Approximating Markov Processes via Averaging

Bisimulation, minimal realization and approximation

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- We can define a notion of smallest process that is bisimilar to a given process.
- We can define a notion of finite approximation and construct a projective limit of the finite approximants.
- This yields the minimal realization.

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- Any cone C defines a order by $u \leq v$ if $v - u \in C$.
- Many of the structures that we want to look at are cones *e.g.* the measures on a space.

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- We will denote these cones by $L_1^+(X, \Sigma, \mu)$ and $L_\infty^+(X, \Sigma)$.

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- We write $\mathcal{M}^p_{\mathsf{UB}}(X)$ for the cone of all measures on (X, Σ) that are uniformly less than a multiple of the measure p: $q \in \mathcal{M}^p_{\mathsf{UB}}$ means that for some real constant $K > 0$ we have $q \leq Kp$.

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The pairing

Pairing function

There is a map from the product of the cones $L_\infty^+(X,p)$ and $L_1^+(X,p)$ to $\mathbb{R}^+$ defined as follows:

$$\forall f \in L_\infty^+(X,p), g \in L_1^+(X,p) \quad \langle f,\ g \rangle = \int fg \mathrm{d}p.$$

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This map is bilinear and is continuous and ω-continuous in both arguments; we refer to it as the pairing.

Duality is the Key

$$\begin{array}{ccccc} \mathcal{M}^{\ll p}(X) & \overset{\sim}{\longleftrightarrow} & L_1^+(X,p) & \overset{\sim}{\longleftrightarrow} & L_\infty^{+,*}(X,p) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{M}_{\mathrm{UB}}^{p} & \overset{\sim}{\longleftrightarrow} & L_\infty^+(X,p) & \overset{\sim}{\longleftrightarrow} & L_1^{+,*}(X,p) \end{array} \qquad (1)$$

Here the vertical arrows represent dualities and the horizontal arrows represent isomorphisms.

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- $p << q$, write $\frac{\mathrm{d}p}{\mathrm{d}q}$ for the Radon-Nikodym derivative $p(A) = \int_A \frac{\mathrm{d}p}{\mathrm{d}q}\mathrm{d}q$.

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- Going between these versions by duality will be very useful.

The "infinity" category

$\mathbf{Rad}_\infty$

The category $\mathbf{Rad}_\infty$ has as objects probability spaces, and as arrows $\alpha : (X, p) \longrightarrow (Y, q)$, measurable maps such that $M_\alpha(p) \leq Kq$ for some real number K.

The reason for choosing the name $\mathbf{Rad}_\infty$ is that $\alpha \in \mathbf{Rad}_\infty$ maps to $d/dqM_\alpha(p) \in L^+_\infty(Y, q)$.

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2. The fact that the category $\mathbf{Rad}_\infty$ embeds in $\mathbf{Rad}_1$ reflects the fact that L_∞^+ embeds in L_1^+.

Pairing function revisited

Recall the isomorphism between $L_\infty^+(X,p)$ and $L_1^{+,*}(X,p)$ mediated by the pairing function:

$$f \in L_\infty^+(X,p) \mapsto \lambda g : L_1^+(X,p).\langle f,\ g\rangle = \int fg\mathrm{d}p.$$

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Expectation value functor

- The **functor** $\mathbb{E}_\infty(\cdot)$ is a functor from $\mathbf{Rad}_\infty$ to ω**CC** which, on objects, maps (X, p) to $L^+_\infty(X, p)$ and on maps is given as follows:

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- Given $\alpha : (X, p) \longrightarrow (Y, q)$ in $\mathbf{Rad}_\infty$ the action of the functor is to produce the map $\mathbb{E}_\infty(\alpha) : L^+_\infty(X, p) \longrightarrow L^+_\infty(Y, q)$ obtained by composing $(P_1(\alpha))^*$ with the isomorphisms between $L_1^{+,*}$ and L^+_∞

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- Given $\alpha:(X,p)\longrightarrow(Y,q)$ in $\mathbf{Rad}_\infty$ the action of the functor is to produce the map $\mathbb{E}_\infty(\alpha):L^+_\infty(X,p)$ $\longrightarrow L^+_\infty(Y,q)$ obtained by composing $(P_1(\alpha))^*$ with the isomorphisms between $L^{+,*}_1$ and L^+_∞

$$\begin{array}{ccc} L^{+,*}_1(X,p) & \longleftarrow\cdots & L^+_\infty(X,p) \\ {\scriptstyle (P_1(\alpha))^*}\downarrow & & \downarrow{\scriptstyle \mathbb{E}_\infty(\alpha)} \\ L^{+,*}_1(Y,q) & \cdots\longrightarrow & L^+_\infty(Y,q) \end{array}$$

