The Mirror of Mathematics

Part I: Classical Stone Duality

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Spring School, Oxford 20 - 22 May 2014

Outline

What am I trying to do?

1. Describe the general notion of duality
2. Work through three specific dualities:
    - Stone,
    - Gelfand and
    - Pontryagin.

What I am not trying to do

- Review all possible dualities.
- Discuss my recent work on automata minimization or Markov processes.
- Prove everything in detail.
- Discuss all the physical connections in detail.

Examples of duality principles

- “and” vs “or” in propositional logic
- Linear programming
- Electric and magnetic fields
- Controllability and observability in control theory: Kalman
- State-transformer and weakest-precondition semantics: Plotkin, Smyth
- Forward and backward dataflow analyses
- Induction and co-induction.

What is duality intuitively ?

- Two types of structures: Foo and Bar.
- Every Foo has an associated Bar and vice versa.
- $V \longrightarrow S$, $S \longrightarrow V'$; V and V' are isomorphic.
- Two *apparently* different structures are actually two different descriptions of the same thing.
- More importantly, given a map: $f : S_1 \longrightarrow S_2$ we get a map $\hat{f} : V_2 \longrightarrow V_1$ and vice versa;
- note the *reversal* in the direction of the arrows.
- The two mathematical universes are *mirror images* of each other.
- Two completely different sets of theorems that one can use.

Examples of such dualities

- Vector spaces and vector spaces.
- Boolean algebras and Stone spaces. [Stone]
- State transformer semantics and weakest precondition semantics. [DeBakker,Plotkin,Smyth]
- Logics and Transition systems. [Bonsangue, Kurz,...]
- Measures and random variables. [Kozen]
- L^P and L^q spaces with $\frac{1}{p} + \frac{1}{q} = 1$.
- Commutative unital C*-algebras and compact Hausdorff spaces. [Gelfand, Stone]

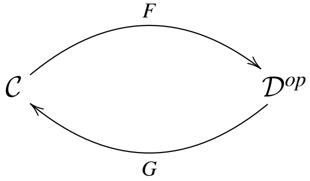
Background

- Basic category theory: functors, natural transformations, adjunctions.
- Elementary algebra: linear algebra, very basic group theory.
- Topology: open neighbourhood, closed sets, connectedness, separation axioms, compactness, continuous functions, homeomorphisms.
- “The reader should not be discouraged if (s)he does not have the prerequisites to read the prerequisites.” - Paul Halmos.

Maps matter!

- An essential aspect of mathematics: *structure-preserving maps between objects*.
- Interesting constructions on objects (usually) have corresponding constructions on the maps.
- Compositions are *preserved* or *reversed*.
- This is *functoriality*.
- From this one can often conclude *invariance properties*.

Duality categorically



Duality categorically

Given

$$\begin{array}{c} A \in \mathcal{C} \\ \downarrow f \\ B \in \mathcal{C} \end{array}$$

Duality categorically

We get

$$\begin{array}{ccc} A \in \mathcal{C} & & F(A) \in \mathcal{D} \\ \Big\downarrow f & & \\ B \in \mathcal{C} & & F(B) \in \mathcal{D} \end{array}$$

Duality categorically

and

$$\begin{array}{ccc} A \in \mathcal{C} & & F(A) \in \mathcal{D} \\ \Big\downarrow f & & \Big\uparrow F(f) \\ B \in \mathcal{C} & & F(B) \in \mathcal{D}. \end{array}$$

Duality categorically

Similarly, given

$$\begin{array}{c} C \in \mathcal{D} \\ \downarrow g \\ D \in \mathcal{D} \end{array}$$

Duality categorically

We get

$$\begin{array}{ccc} G(C) \in \mathcal{C} & & C \in \mathcal{D} \\ & & \downarrow g \\ G(D) \in \mathcal{C} & & D \in \mathcal{D} \end{array}$$

