CAUSALITY CONDITIONS IN SPACETIMES

OTTAWA, 26 JUNE '03

SUMATI SURYA
(UNIV. OF ALBERTA)

CAUSAL LORENTZIAN GEOMETRY $\rightleftarrows$ PARTIAL ORDER

CONTINUUM

DISCRETENESS

A DISCRETE TOPOLOGY ON A PARTIAL ORDER IS INSUFFICIENT TO CAPTURE CERTAIN STRICTER "GLOBAL" CAUSALITY CONDITIONS ON THE SPACETIME WHICH ARE DEFINED USING THE MANIFOLD TOPOLOGY.

QN: CAN ONE DEFINE AN APPROPRIATE TOPOLOGY ON A POSET WHICH WILL CAPTURE SOME FEATURES OF THE GLOBAL CAUSALITY CONDITIONS?

- BASICS OF CAUSAL STRUCTURE
- GLOBAL CAUSALITY CONDITIONS & THE CAUSAL HIERARCHY.

SPACE TIME, (M, g)

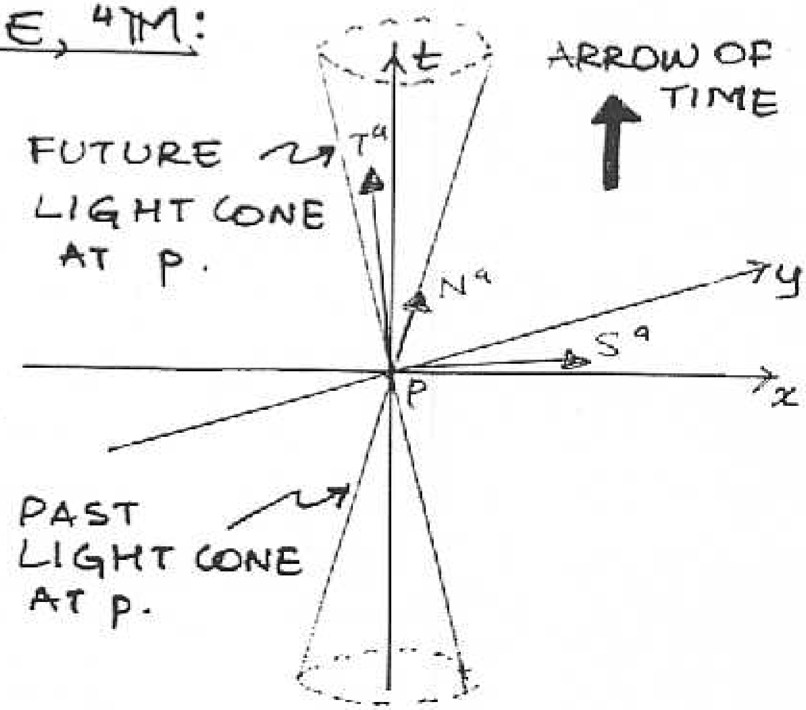
M: REAL, 4-DIMENSIONAL, C^∞, CONNECTED, HAUSDORFF, PARACOMPACT MANIFOLD

g: C^k $(k \geq 2)$ TYPE $(0,2)$ SYMMETRIC, NON-DEGENERATE TENSOR WITH SIGNATURE $(-,+,+,+)$ $\}$ "LORENTZIAN METRIC"

LORENTZIAN STRUCTURE CAPTURES THE IDEA OF LOCAL CAUSALITY (i.e. no faster than light travel!)

