Nuclear ideal systems in tensor-∗ categories

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Topos Institute Colloquium 9th February 2023
Collaborators

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This talk is based on work by Abramsky, Blute and me:
Later we formalized conformal field theory:
Outline

1. Introduction
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2. Compact closed categories
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3. The search for quantitative relations
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4. A bit of functional analysis
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5. Nuclear ideals
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6. PRel and SRel
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7. Other examples
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6. PRel and SRel
7. Other examples
8. Conclusions
Simply subsets of $R \subseteq A_1 \times \ldots \times A_n$, the basic ingredients of relational databases and many many mathematical structures.
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- Binary relations: $R \subseteq A \times B$, write $aRb$ instead of $(a, b) \in R$
- $n$-ary relations can be seen as binary ones
  $R \subseteq (A_1 \times \ldots \times A_m) \times (B_1 \times \ldots \times B_n)$ so we can write
  $R(x_1, \ldots, x_m; y_1, \ldots, y_n)$. 
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- Sets and relations form a category: $R : A \rightarrow B$ means $R \subseteq A \times B$. 
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- Composition: $R : A \to B, S : B \to C$, define $S \circ R : A \to C$ by $a(S \circ R)c = \exists b \in B \ aRb \land bSc$. 
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One can define a trace: $Tr_U^{A,B} : \text{Hom}(A \times U, B \times U) \rightarrow \text{Hom}(A, B)$ by $aTr(R)b = \exists u \ R(a, u; b, u)$.
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We can repartition the interface: $R(x_1, \ldots, x_m; y_1, \ldots, y_n)$ can be transposed to give $R'(x_1, \ldots, x_{m-1}; x_m, y_1, \ldots, y_n)$.
finite-dimensional vector spaces over some field $k$. 

Multilinear algebra

- Finite-dimensional vector spaces over some field $k$. 

We have a category with morphisms the $k$-linear maps. We have notion of multi-linear map $f: V_1 \times \ldots \times V_n \to k$. 

Introduce the concept of tensor product $V_1 \otimes V_2 \otimes \ldots \otimes V_n$, to make the multi-linear maps proper linear maps: $f: V_1 \otimes \ldots \otimes V_n \to k$. 

We have matrices as concrete (basis-dependent) representations of linear maps. We have higher tensors for multilinear maps. Index notations, diagrammatic notations.

One can define a partial trace $\text{Tr}_{V,W,U}: \text{Hom}(V \times U, W \times U) \to \text{Hom}(V, W)$ by well-known formulas. 

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Monoidal categories

- Categories $\mathcal{C}$ equipped with a “multiplication”: $\otimes$, a bifunctor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$.

- Equipped with a unit, written $I$, and $\otimes$ is associative, “up to a natural isomorphism.”

- Any diagram constructed from the natural isomorphisms must commute.

- Fortunately this follows from the requirement that a few specific diagrams must commute.

- Vector spaces and linear maps form a monoidal category with $\otimes$ the usual tensor product.

- Sets and relations also form a monoidal category with the cartesian product playing the role of the monoidal product.

- If there is a natural iso $A \otimes B \sim B \otimes A$ (plus some conditions) we have a symmetric monoidal category.
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- A symmetric monoidal closed category, has for each functor $\cdot \otimes X$ a right adjoint written $X \multimap (\cdot)$. 

$$\text{hom}(A \otimes X, B) = \text{hom}(A, X \multimap B).$$

Think of $A \multimap B$ as the space of “linear maps” from $A$ to $B$. 
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Nuclear Ideals

Topos Feb 2023
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Figure: A morphism from $A \otimes B \otimes C$ to $D \otimes E \otimes F$
Figure: A morphism from $A \otimes B$ to $D \otimes E \otimes F \otimes C$
Compact closed categories

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When do we have $A \circ B \cong A^* \otimes B$?

A **compact closed category** $C$ has, for every object $A$ a **dual** object $A^*$ and isos: $\nu : I \to A \otimes A^*$, $\psi : A^* \otimes A \to I$. 

Finite-dimensional vector spaces and linear maps are the classic example of a compact closed category. The other basic example is sets and binary relations.
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- Unit object: (any) one-point set
  $$(\cdot)^* \text{ is given by } X^* = X \text{ and } R^* = R^c.$$  
- If we write $I = \{\bullet\}$ then $\nu : I \to X \otimes X^*$ is $\bullet \nu(x, x)$ for all $x$; similarly for $\psi$. 
The naïve idea

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  \[ (g \circ f)(x, z) = \int_Y f(x, y)g(y, z)dy. \]
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$(g \circ f)(x, z) = \int_Y f(x, y)g(y, z)dy$.

