Two crucial issues
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• Correctness of software:
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• with respect to precise specifications.
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- Efficiency of code:
Two crucial issues

- Correctness of software:
  - with respect to precise specifications.

- Efficiency of code:
  - based on well-designed algorithms.
Logic is the key
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- Precise specifications have to be made in a formal language
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• Precise specifications have to be made in a formal language

• with a rigorous definition of meaning.
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• Logic is the “calculus” of computer science.
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• It comes with a framework for reasoning.
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• Precise specifications have to be made in a formal language

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• Logic is the “calculus” of computer science.

• It comes with a framework for reasoning.

• Many kinds of logic: propositional, predicate, modal, .....
Probability
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- Probability is also a framework for reasoning
Probability

• Probability is also a framework for reasoning quantitatively.
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• But is this relevant for computer programmers?
Probability

- Probability is also a framework for reasoning quantitatively.
- But is this relevant for computer programmers?
  - Yes!
Probability

- Probability is also a framework for reasoning quantitatively.
- But is this relevant for computer programmers?
- Yes!
- Probabilistic reasoning is everywhere.
Some quotations
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• The true logic of the world is the calculus of probabilities — James Clerk Maxwell
Some quotations

- The true logic of the world is the calculus of probabilities — James Clerk Maxwell

- The theory of probabilities is at bottom nothing but common sense reduced to calculus — Pierre Simon Laplace
Why does one use probability?
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• Some algorithms use probability as a computational resource: randomized algorithms.
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• Software for interacting with physical systems have to cope with noise and uncertainty: telecommunications, robotics, vision, control systems, ....
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• Software for interacting with physical systems have to cope with noise and uncertainty: telecommunications, robotics, vision, control systems, ....

• Big data and machine learning: probabilistic reasoning has had a revolutionary impact.
Basic Ideas
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Sample space $X$: the set of things that can possibly happen.
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Subprobability: $\sum_{x \in X} \Pr(x) \leq 1$. 
A Puzzle
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Imagine a town where every birth is equally likely to give a boy or a girl. \( \Pr(\text{boy}) = \Pr(\text{girl}) = \frac{1}{2} \).
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Each birth is an independent random event.
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Each birth is an *independent* random event.

There is a family with two children.
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One of them is a boy (not specified which one), what is the probability that the other one is a boy?
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Each birth is an independent random event.

There is a family with two children.

One of them is a boy (not specified which one), what is the probability that the other one is a boy?

Since the births are independent, the probability that the other child is a boy should be $\frac{1}{2}$. Right?
Puzzle (continued)
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Wrong!
Wrong!

Initially, there are 4 *equally likely* situations: bb, bg, gb, gg.
Puzzle (continued)

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Initially, there are 4 equally likely situations: bb, bg, gb, gg.

The possibility gg is ruled out with the additional information.
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So of the three *equally likely* scenarios: bb, bg, gb, only one has the other child being a boy.
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So of the three *equally likely* scenarios: \(bb, bg, gb\), only one has the other child being a boy.

The correct answer is \(\frac{1}{3}\).
Puzzle (continued)

Wrong!

Initially, there are 4 equally likely situations: bb, bg, gb, gg.

The possibility gg is ruled out with the additional information.

So of the three equally likely scenarios: bb, bg, gb,
only one has the other child being a boy.

The correct answer is $\frac{1}{3}$.

If I had said, “The elder child is a boy”, then the probability that the other child is a boy is indeed $\frac{1}{2}$.
Conditional probability
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Conditioning = revising probability in the presence of new information.
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Conditional probability/expectation is the heart of probabilistic reasoning.
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Conditional probability is tricky!
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Conditional probability/expectation is the heart of probabilistic reasoning.

Conditional probability is tricky!

Analogous to inference in ordinary logic.
Conditional probability
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Definition: if $A$ and $B$ are events, the **conditional probability** of $A$ given $B$, written $\Pr(A \mid B)$ is defined by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$
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We are told that the outcome is one of the possibilities in $B$. We now need to change our guess for the outcome being in $A$. 