MINKOWSKI SPACETIME, ${}^4\mathbb{M}$:

$M \simeq {}^4\mathbb{R}$; $g = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$

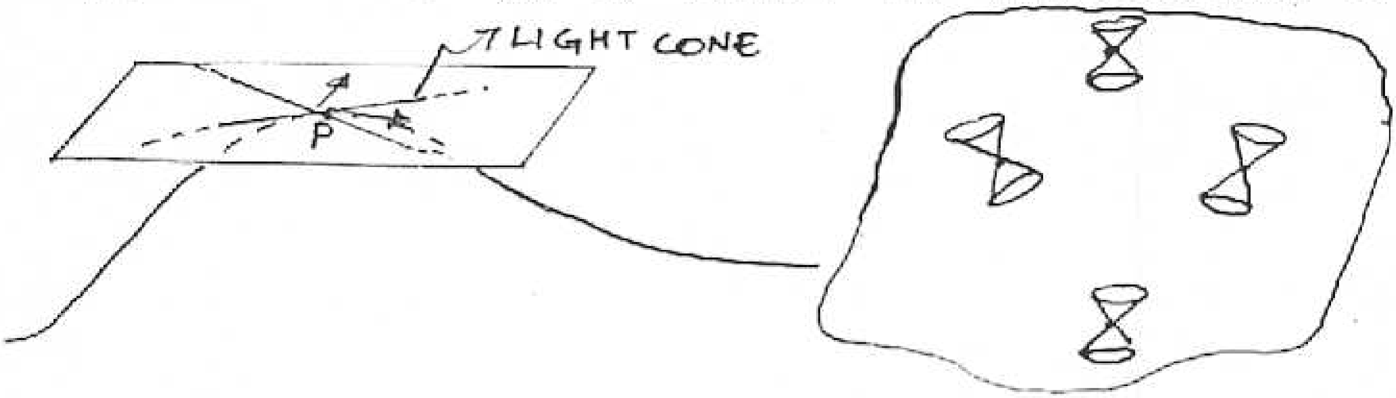


T^a: TIME-LIKE VECTOR "INSIDE" LIGHT CONE
$g(T,T) < 0$

N^a: NULL VECTOR "ON" LIGHTCONE
$g(N,N) = 0$

S^a: SPACE-LIKE VECTOR "IN" LIGHT CONE
$g(S,S) > 0$

FOR ANY SPACETIME, (M, g), THE TANGENT SPACE T_pM AT A POINT $p \in M$ IS ISOMORPHIC TO THE TANGENT SPACE OF A POINT IN MINKOWSKI SPT.



∴ AT EVERY POINT $p \in M$ THE TANGENT SPACE CAN BE DIVIDED UP INTO TIME-LIKE, SPACE-LIKE & NULL VECTORS

EVERY POINT $p \in M$ HAS A "LIGHT CONE"

IN ${}^4M$: $T_pM \longrightarrow {}^4M$ IS AN ISOMORPHISM.

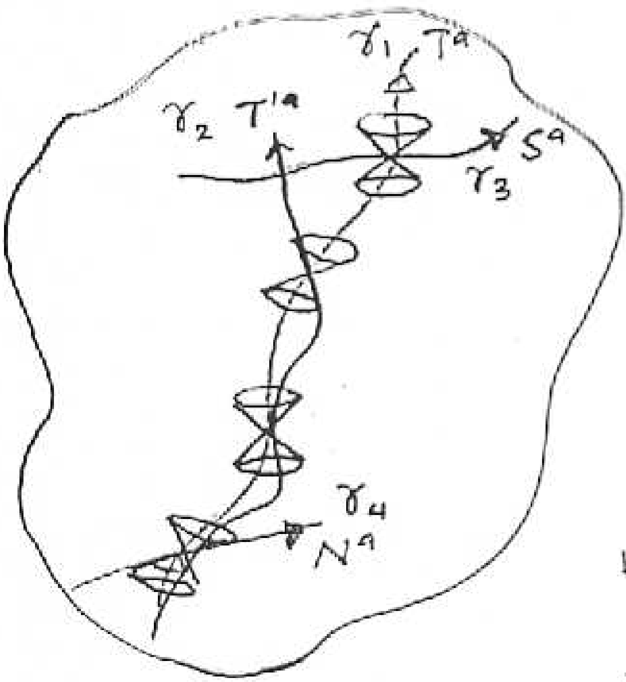
IN GENERAL, $\exists$ AN OPEN SUBSET $\mathcal{U}$ OF T_pM DIFFEOMORPHIC TO A NEIGHBOURHOOD OF p IN M.
EXPONENTIAL MAP: $\exp_p : \mathcal{U} \to M$

$\exp_p(\mathcal{U})$: CONVEX NORMAL NEIGHBOURHOOD OF p.

