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- M'' is the minimal automaton accepting L !

Examples of duality principles

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- Induction and co-induction

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- The two mathematical universes are *mirror images* of each other.
- Two completely different sets of theorems that one can use.

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- Commutative unital C^* -algebras and compact Hausdorff spaces. [Gelfand, Stone]

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- V^* has the same dimension as V and a (basis-dependent) isomorphism between V and V^* .
- The double dual V^{**} is also isomorphic to V
- with a “nice” canonical isomorphism: $v \in V \mapsto \lambda\sigma \in V^*.\sigma(v)$.

Duality for high school students II

$$U \xrightarrow{\theta} V$$

$$U^* \xleftarrow{\theta^*} V^*$$

Given a linear maps θ between vector spaces U and V we get a map θ^* *in the opposite direction* between the dual spaces:

$$\theta^*(\sigma \in V^*)(u \in U) = \sigma(\theta(u)).$$

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- Elegant and (almost) compositional version: Plotkin's *structured operational semantics*.
- Denotational semantics: compositional, equivalent to operational semantics.

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- Backward flow in **wp** semantics.
- D and E domains, viewed as topological spaces, open sets: \mathcal{O}_D and \mathcal{O}_E . A **predicate transformer** is a *strict, continuous and multiplicative* map $p : \mathcal{O}_E \rightarrow \mathcal{O}_D$.

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- Relate predicate-transformer semantics to state-transformer semantics: Jaco De Bakker (1978).
- Duality: The category of state transformers is equivalent to the (opposite of) the category of predicate transformers: Plotkin (1979).

Duality for probabilistic programs: Kozen

Probabilistic programs and *expectation transformers*: Kozen (1981)

Logic	Probability
States s	Distributions μ
Formulas P	Random variables f
Satisfaction $s \models P$	Integration $\int f d\mu$

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- This gives the minimal DFA recognizing the same language!
- The intermediate step can blow up the size of the automaton exponentially before minimizing it.
- But experimental results seem to indicate that it often works well in practice.

Deterministic Automata

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$: a deterministic finite (Moore) automaton. S is the set of **states**, \mathcal{A} an **input alphabet** (actions), \mathcal{O} is a set of **observations**.
- $\delta : S \times \mathcal{A} \rightarrow S$ is the **state transition function**.
- $\gamma : S \rightarrow \mathbf{2}^{\mathcal{O}}$ or $\gamma : S \times \mathcal{O} \rightarrow \mathbf{2}$ is a labeling function.

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- *except* that we do not have a start state,
- which means that reachability makes no sense.
- We will worry about that in a minute.

A Simple Modal Logic

- View \mathcal{O} as propositions, define a simple modal logic. A *formula* φ is:

$$\varphi ::= \omega \in \mathcal{O} \mid (a)\varphi$$

where $a \in \mathcal{A}$.

- We say $s \models \omega$, if $\omega \in \gamma(s)$ (or $\gamma(s, \omega) = T$).
We say $s \models (a)\varphi$ if $\delta(s, a) \models \varphi$.
- Now we define $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{s \in S \mid s \models \varphi\}$.

An Equivalence Relation on Formulas

- We write sa as shorthand for $\delta(s, a)$.
- Define $\sim_{\mathcal{M}}$ between *formulas* as $\varphi \sim_{\mathcal{M}} \psi$ if $\llbracket \varphi \rrbracket_{\mathcal{M}} = \llbracket \psi \rrbracket_{\mathcal{M}}$.

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- Equivalence class for φ same as of states $\llbracket \varphi \rrbracket_{\mathcal{M}}$ that satisfy φ .

A Dual Automaton

- Given a finite automaton $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$.
Let T be the set of $\sim_{\mathcal{M}}$ -equivalence classes of formulas on \mathcal{M} .
- We define $\mathcal{M}' = (S', \mathcal{A}, \mathcal{O}', \delta', \gamma')$ as follows:
 - $S' = T = \{[\varphi]_{\mathcal{M}}\}$
 - $\mathcal{O}' = S$
 - $\delta'([\varphi]_{\mathcal{M}}, a) = [(a)\varphi]_{\mathcal{M}}$
 - $\gamma'([\varphi]_{\mathcal{M}}) = [\varphi]_{\mathcal{M}}$ or $\gamma'([\varphi]_{\mathcal{A}}, s) = (s \models \varphi)$.

The intuition

Interchange states and observations.

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- For deterministic machines bisimulation is the same as trace equivalence.
- This gives an intuition for why Brzozowski's algorithm works,
- but it does not really address the role of reachability properly.

Probabilistic systems

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- Dual automaton from tests: probabilistic analogues of modal formulas.
- Main point: not minimization, but can learn systems from data even when the state is not directly observable
- because the double-dual serves as a substitute for the original machine.

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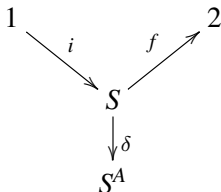
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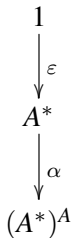
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- Double dual: state can be regarded as the summary of the outcomes of experiments.