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Bayes’ Rule

\[ \Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} \, . \]

How to revise probabilities.
Bayes’ Rule

\[ \Pr(A \mid B) = \frac{\Pr(B \mid A) \cdot \Pr(A)}{\Pr(B)} . \]

How to revise probabilities.

Proof is just from the definition.
Example
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Two coins, one fake (two heads) one OK.
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One coin chosen with equal probability and then tossed to yield a H.
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What is the probability the coin was fake?   Answer: \( \frac{2}{3} \).
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One coin chosen with equal probability and then tossed to yield a H.

What is the probability the coin was fake? Answer: $\frac{2}{3}$.

Pr($H$ | Fake) = 1, Pr(Fake) = $\frac{1}{2}$, Pr($H$) = $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$. 
Example

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What is the probability the coin was fake? Answer: \( \frac{2}{3} \).

\[ \Pr(H \mid \text{Fake}) = 1, \Pr(\text{Fake}) = \frac{1}{2}, \Pr(H) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}. \]

Hence \( \Pr(\text{Fake} \mid H) = \frac{1/2}{3/4} = \frac{2}{3}. \)
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$\Pr(H \mid \text{Fake}) = 1$, $\Pr(\text{Fake}) = \frac{1}{2}$, $\Pr(H) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

Hence $\Pr(\text{Fake} \mid H) = \frac{(\frac{1}{2})/(\frac{3}{4})}{(\frac{3}{4})} = \frac{2}{3}$.

Similarly $\Pr(\text{Fake} \mid HHHH) = \frac{8}{9}$. 
Example

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What is the probability the coin was fake? Answer: $\frac{2}{3}$.

$\Pr(H \mid \text{Fake}) = 1$, $\Pr(\text{Fake}) = \frac{1}{2}$, $\Pr(H) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

Hence $\Pr(\text{Fake} \mid H) = \left(\frac{1}{2}\right) / \left(\frac{3}{4}\right) = \frac{2}{3}$.

Similarly $\Pr(\text{Fake} \mid HHHH) = \frac{8}{9}$. $\Pr(\text{Fake} \mid H \ldots H) = \frac{1}{1 + \left(\frac{1}{2}\right)^n}$.
Bayes’ rule shows how to update the *prior* probability of $A$ with the new information that the outcome was $B$: this gives the *posterior* probability of $A$ given $B$. 
Expectation values
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A random variable $r$ is a real-valued function on $X$. 
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The conditional expectation value of $r$ given $A$ is:
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The conditional expectation value of $r$ given $A$ is:

$$\mathbb{E}[r \mid A] = \sum_{x \in X} r(x) \Pr(\{x\} \mid A).$$
Expectation values

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The expectation value of \( r \) is

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E[r] = \sum_{x \in X} Pr(x)r(x).
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The conditional expectation value of \( r \) given \( A \) is:

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E[r \mid A] = \sum_{x \in X} r(x)Pr(\{x\} \mid A).
\]

Conditional probability is a special case of conditional expectation.
Calculating expectations through conditioning
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Two people roll dice independently.
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The first one keeps rolling until he gets a 1 \textit{immediately} followed by a 2.
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Do they have the same expected number of rolls?
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If not, who is expected to finish first?
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The second one keeps rolling until she gets a 1 and a 1.

Do they have the same \textit{expected} number of rolls?

If not, who is expected to finish first?

What is the expected number of rolls for each one?
For the first: \( \frac{1}{6} \cdot (1 + \frac{1}{6} \cdot 1 + \ldots) + \ldots \)
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Is there a better way?
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Is there a better way?

Use conditional expectations and think in terms of state-transition diagrams:
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Is there a better way?