Duality categorically

and

$$
\begin{array}{ccc}
G(C) \in \mathcal{C} & \qquad & C \in \mathcal{D} \\
\uparrow{\scriptstyle G(g)} & & \downarrow{\scriptstyle g} \\
G(D) \in \mathcal{C} & \qquad & D \in \mathcal{D}.
\end{array}
$$

Isomorphisms

We have isomorphisms

$$A \simeq G(F(A)) \text{ and } C \simeq F(G(C)).$$

Duality categorically

Categorical Duality

We have a (contravariant) adjunction between categories $\mathcal{C}$ and $\mathcal{D}$, which is an *equivalence* of categories.

Often obtained by looking at maps into an object living in both categories: a schizophrenic object.

A duality that you know and love (I)

- Finite-dimensional vector space V over, say, $\mathbb{C}$.
- *Dual space* V^* of linear maps from V to $\mathbb{C}$.
- V^* has the same dimension as V and a (basis-dependent) isomorphism between V and V^*.
- The double dual V^{**} is also isomorphic to V
- with a “nice” canonical isomorphism:
  $v \in V \mapsto \lambda\sigma \in V^*.\sigma(v)$.

A duality that you know and love (II)

$$U \xrightarrow{\theta} V$$

$$U^* \xleftarrow[\theta^*]{} V^*$$

Given a linear maps θ between vector spaces U and V we get a map θ^* *in the opposite direction* between the dual spaces:

$$\theta^*(\sigma \in V^*)(u \in U) = \sigma(\theta(u)).$$

Boolean algebras

A Boolean algebra is a set A equipped with two constants, $0, 1$, a unary operation $(\cdot)'$ and two binary operations $\vee, \wedge$ which obey the following axioms, p, q, r are arbitrary members of A:

$$\begin{aligned} 0' &= 1 & 1' &= 0 \\ p \wedge 0 &= 0 & p \vee 1 &= 1 \\ p \wedge 1 &= p & p \vee 0 &= p \\ p \wedge p' &= 0 & p \vee p' &= 1 \\ p \wedge p &= p & p \vee p &= p \end{aligned}$$

Boolean algebras II

$$
\begin{aligned}
p'' &= p \\
(p \wedge q)' &= p' \vee q' \\
(p \vee q)' &= p' \wedge q' \\
p \wedge q &= q \wedge p \\
p \vee q &= q \vee p \\
p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\
p \vee (q \vee r) &= (p \vee q) \vee r \\
p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) \\
p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r)
\end{aligned}
$$

The operation $\vee$ is called *join*, $\wedge$ is called *meet* and $(\cdot)'$ is called *complement*. Maps are Boolean algebra homomorphisms.

Order

A Boolean algebra has a natural order.

Order from the algebraic structure

$p \leq q$ iff $p = p \wedge q$ (or $q = p \vee q$).

Boolean algebra homomorphisms are order preserving.

Examples of Boolean algebras

- All subsets of a set X, the *powerset* : $\mathcal{P}(X)$.
- The *regular* (int$($cl$(A)) = A$) open sets of a topological space.
- The collection of (equivalence classes of) formulas of classical propositional logic.
- A non-example: *all* the open sets of $\mathbb{R}$.

Atoms

Definition

An element a of a Boolean algebra B that satisfies (i) $0 < a$ and (ii) if $0 \leq p \leq a$ then $0 = p$ or $a = p$ is called an **atom.**

Example

Singleton set in $\mathcal{P}(X)$.

Definition

A Boolean algebra in which every element is the join of atoms below it is called **atomic.**

Example

A non-example: the Boolean algebra generated by the half-closed intervals of $\mathbb{R}$ is not atomic.

CABAs

Definition

An *atomic* Boolean algebra that is complete (every subset has a meet and a join) is called a **CABA**.

Every finite Boolean algebra is a CABA.

Representation theorem for CABAs

Theorem

A CABA is isomorphic to the set of all subsets of some set with the usual set-theoretic operations as the Boolean algebra structure.