CURVES IN (M,g):

$C^\infty \quad \mu : [a,b] \subset \mathbb{R} \longrightarrow M$.

$\mathrm{Im}(\mu) = \gamma \subset M$ IS A CURVE IN (M,g).



IF THE TANGENT VECTOR TO γ AT EVERY POINT IS

(a) $g(T,T) < 0$: γ IS TIME-LIKE

(b) $g(N,N) = 0$: γ IS NULL

(c) $g(S,S) > 0$: γ IS SPACE-LIKE

IF IT IS TIME-LIKE <u>OR</u> NULL

THEN IT IS CALLED A

CAUSAL CURVE.

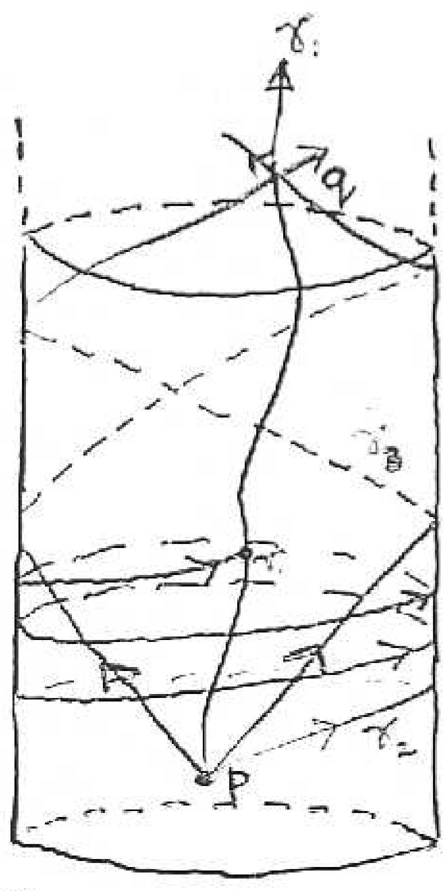
γ_1 : PURELY TIMELIKE
γ_2 : CAUSAL
γ_3 : SPACE-LIKE
γ_4 : PURELY NULL

γ_1, γ_3 & γ_4 ARE FUTURE DIRECTED CAUSAL CURVES.

ORDER FROM LORENTZIAN GEOMETRY.

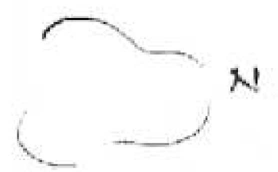
CHRONOLOGICAL RELATION : $x \ll y$, i.e. $\exists$ A TIME-LIKE CURVE FROM x TO y.

CAUSAL RELATION : $x < y$, i.e. $\exists$ A CAUSAL CURVE FROM x TO y.



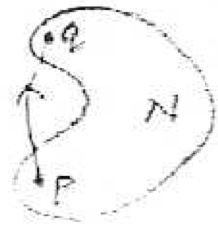
$p \ll q$; $p \ll r$

$N \subset M$



$x \ll_N y$ IF $\exists$ A TIME-LIKE CURVE FROM x TO y IN N

& $x <_N y$ IF $\exists$ A CAUSAL CURVE FROM x TO y IN N.



$p \ll q$, BUT

$p \not\ll_N q$.

AXIOMATIC DEFINITION OF CAUSAL STRUCTURE

(KRONHEIMER & PENROSE)

(a) $x < x \quad \forall \; x \in M$

(b) $x \not\ll x$

(c) IF $x < y$ AND $y < x \Rightarrow x = y$ $\Big\}$ CAUSALITY CONDITION.