Use conditional expectations and think in terms of state-transition diagrams:
Let $x = \mathbb{E}[\text{Finish} \mid \text{Start}]$
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Let $x = \mathbb{E}[\text{Finish} \mid \text{Start}]$ \quad \text{Let } y = \mathbb{E}[\text{Finish} \mid 1]$

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$$y = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot (1 + y) + \frac{2}{3} \cdot (1 + x)$$

Easy to solve: $x = 30, y = 36.$
Let $x = \mathbb{E}[\text{Finish} \mid \text{Start}]$
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\[2, 3, 4, 5, 6\]
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Easy to solve: $x = 42, y = 36$. 
Let $x = \mathbb{E}[\text{Finish} \mid \text{Start}]$  \quad \text{Let } y = \mathbb{E}[\text{Finish} \mid 1]$

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Easy to solve: $x = 42, y = 36$.

Did you expect this to be the slower one?
Understanding programs
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The *state* of a program is the correspondence between names and values.
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The *state* of a program is the correspondence between names and values. \([X \mapsto 3, Y \mapsto 4, Z \mapsto -2.5]\)
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Running a part of a program changes the state.
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\text{if } X > 1 \text{ then } Y = Y + Z \text{ else } Y = Z
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[X \mapsto 3, Y \mapsto 4, Z \mapsto -2.5] \rightarrow [X \mapsto 3, Y \mapsto 1.5, Z \mapsto -2.5]
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Ordinary programs define state-transformer *functions*. 
Understanding programs

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Ordinary programs define state-transformer functions. When one combines program pieces one can compose the functions to find the combined effect.
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How do we understand probabilistic programs?
Understanding programs

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Ordinary programs define state-transformer functions. When one combines program pieces one can compose the functions to find the combined effect.

How do we understand probabilistic programs?

As distribution transformers.
$X = 0$; $C = \text{toss}$; if $C = 1$ then $X = X + 1$ else $X = X - 1$. 
$X = 0; \ C = \text{toss}; \ \text{if} \ C = 1 \ \text{then} \ X = X + 1 \ \text{else} \ X = X - 1.$

Initial distribution: $[X \mapsto (0, 1.0), C \mapsto (0, 1.0)]$
\( X = 0; \ C = \text{toss}; \) if \( C = 1 \) then \( X = X + 1 \) else \( X = X - 1. \)

Initial distribution: \([X \mapsto (0, 1.0), C \mapsto (0, 1.0)]\)

Final distribution: \([X \mapsto (1, 0.5)(-1, 0.5), C \mapsto (0, 0.5)(1, 0.5)]\)
\[ X = 0; \quad C = \text{toss}; \quad \text{if } C = 1 \text{ then } X = X + 1 \text{ else } X = X - 1. \]

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A Markov chain has \(S\): states and a probability distribution transformer \(T\).
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A Markov chain has $S$: states and a probability distribution transformer $T$.

$$T: S_t \times S_t \rightarrow [0, 1] \text{ or } T: S_t \rightarrow \text{Dist}(S_t).$$
\( X = 0;\ C = \text{toss}; \) if \( C = 1 \) then \( X = X + 1 \) else \( X = X - 1 \).

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A Markov chain has \( S \): states and a probability distribution transformer \( T \).

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T : St \times St \rightarrow [0, 1] \text{ or } T : St \rightarrow Dist(St).
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\( T(s_1, s_2) \) is the conditional probability of being in state \( s_2 \) after the transition given that the state was \( s_1 \) before.
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\( T(s_1, s_2) \) is the conditional probability of being in state \( s_2 \) after the transition given that the state was \( s_1 \) before.

Markov property: the transition probability only depends on the current state, not on the whole history.
Because of the Markov property, one can describe the effect of a transition by a matrix.
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When one combines probabilistic program pieces one can multiply the transition matrices to find the combined effect.
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We are understanding the program by stepping forwards.
Because of the Markov property, one can describe the effect of a transition by a matrix.

When one combines probabilistic program pieces one can multiply the transition matrices to find the combined effect.