The other expectation value functor

The **functor** $\mathbb{E}_\mathbf{1}(\cdot)$ is a functor from $\mathbf{Rad}_1$ to ω**CC** which maps the object (X, p) to $L_1^+(X, p)$ and on maps is given as follows:
Given $\alpha : (X, p) \longrightarrow (Y, q)$ in $\mathbf{Rad}_1$ the action of the functor is to produce the map $\mathbb{E}_1(\alpha) : L_1^+(X, p) \longrightarrow L_1^+(Y, q)$ obtained by composing $(P_\infty(\alpha))^*$ with the isomorphisms between $L_\infty^{+,*}$ and L_1^+ as shown in the diagram below

$$\begin{array}{ccc}
L_\infty^{+,*}(X,p) & \longleftarrow & L_1^+(X,p) \\
{\scriptstyle (P_\infty(\alpha))^*}\downarrow & & \downarrow{\scriptstyle \mathbb{E}_1(\alpha)} \\
L_\infty^{+,*}(Y,q) & \longrightarrow & L_1^+(Y,q)
\end{array}$$

The approximation map

The expectation value functors project a probability space onto another one with a possibly coarser σ-algebra.
Given an AMP on (X, p) and a map $\alpha : (X, p) \to (Y, q)$ in $\mathbf{Rad}_\infty$, we have the following approximation scheme:

Approximation scheme

$$\begin{array}{ccc} L^+_\infty(X,p) & \xrightarrow{\tau_a} & L^+_\infty(X,p) \\ {\scriptstyle P_\infty(\alpha)}\uparrow & & \downarrow{\scriptstyle \mathbb{E}_\infty(\alpha)} \\ L^+_\infty(Y,q) & \xrightarrow{\alpha(\tau_a)} & L^+_\infty(Y,q) \end{array}$$

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- Take (X, Σ) and (X, Λ) with $\Lambda \subset \Sigma$ and use the measurable function $id : (X, \Sigma) \longrightarrow (X, \Lambda)$ as α.

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$$

- Thus $id(\tau_a)$ is the approximation of τ_a obtained by averaging over the sets of the coarser σ-algebra Λ.

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The category $\mathbf{AMP}$

Objects: probability spaces (X, Σ, p), along with an abstract Markov process τ_a on X.
Arrows: $\alpha : (X, \Sigma, p, \tau_a) \longrightarrow (Y, \Lambda, q, \rho_a)$ are surjective measure-preserving maps from X to Y such that $\alpha(\tau_a) = \rho_a$.

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- We define the category $\mathbf{Rad}_=$ to have the same objects as $\mathbf{AMP}$ but the maps are only measure preserving (and, of course, measurable).

Bisimulation traditionally

Larsen-Skou definition

Given an **LMP** (S, Σ, τ_a) an equivalence relation R on S is called a *probabilistic bisimulation* if sRt then for every *measurable* R-closed set C we have for every a

$$\tau_a(s, C) = \tau_a(t, C).$$

This variation to the continuous case is due to Josée Desharnais and her Indian friends.

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- This means τ_a sends the subspace $L^+_\infty(X, \Lambda, p)$ to itself; where we are now viewing τ_a as a map on $L^+_\infty(X, \Lambda, p)$.

The bisimulation diagram

$$\begin{array}{ccc} L^+_\infty(X,\Sigma,p) & \xrightarrow{\tau_a} & L^+_\infty(X,\Sigma,p) \\ \Big\uparrow & & \Big\uparrow \\ L^+_\infty(X,\Lambda,p) & \xrightarrow{\tau_a} & L^+_\infty(X,\Lambda,p) \end{array}$$

Zigzag maps

We can generalize the notion of event bisimulation by using maps other than the identity map on the underlying sets. This would be a map α from (X, Σ, p) to (Y, Λ, q), equipped with LMPs τ_a and ρ_a respectively, such that the following commutes:

$$\begin{array}{ccc} L_\infty^+(X,\Sigma,p) & \xrightarrow{\tau_a} & L_\infty^+(X,\Sigma,p) \\ P_\infty(\alpha)\uparrow & & \uparrow P_\infty(\alpha) \\ L_\infty^+(Y,\Lambda,q) & \xrightarrow{\rho_a} & L_\infty^+(Y,\Lambda,q) \end{array} \quad (2)$$

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- With the cospan definition we get logical characterization on *all* measurable spaces.
- On analytic spaces the two concepts co-incide.
- Recent results show that the theory cannot be made to work with spans on general measurable spaces.