(d) IF $x < y$ AND $y < z \Rightarrow x < z$ $\Big\}$ TRANSITIVITY

(e) IF $x \ll y$ THEN $x < y$

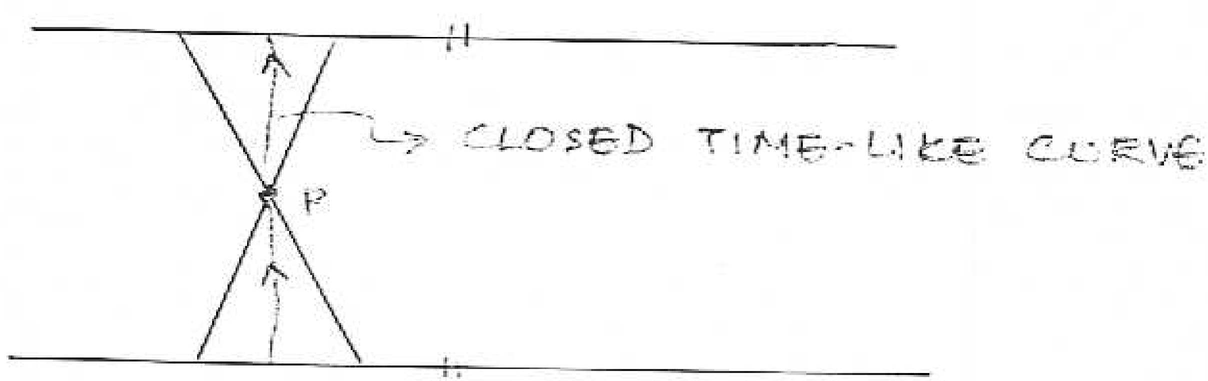
(f) IF $x < y$ AND $y \ll z \Rightarrow x \ll z$

(f') IF $x \ll y$ AND $y < z \Rightarrow x \ll z$.

(g) $x \rightarrow y$ IFF $x < y$ AND NOT $x \ll y$.

LOCAL CAUSALITY: FOR EVERY $p \in M$: $\exists$ A NEIGHBOURHOOD $N \ni p$ S.T. (a) - (g) IS SATISFIED FOR $\ll_N$ AND $<_N$.

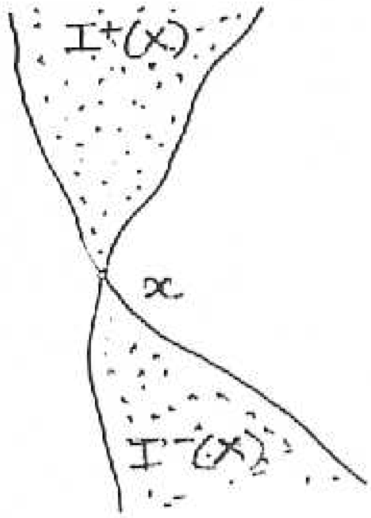
GLOBALLY, HOWEVER, (b) & (c) CAN BE VIOLATED, GIVING RISE TO CLOSED TIME-LIKE &/OR CAUSAL CURVES:



PAST & FUTURE SETS

$$I^+(x) = \{y \in M \mid x \ll y\} \; ; \; I^-(x) = \{y \in M \mid y \ll x\}$$

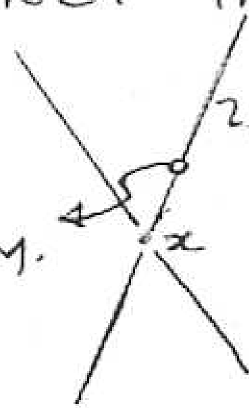
$$J^+(x) = \{y \in M \mid x < y\} \; ; \; J^-(x) = \{y \in M \mid y < x\}$$



$I^{\pm}(x)$ ARE OPEN SETS IN M.

$J^{\pm}(x)$ ARE CLOSED SETS IN MINKOWSKI SPACETIME, BUT NOT IN GENERAL:

EXAMPLE: REMOVE A POINT IN ${}^4M$.