Proof idea: If B is a Boolean algebra and A is its set of atoms then B is isomorphic to the power set of A.

Corollary

A Boolean algebra is isomorphic to the power set of some set iff it is complete and atomic.

Compact Hausdorff space

Compact

A topological space is said to be **compact** if every open cover has a finite subcover.

Closed and bounded subsets of $\mathbb{R}^n$ are compact.

Hausdorff

A topological space X is said to be **Hausdorff** (T_2) if for every pair of distinct points x, y there are *disjoint* open sets U, V with $x \in U$ and $y \in V$.

Compact Hausdorff spaces are "on the edge": if you add more open sets the topology fails to be compact and if you remove some open sets it fails to be Hausdorff.

Separation axioms

T_0

A topological space X is said to be $\mathbf{T_0}$ if for every pair of distinct points x and y there is an open set containing one of them but not the other.

T_1

A topological space X is said to be $\mathbf{T_1}$ if for every pair of distinct points x, y there is an open set that contains x but not y and another open set that contains y but not x.

The conditions T_0, T_1 and T_2 form a natural progression: each is strictly more stringent that its predecessor.

Regular and normal spaces

Regular (T_3)

A topological space X is said to be **regular** if it is T_1 and for every point x and *closed* set C with $x \notin C$ there are *disjoint* open sets U and V such that $x \in U$ and $C \subset V$.

Normal (T_4)

A topological space X is said to be **normal** if for every pair of *disjoint* closed subsets C, D there are *disjoint* open subsets U, V such that $C \subset U$ and $D \subset V$.

A compact Hausdorff space is automatically normal, hence also regular.

Connectedness

Connected

A topological space is said to be **connected** if there is no proper subset that is both open and closed. Equivalently, there are not two disjoint open sets whose union is the whole space.

A maximal connected subset (in the subspace topology) is called a *connected component*.

Totally disconnected

A topological space is said to be **totally disconnected** if the only connected components are the singletons.

The Cantor set is totally disconnected. The irrational numbers are another example of a totally disconnected space.

Zero dimensional spaces

A set that is both closed and open is called *clopen*. The existence of a non-trivial clopen set means that the space is not connected.

Zero dimensional space

A topological space X is said to be **zero dimensional** if there is a base for the topology consisting of clopen sets.

For (locally) compact Hausdorff spaces we have

Proposition

A locally compact Hausdorff space is totally disconnected iff it is zero dimensional.

Stone spaces

Stone spaces

A **Stone space** is a zero-dimensional compact Hausdorff space (hence totally disconnected).

These were called Boolean spaces by early authors.

Profinite groups are an example of a Stone space.

Note that the collection of clopen sets forms a Boolean algebra.

Stone spaces with continuous maps as the morphisms form a category called **Stone**.

A useful fact

Lemma

A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

Corollary

A continuous bijection from a Stone space to a compact Hausdorff space is a homeomorphism and hence, maps clopens to clopens.

The grand theorem

$$\mathbf{BA} \underset{\mathcal{C}l}{\overset{\mathcal{U}}{\rightleftarrows}} \mathbf{Stone}^{op}$$

We need to describe the functors $\mathcal{U}$ and $\mathcal{C}l$ and establish the existence of natural isomorphisms.

Filters and ultrafilters

Filter

A **filter** F in a Boolean algebra is a subset of B such that:

1. $1 \in F$,
2. $p, q \in F$ implies that $p \wedge q \in F$ and
3. $p \in F$ and $p \leq q$ implies that $q \in F$.

Ultrafilters

An **ultrafilter** U in a Boolean algebra B is a filter of B such that for every element $b \in B$, either $b \in U$ or $b' \in U$.

Observation

If B is a Boolean algebra and $\mathbf{2}$ is the two-element Boolean algebra and $h : B \longrightarrow \mathbf{2}$ is a homomorphism then $h^{-1}(\{1\})$ is an ultrafilter. All ultrafilters can be described this way.