If all works well we hope to get a compact closed category.
Small Problem

Schwartz

There is no function that can serve as an identity for this operation. There is no “function” $\delta$ such that:

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Schwartz, Gelfand
OK, we’ll invent distributions.
What happened to us

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- But in the end we failed to construct a compact closed category.
- Then we tried using measure theory and thinking of the Dirac delta “function” as a measure. Again we failed to construct a compact closed category.
- Finally Rick Blute realized this was a pattern and formulated the notion of nuclear ideals and realized that there was a well-known example from Hilbert space theory.
Summary

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- The maps in the nuclear ideal do behave strikingly like they were part of a compact closed category: one can transpose freely.
Summary

- There are situations where one does not have a category because the things that want to be the identity maps are too “singular”.
- Nevertheless, the maps of interest can sit as ideals inside a bona-fide monoidal category.
- The maps in the nuclear ideal do behave strikingly like they were part of a compact closed category: one can transpose freely.
- This is what Grothendieck was doing with Banach spaces: when can the maps be thought of as “matrices”? 
Hilbert spaces and tensor products

- Hilbert spaces are vector spaces with an inner product, which induces a norm which induces a metric.
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- For complex Hilbert spaces we also have conjugation or equivalently a “dagger” (more later).
Universal property of tensor products?

There is a unique map, $!$, from $U \times V$ to $U \otimes V$ such that: given a bilinear map from $U \times V$ to $W$, there is a unique linear map from $U \otimes V$ to $W$ making the diagram commute.
Hilbert-Schmidt maps

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A bit of functional analysis

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Grothendieck discovered **nuclear spaces** and **nuclear maps** when he was trying to explain why spaces of distributions had nice properties with respect to tensor product.
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The definition(s) of nuclear space are complicated and filled with analysis details about topological vector spaces and various types of tensor products.

Let $f : A \to B$, where $A$ and $B$ are Banach spaces. Being nuclear is equivalent to saying there is an element $\sum_i f_i \otimes b_i \in A^* \otimes B$ with for all $a \in A$ we have $f(a) = \sum_i f_i(a)b_i$. (Some details elided)
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- Nuclear spaces are typically not describable as normed vector spaces; the only spaces that are nuclear and normed are finite dimensional.
Given a HS map $f : \mathcal{H}_1 \to \mathcal{H}_2$ and any bounded linear maps $g : \mathcal{H}_2 \to \mathcal{H}_3$ and $h : \mathcal{H}_0 \to \mathcal{H}_1$, the composites $f \circ h$ and $g \circ f$ are both HS.
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- Trace class maps also form a two-sided ideal.

- The composite \( g \circ f : \mathcal{H} \to \mathcal{H} \) of two nuclear maps \( f : \mathcal{H} \to \mathcal{K} \) and \( g : \mathcal{K} \to \mathcal{H} \) is always trace class.
Some morphisms

Let $\mathcal{C}$ be a symmetric monoidal closed category.
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If $f : A \to B$ in $\mathcal{C}$, we call $n(f) : I \to A \to B$ the name of $f$. 

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Nuclear morphisms

We say that $f$ is *nuclear* if there exists $p(f) : I \rightarrow B \otimes A^*$ such that the following diagram commutes:

![Diagram](image-url)
Preservation properties

Suppose that $f : A \to B$ and $g : C \to D$ are nuclear, then so are:

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- \( f \otimes g : A \otimes C \rightarrow B \otimes D \)
Nuclear ideals

Nuclearity and compact closure

We say that an object of $C$ is *nuclear* if its identity map is nuclear.
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Nuclearity and compact closure

We say that an object of $\mathcal{C}$ is nuclear if its identity map is nuclear. For any symmetric monoidal closed category, the full subcategory of nuclear objects is compact-closed.
Tensor-$\ast$ categories

$\mathcal{C}$ is a $\ast$-category if it is equipped with a functor: $(−)^\ast : \mathcal{C}^{op} \to \mathcal{C}$, which is strictly involutive and the identity on objects.
Tensor-\(*\) categories

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satisfying the usual monoidal equations, and some other simple equations.
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- $\overline{\overline{A}} \cong A.$
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- $\overline{I} \cong I$.

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\[\text{(Panangaden Nuclear Ideals Topos Feb 2023 26 / 42)}\]
Nuclear Ideals

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  - closed under $(\cdot)^\ast$,
  - and the conjugate functor.
A bijection $\theta : \mathcal{N}(A, B) \rightarrow Hom(I, \overline{A} \otimes B)$. 
Nuclear Ideal - II

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If $f: A \rightarrow B$ is a nuclear morphism, we can use the bijection $\theta$ and the $*$-functor to construct various *transposes*.
A bijection \( \theta : \mathcal{N}(A, B) \rightarrow Hom(I, \overline{A} \otimes B) \).

If \( f : A \rightarrow B \) is a nuclear morphism, we can use the bijection \( \theta \) and the \(*\)-functor to construct various *transposes*. The bijection \( \theta \) must preserve all of the tensored \(*\)-structure.
Finally, $\theta$ has to satisfy a naturality property and a “compactness” property.
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- Blute, P. and Pronk (2007) gave an alternate definition of nuclear ideals in terms of dagger compact categories.
Lawvere’s category of probabilistic mappings

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These are well known in probability as Markov kernels.
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Lawvere’s category of probabilistic mappings

- **Mes** is the category of sets equipped with $\sigma$-algebras; morphisms are measurable functions.