THESE POINTS DO NOT BELONG TO $J^+(x)$.

a) $I^{\pm}(S) = \bigcup_{x \in S} I^{\pm}(x)$ (b) $J^{\pm}(S) = \bigcup_{x \in S} J^{\pm}(x)$

c) $I^{\pm}(S) = I^{\pm}(\overline{S})$ & (d) $I^{\pm}(S) = I^{\pm}(I^{+}(S))$.

$$J^{\pm}(S) = J^{\pm}(J^{\pm}(S)).$$

GLOBAL CAUSALITY CONDITIONS: THE CAUSAL HIERARCHY

- CHRONOLOGY CONDITION:

$$x \ll y \Rightarrow y \not\ll x \quad \forall\, x, y \in M.$$

CHRONOLOGY VIOLATING SETS: $I^+(q) \cap I^-(q)$

CAUSALITY CONDITION

$$\left.\begin{array}{l} \\ x < y,\ y < x \Rightarrow x = y. \end{array}\right\} \text{PARTIAL ORDER FOR CAUSAL SETS.}$$

CAUSALITY VIOLATING SETS: $J^+(q) \cap J^-(q)$.

WHILE IMPOSING THE CAUSALITY CONDITION SEEMS REASONABLE FOR A PHYSICAL THEORY, IT MAY NOT BE SUFFICIENT IN THE CONTINUUM, WHERE MORE COMPLICATED CAUSAL ANOMALIES CAN OCCUR.

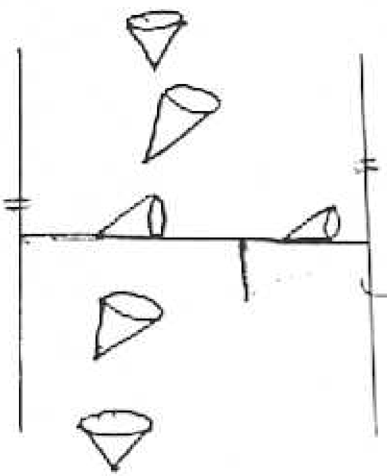
CAUSAL HIERARCHY: A LIST OF CAUSALITY CONDITIONS EACH OF WHICH IS A STRONGER RESTRICTION ON THE CLASS OF SPACETIMES THAN THE ONE BEFORE.

CAUSALITY COND $\subset \ldots\ldots \subset (\quad) \subset \ldots \subset$ GLOBAL HYP.

FUTURE / PAST DISTINGUISHING CONDITION

$$I^{\pm}(p) = I^{\pm}(q) \Rightarrow p = q \quad \forall\, p, q \in M.$$

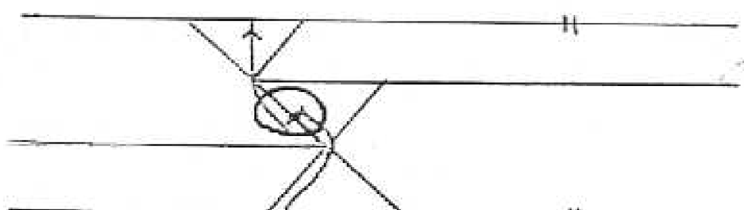
EXAMPLE:



CAUSALITY CONDITION IS VALID.
IT IS PAST DISTINGUISHABLE BUT NOT FUTURE DISTING.

STRONG CAUSALITY CONDITION

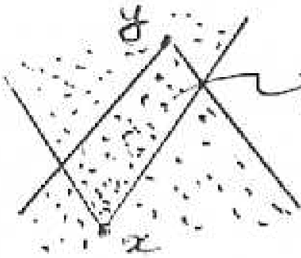
$\forall\, p \in M$: EVERY NBD. OF p CONTAINS A NBD OF p S.T. NO CAUSAL CURVE INTERSECTS IT MORE THAN ONCE



IMPOSING STRONG CAUSALITY ON A COMPACT SET MEANS THAT NO CAUSAL CURVE CAN GET "IMPRISONED" IN THIS SET.