From Boolean algebras to Stone spaces

1. View $\mathbf{2}$ as a topological space with the discrete topology.
2. Let B be a Boolean algebra; the space $\mathbf{2}^B$ of *arbitrary functions*, endowed with the product topology, is a Stone space.
3. The basic clopens are of the form $\{f \mid f(b) = \delta(b), b \in L\}$, where L is a *finite subset* of B and $\delta : L \longrightarrow \mathbf{2}$ is any function.
4. The subset $S \subset \mathbf{2}^B$ of *homomorphisms* forms a closed subset and hence is a Stone space in its own right.
5. We can identify S with the space of ultrafilters of B.

And back

1. The basic clopens of S are $\forall b \in B.\ U_b = \{u \mid b \in u\}$.
2. In short, the clopens of S correspond to the elements of B.
3. Not all opens are clopen of course.
4. Note that clopens always form a Boolean algebra.
5. S is called the *dual space* of B.
6. Given a Stone space S the Boolean algebra of its clopens is called the *dual algebra* of S.

Isomorphisms

Isomorphism theorem 1

If B is a Boolean algebra and S its dual space and A is the dual algebra of S, then B and A are isomorphic as Boolean algebras.

Isomorphism theorem 2

If S is a Stone space and A its dual algebra and X is the dual space of A, then S and X are homeomorphic as topological spaces.

Uses the “useful fact.”

Functorial version

1. $\mathcal{C}l$ from **Stone** to **BA**: given $f : X \longrightarrow Y$ in **Stone**, define $\mathcal{C}l(f) = f^{-1} : \mathcal{C}l(Y) \longrightarrow \mathcal{C}l(X)$.
2. $\mathcal{U} : \mathbf{BA} \longrightarrow \mathbf{Stone}$: given $h : A \longrightarrow B$ in **BA**, define $\mathcal{U}(h) : \mathcal{U}(B) \longrightarrow \mathcal{U}(A)$ by $g : B \longrightarrow \mathbf{2} \mapsto g \circ h(: A \longrightarrow \mathbf{2})$.

Stone duality

Let S be a Stone space and B a Boolean algebra. There is a natural bijection between the hom-sets $\mathbf{BA}(B, \mathcal{C}l(S))$ and $\mathbf{Stone}(S, \mathcal{U}(B)))$ (or $\mathbf{Stone}^{op}(\mathcal{U}(B), S)$.

We have an adjunction $\mathcal{U}^{op} \dashv \mathcal{C}l$, in fact we have an equivalence of categories, because the natural transformations associated with the adjunction are isomorphisms.

Schizophrenia

- What is $\mathbf{2}$?
- It is a two-element Boolean algebra: $\mathcal{U}(B) = \mathbf{BA}(B, 2)$.
- It is also a two-point topological space, in fact a Stone space
- and $\mathcal{C}l(S) = \mathbf{Stone}(S, \mathbf{2})$.
- Many dualities are mediated by such “schizophrenic” objects.

Prime spectrum

- In a lattice a filter, F, is said to be **prime** if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.
- Unlike in Boolean algebras, prime filters are not the same thing as *maximal filters*.
- The prime spectrum of a lattice is the collection of prime filters of the lattice.

Priestley duality

- Priestley defined a new topology on the prime spectrum of a bounded distributive lattice.
- This topology is both compact and Hausdorff.
- However, there is also an order structure that plays a crucial role.
- A Priestley space is a compact ordered topological space where the clopen *down-sets* separate points.
- One gets a duality theorem between Priestley spaces and bounded distributive lattices.

Stonean spaces

- A compact Hausdorff space is said to be **Stonean** if the closure of every open set is open (hence clopen).
- Every Stonean space is a Stone space but not *vice versa*.
- There is a duality between Stonean spaces and *complete* Boolean algebras: important in the theory of C^* algebras.