We are understanding the program by stepping forwards.

This is called “forwards” or state-transformer semantics.
Backwards semantics
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Maybe we do not want to track every detail of the state as it changes.
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Perhaps we want to know if a property holds, e.g. $X > 0$. 

**Backwards semantics**
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Maybe we do not want to track every detail of the state as it changes.

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We write $\{P\}$ step $\{Q\}$ to mean that $P$ holds before the step and $Q$ holds after the step.
Backwards semantics

Maybe we do not want to track every detail of the state as it changes.

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We write \( \{P\} \) step \( \{Q\} \) to mean that \( P \) holds before the step and \( Q \) holds after the step.

\[
\{X > 0\} \ X = X - 5 \ \{X > 0\} \ ??
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Backwards semantics

Maybe we do not want to track every detail of the state as it changes.

Perhaps we want to know if a property holds, e.g. $X > 0$.

We write $\{P\}$ step $\{Q\}$ to mean that $P$ holds before the step and $Q$ holds after the step.

$$\{X > 0\} \; X = X - 5 \; \{X > 0\} \; ??$$

We cannot say anything for sure after the step!
But we can go backwards!
But we can go backwards!

\[ \{X > 5\} \quad X = X - 5 \quad \{X > 0\} \]
But we can go backwards!

\{ X > 5 \} \quad X = X - 5 \quad \{ X > 0 \}
But we can go backwards!

\[
\{ X > 5 \} \quad X = X - 5 \quad \{ X > 0 \}
\]

We must read this differently: if we want \( X > 0 \) after the step, we must make sure \( X > 5 \) before the step.
But we can go backwards!

\[ \{X > 5\} X = X - 5 \{X > 0\} \]

We must read this differently: if we want \( X > 0 \) after the step, we must make sure \( X > 5 \) before the step.

This is called “predicate-transformer” semantics.
But we can go backwards!

\[ \{X > 5\} \ X = X - 5 \ {\{X > 0\}} \]

We must read this differently: if we want \( X > 0 \) after the step, we must make sure \( X > 5 \) before the step.

This is called “predicate-transformer” semantics.

Can we do something like this for probabilistic programs?
<table>
<thead>
<tr>
<th>Classical logic</th>
<th>Generalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truth values {0, 1}</td>
<td>Probabilities ([0, 1])</td>
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<tr>
<td>Predicate</td>
<td>Random variable</td>
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<tr>
<td>State</td>
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<td>The satisfaction relation (\models)</td>
<td>Integration (\int)</td>
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</table>
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(\hat{T}r)(s) = \sum_{s' \in S} T(s, s')r(s').
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This tells you the expected reward \textit{before} the transition assuming that \( r \) is the reward after the transition.
Probabilistic Models
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• A general framework to reason about situations computationally and quantitatively.
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• We want to represent several random variables that may be connected in different ways.
A simple graphical model
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Trying to analyze what is wrong with a web site:
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Symptoms: crashed or slow

There are 64 states.
The arrows show probabilistic dependencies.
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Perhaps too simple: if attacked then the traffic should be heavy.
This version: traffic is affected by being attacked.
Medical example (from Koller and Friedman)
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Flu and allergy are correlated through season.
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Given the season, they are independent: \((A \perp F \mid S)\)
Independence relations:
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\[(F \perp A \mid S), (Sn \perp S \mid F, A), (P \perp A, Sn \mid F), (P \mid Sn \mid F)\]
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\(Sn\) depends on \(S\) but it is \textit{conditionally} independent given \(A\) and \(F\).
Independence and Factorization
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Because we can factorize the joint distributions.
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A huge advantage for representing, computing and reasoning.
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Continuous state spaces: robotics, telecommunication, control systems, sensor systems.
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There is a rich and fascinating theory of programming and reasoning about probabilistic systems.
Logic and Probability are your weapons.
Go forth and conquer the software world!
Thank you!