DIGRESSION: THE ALEXANDROV TOPOLOGY

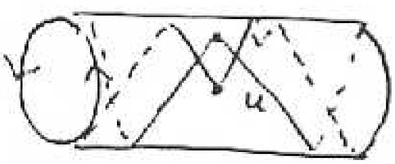
DEFN: $\langle x, y\rangle = I^+(x) \cap I^-(y)$



$\to$ OPEN SET. IN MANIFOLD TGY.

FOR ANY $x \in \langle p,q\rangle \cap \langle r,s\rangle \ \exists \ u, v \in M$, s.that
$x \in \langle u,v\rangle \subset \langle p,q\rangle \cap \langle r,s\rangle$.

$\langle x,y\rangle$: BASE FOR A TOPOLOGY ON M:
ALEXANDROV TOPOLOGY



$\langle u,v\rangle = M$ $\Rightarrow$ A. TGY $\neq$ MANIFOLD TGY
($\forall\, u,v \in M$)

THEOREM: THE FOLLOWING THREE RESTRICTIONS ON A SPACETIME ARE EQUIVALENT:

(a) (M, g) : STRONGLY CAUSAL.
(b) ALEXANDROV TOPOLOGY AGREES WITH MANIFOLD TOPOLOGY
(c) ALEXANDROV TOPOLOGY IS HAUSDORFF.

- STABLE CAUSALITY

A 'VARIATION' OR 'SMALL FLUCTUATION' IN THE SPACETIME METRIC COULD ALSO YIELD CAUSALITY VIOLATIONS.

LOR(M) : SPACE OF ALL LORENTZIAN METRICS ON M

"FINE C^r TOPOLOGY" ON LOR(M):

$B = \{\mathcal{B}_i\}$: FIXED COUNTABLE COVERING OF M S.T. $\overline{\mathcal{B}_i}$ LIES IN A COORDINATE CHART $\subseteq$ M & ANY COMPACT SUBSET OF M INTERSECTS ONLY A FINITE # OF $\mathcal{B}_i$s.

C^0 FUNCTION, $\delta : M \to (0, \infty)$

$g_1, g_2 \in$ LOR(M) ARE δ CLOSE IN THE C^r TOPOLOGY, $|g_1 - g_2| < \delta$, IF FOR EACH $p \in M$, IF IN THE FIXED COORDINATES OF ALL $\mathcal{B}_i \ni p$, g_1, g_2 ARE $\delta(p)$ CLOSE AT p & SO ARE $\left(\frac{\partial}{\partial x_i}\right)^m g_1, \left(\frac{\partial}{\partial x_i}\right)^m g_2 \; \forall \; m \leq r$.

TWO LORENTZIAN METRICS WHICH ARE CLOSE IN THE C^0 TOPOLOGY HAVE LIGHT CONES WHICH ARE CLOSE.

(M,g) IS STABLY CAUSAL IF THERE IS A FINE C^0 NEIGHBOURHOOD OF g, $\mathcal{U}(g) \subset \mathrm{LOR}(M)$ SUCH THAT EACH $g_1 \in \mathcal{U}(g)$ IS CAUSAL

A PARTIAL ORDERING FOR LOR(M):

$g_1 \leq g_2$ IF FOR EACH $p \in M$, & $v \in T_pM$, $v \neq 0$

$g_1(v,v) \leq 0 \Rightarrow g_2(v,v) \leq 0$

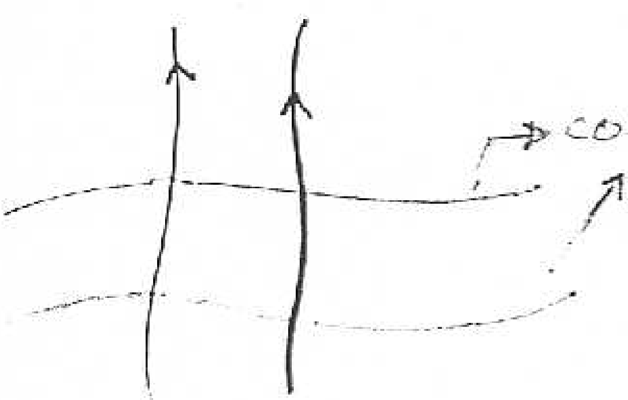
$g_1 < g_2$ IF FOR EACH $p \in M$ & $v \in T_pM$, $v \neq 0$