An automaton in diagrams



- Here S is the state space, A is the set of actions, 1 is the one-element set and 2 is a two-element set.
- The map i defines an initial state and f defines a set of final states. I will write i for the map and for the initial state itself.
- the transition function $\delta : S \times A \rightarrow S$ has been written as $\delta : S \rightarrow S^A$.
- There is a natural extension $\delta^* : S \rightarrow S^{A^*}$.

A very special (infinite) automaton



- This automaton has all words as its state space.
- The initial state is the empty word ε .
- The transition function α acts by $\alpha(w) = \lambda a : A.w \cdot a$.
- We do not bother to define “final” states in this machine.

Reachability

$$\begin{array}{ccc}
 \mathbf{1} & & \\
 \downarrow \varepsilon & \searrow i & \\
 A^* & \overset{r}{\dashrightarrow} & S \\
 \downarrow \alpha & & \downarrow \delta \\
 (A^*)^A & \overset{r^A}{\dashrightarrow} & S^A
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- Given any function between sets $f : V \rightarrow W$, we have a map $f^A : V^A \rightarrow W^A$, given by $f^A(\phi) = \lambda a : A. f(\phi(a)) = f \circ \phi$.

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- Note, final states play no role.

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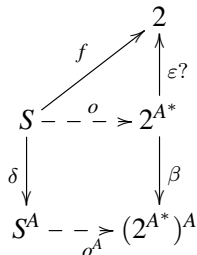
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Observability



- Here o is the map that takes a state to the language recognized starting from that state.

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 S & \overset{o}{\dashrightarrow} & 2^{A^*} \\
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 S^A & \overset{o^A}{\dashrightarrow} & (2^{A^*})^A
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- Here o is the map that takes a state to the language recognized starting from that state.
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- Here o is the map that takes a state to the language recognized starting from that state.
- It is the unique map making the upper triangle and the lower square commute.
- Think of o as giving the observable behaviour of a state.
- A machine is *observable* exactly when distinct states recognize different languages, i.e. when o is an injection.

The butterfly

$$\begin{array}{ccccc}
 1 & & & & 2 \\
 \varepsilon \downarrow & \searrow i & & \nearrow f & \\
 A^* & \overset{r}{\dashrightarrow} & S & \overset{o}{\dashrightarrow} & 2^{A^*} \\
 \alpha \downarrow & & \downarrow \delta & & \downarrow \beta \\
 (A^*)^A & \overset{r^A}{\dashrightarrow} & S^A & \overset{o^A}{\dashrightarrow} & (2^{A^*})^A
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A deterministic automaton (S, δ, i, f) is minimal if it is both reachable and observable.

The power-set construction

Given sets U, V and a function $f : U \rightarrow V$ we define

$$\mathcal{P}(f) : \mathcal{P}(V) \rightarrow \mathcal{P}(U)$$

by

$$\mathcal{P}(f)(P \subseteq V) = f^{-1}(P).$$

Reverse in terms of power-set

$$\begin{array}{c}
 S \\
 \delta \downarrow \\
 S^A
 \end{array}
 \parallel \parallel
 \begin{array}{c}
 S \times A \\
 \downarrow \\
 S
 \end{array}
 \parallel
 \begin{array}{c}
 2^{S \times A} \\
 \uparrow \\
 2^S
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- Initial becomes final under power-set. The final state $S \rightarrow \mathbf{2}$ becomes the new initial state by observing that such a function is the same thing as a subset.
- It makes reachable into observable, *but not vice versa*.

Why Brzozowski's algorithm works

Theorem

If (S, δ, i, f) is a reachable deterministic automaton accepting L , then $(2^S, 2^\delta, f, 2^i)$ is an observable deterministic automaton accepting $rev(L)$.

If, we take its reachable part again and reverse it again we again get an observable automaton this time recognizing L . If we take the reachable part we get a minimal automaton recognizing L .

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- “Surely, this is abstract nonsense; categorical mumbo-jumbo for something that can be explained simply!”

No, it is generalized abstract nonsense!

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- Weighted automata (i.e. automata over vector spaces) can be minimized by using the same idea with the self duality of vector spaces.
- Belief automata can be minimized using Gelfand duality.

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- η is an observation function: an element of V^* , the dual space.

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- The minimal automaton has size equal to the rank of the Hankel matrix.

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- One can show $L_{\mathcal{A}} = L_{\vec{\mathcal{A}}} = L_{\overleftarrow{\mathcal{A}}}$.
- One can also show that $\overleftrightarrow{\mathcal{A}}$ is the minimal automaton with the same (weighted) language as \mathcal{A} by using the rank criterion.

Belief automata

- A **probabilistic automaton with observations** is

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- Minimization via Stone duality \longrightarrow minimization via Gelfand duality.

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- Completeness theorems, which typically work by constructing transition systems from consistent sets of formulas embody a key aspect of duality results *but*,
- the arrow part of the duality is crucial for proving our minimization results.

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- Convex automata: exploit convex duality.

Thank you!