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- These are well known in probability as Markov kernels.
The Giry Monad

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The Radon-Nikodym theorem

Given $(\sigma)$-finite measures $\mu, \nu$ on a measurable space $X$, we say $\nu$ is **absolutely continuous** with respect to $\mu$, if for every measurable set $A$, $\mu(A) = 0$ implies $\nu(A) = 0$. 

Notation: $\nu \ll \mu$.

If $\nu \ll \mu$ then there is a measurable function $h: X \to \mathbb{R}$ such that $\forall A \subset X$, $\nu(A) = \int_A h \, d\mu$.

This $h$ is "essentially unique": if $h'$ satisfies the same equation then $h$ and $h'$ differ on a set of $\mu$-measure 0.

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Probabilistic relations: PRel

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- **Morphisms**: \(\alpha: (X, \Sigma, \mu) \to (X', \Sigma', \mu')\) are probability measures on \(X \times X'\) (actually on \(\Sigma \otimes \Sigma'\)) such that:

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  - its marginals are absolutely continuous with respect to \(\mu\) and \(\mu'\).
- How do we compose these things?
Let \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) be measurable spaces.
From joint measures to Markov kernels

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- Let \(P_X\) be a probability measure on \((X, \Sigma_X)\).
- Let \(h(x, B) : X \times \Sigma_Y \rightarrow [0, 1]\) be a stochastic kernel.
- Then we have a unique measure \(P\) on the product such that for all \(A \in \Sigma_X\):
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P(A \times B) = \int_A h(x, B) dP_X(x).
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So we can go back and forth between distributions on the product space \(X \times Y\) and a pair consisting of a kernel \(h : X \rightarrow \Sigma_Y\) and a measure on \(X\).
Let \((X, \Sigma_X)\) and \((Y, \Sigma_Y)\) be measurable spaces. Let \(P_X\) be a probability on \((X, \Sigma_X)\). Let \(h(x, B) : X \times \Sigma_Y \to [0, 1]\) be a stochastic kernel. Then we have a unique measure \(P\) on the product such that for all \(A \in \Sigma_X\):

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So we can go back and forth between distributions on the product space \(X \times Y\) and a pair consisting of a kernel \(h : X \to \Sigma_Y\) and a measure on \(X\). And, of course we could instead use a kernel \(k : Y \to \Sigma_X\) and a measure on \(Y\).
Composing probabilistic relations

To compose morphisms we calculate their associated stochastic kernels $F(x, B)$ and $G(y, C)$ using the Radon-Nikodym theorem.
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$$\gamma(A \times C) = \int_A H(x, C) d\mu(x)$$
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The identity on $(X, \Sigma_X, \mu)$ is $\Delta(A \times B) = \mu(A) \cdot \mu(B)$ which can be extended to all the measurable sets of $X \times X$. The associated kernel is the Dirac delta “function”.

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A nuclear ideal for PRel

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- While \(f\) itself is only unique almost everywhere, the measure with which \(f\) is associated is easily viewed - in a canonical way - both as a member of \(\text{Hom}(X, Y)\) and as a member of \(\text{Hom}(I, X \times Y)\).
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- While $f$ itself is only unique almost everywhere, the measure with which $f$ is associated is easily viewed - in a canonical way - both as a member of $\text{Hom}(X, Y)$ and as a member of $\text{Hom}(I, X \times Y)$.
- Thus every element of the set $\text{Hom}(I, X \otimes Y)$ is associated with a measure that has a functional kernel which is in turn one of the members of the set $\mathcal{N}(X, Y)$. 

What have we got?

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- The putative identity is too singular to be a function, but we can realize it as a measure.
- The category we get by including such measures is not compact closed.
- But the original functions do form a nuclear ideal.
Simple examples

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- One can easily construct a category of injective partial functions.
- It is easy to make it a $\ast$-tensor category.
- One can construct a nuclear ideal by looking at functions whose domain consists of exactly one element and throw in the everywhere undefined function as well.
Using Schwartz distributions

Schwartz, and independently Gelfand and Shilov, were trying to make sense of “generalized functions” like the Dirac delta function and its derivatives.
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- For example $\delta'(f) = -f'(0)$. Now we can differentiate these things!
- These distributions are perfect for studying differential equations.
- We developed another $\ast$-tensor category based on a special kind of distribution and showed that the functional versions of these distributions give a nuclear ideal.
Formalizing conformal field theory

- Segal gave a categorical formulation of conformal field theory and remarked in passing that his category lacked identity morphisms.
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- We showed that his “category” was actually a nuclear ideal inside a $\ast$-tensor category.
- This involved some interesting mathematics: cobordisms, Riemann surfaces etc.
Conclusions

- There are many natural examples of nuclear ideals in mathematics.
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- Would they be useful for formalizing infinite-dimensional quantum mechanics?

- We defined trace ideals in terms of nuclear ideals. Is there a more intrinsic way?