$g_1(v,v) \leq 0 \Rightarrow g_2(v,v) < 0$

g IS STABLY CAUSAL IFF $\exists\, g_1 \in \mathrm{LOR}(M)$, $g < g_1$

CAUSAL STABILITY $\Longleftrightarrow$ (M,g) ADMITS A GLOBAL TIME FUNCTION, f.

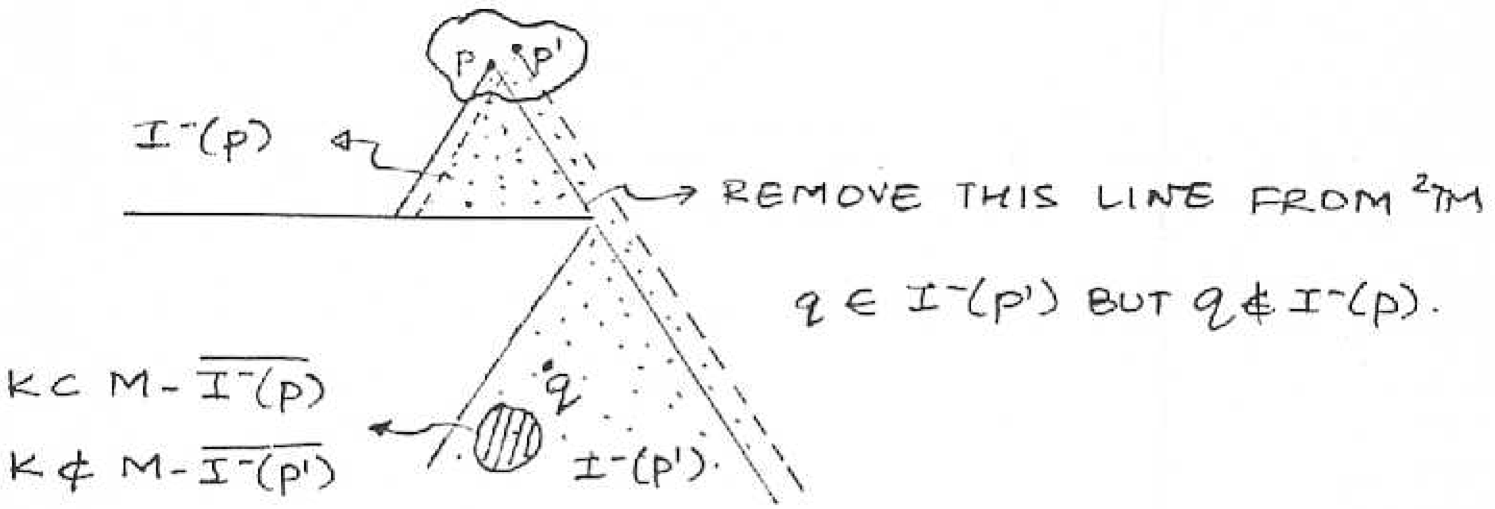
$f: M \to \mathbb{R}$, ∇f IS TIMELIKE



CONST. f SLICES: THESE MAY NOT BE DIFFEOMORPHIC TO EACH OTHER.

- CAUSAL CONTINUITY

VOLUME OF PAST & FUTURE SETS, $I^{\pm}(x)$ SHOULD VARY CONTINUOUSLY WITH x.



INNER CONTINUITY: $F: x \in M \to$ OPEN SET IN M, $F(x)$.
$F(x)$ IS INNER CONTINUOUS IF $\forall x$ AND ANY COMPACT $K \subset F(x)$, $\exists$ AN OPEN NEIGHBOURHOOD $\mathcal{U}$ OF x, S.T. $K \subset F(x') \ \forall x' \in \mathcal{U}$.

$I^{\pm}(x)$ IS INNER CONTINUOUS.

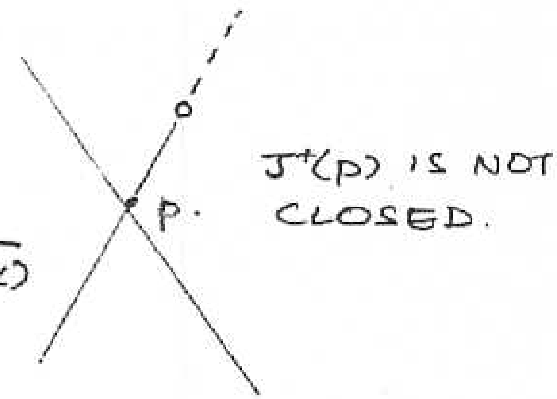
OUTER CONTINUITY: IF FOR ALL $x \in M$, AND ANY COMPACT $K \subset M - \overline{F(x)}$, $\exists$ AN OPEN NBD. $\mathcal{U}$ OF x S.T. $K \subset M - \overline{F(x')} \ \forall x' \in \mathcal{U}$.

$I^{\pm}(x)$ NEED NOT BE OUTER CONTINUOUS:

CAUSAL CONTINUITY: $I^{\pm}(x)$ IS OUTER CONT. $\forall x \in M$.

CAUSAL SIMPLICITY

(M, g) IS CAUSALLY SIMPLE IF $J^+(x)$ AND $J^-(x)$ IS CLOSED $\forall\, x \in M$.



NO "POINTS REMOVED":

CAUSAL SIMPLICITY $\Rightarrow J^\pm(x) = \overline{I^\pm(x)}$ $\forall\, x \in M$.

GLOBAL HYPERBOLICITY

(M, g) IS GLOBALLY HYPERBOLIC IF FOR EACH $p, q \in M$, $J^+(p) \cap J^-(q)$ IS COMPACT AND THE SPACETIME IS STRONGLY CAUSAL.

IN SUCH A SPACETIME, "CONSTANT TIME" SLICES ARE DIFFEOMORPHIC TO EACH OTHER.

CAUSAL HIERARCHY

$\underbrace{I^{\pm}(x),\ J^{\pm}(x)}$

CLASSICAL,
WELL-BEHAVED
SPACETIME

$\underbrace{I^{\pm}(x),\ K^{\pm}(x)}$

(SORKIN & WOOLGAR

$\boxed{\text{GLOBAL HYPERBOLICITY}}$

$\Downarrow$

$\boxed{\text{CAUSAL SIMPLICITY}}$

$\Downarrow$

PATIAL
OPOLOGY
LUCTUATES

$\boxed{\text{CAUSAL CONTINUITY}}$

AXIOMATIC
DEFINITION

$\Downarrow$

$\boxed{\text{CAUSAL STABILITY}}$

$\Downarrow$

$\boxed{\text{STRONG CAUSALITY}}$

$\Downarrow$

XIOMATIC
EFINITION:
PURELY
ORDER THEORETIC

$\boxed{\text{CAUSALITY}}$

REFERENCES

- R. PENROSE : "TECHNIQUES OF DIFFERENTIAL TOPOLOGY IN RELATIVITY"
- BEEM & EHRLICH : "GLOBAL LORENTZIAN GEOMETRY"
- HAWKING & ELLIS : "THE LARGE SCALE STRUCTURE OF SPACE-TIME"
- KRONHEIMER & PENROSE : "ON THE STRUCTURE OF CAUSAL SPACES": Proc. Cambr. Phil. Soc. (1967), 63, 481
- HAWKING & SACHS : "CAUSALLY CONTINUOUS SPACETIMES": Comm. Math. Phys. 35, 287-296 (1974).
- SORKIN & WOOLGAR : "A CAUSAL ORDER FOR SPACETIMES WITH C^0 LORENTZIAN METRICS": Class. & Quant. Gravity, (1996) 13, 1971-1994.