The official definition of bisimulation

Bisimulation

We say that two objects of $\mathbf{AMP}$, (X, Σ, p, τ) and (Y, Λ, q, ρ), are *bisimilar* if there is a third object (Z, Γ, r, π) with a pair of zigzags

$$\alpha : (X, \Sigma, p, \tau) \longrightarrow (Z, \Gamma, r, \pi)$$
$$\beta : (Y, \Lambda, q, \rho) \longrightarrow (Z, \Gamma, r, \pi)$$

giving a cospan diagram

$$\begin{array}{ccc} (X, \Sigma, p, \tau) & & (Y, \Lambda, q, \rho) \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & (Z, \Gamma, r, \pi) & \end{array} \tag{3}$$

Note: identity is a zigzag.

Fundamental categorical result

The category $\mathbf{AMP}$ has pushouts

Furthermore, if the morphisms in the span are zigzags then the morphisms in the pushout diagram are also zigzags.

Pushouts explicitly

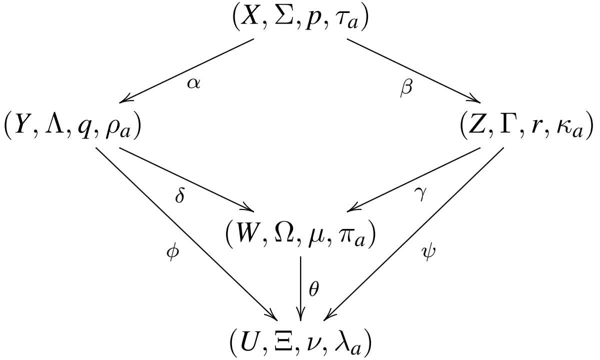
More explicitly, let $\alpha : (X, \Sigma, p, \tau_a) \longrightarrow (Y, \Lambda, q, \rho_a)$ and $\beta : (X, \Sigma, p, \tau_a) \longrightarrow (Z, \Gamma, r, \kappa_a)$ be a span in $\mathbf{AMP}$. Then there is an object (W, Ω, μ, π_a) of $\mathbf{AMP}$ and $\mathbf{AMP}$ maps $\delta : Y \longrightarrow W$ and $\gamma : Z \longrightarrow W$ such that the diagram

$$\begin{array}{ccc} & (X, \Sigma, p, \tau_a) & \\ \alpha \swarrow & & \searrow \beta \\ (Y, \Lambda, q, \rho_a) & & (Z, \Gamma, r, \kappa_a) \\ \delta \searrow & & \swarrow \gamma \\ & (W, \Omega, \mu, \pi_a) & \end{array} \tag{4}$$

commutes.

Couniversality

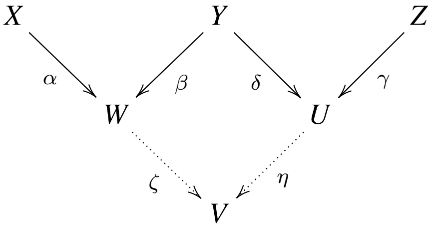
If (U, Ξ, ν, λ_a) is another **AMP** object and $\phi : Y \longrightarrow U$ and $\psi : Z \longrightarrow U$ are **AMP** maps such that α, β, ϕ and ψ form a commuting square, then there is a unique **AMP** map $\theta : W \longrightarrow U$ such that the diagram



(5)

commutes.

Bisimulation is an equivalence



(6)

The pushouts of the zigzags β and δ yield two more zigzags ζ and η (and the pushout object V). As the composition of two zigzags is a zigzag, X and Z are bisimilar. Thus bisimulation is transitive.

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- Obviously, the concept cannot be based on counting states.
- We want to look for a bisimulation equivalent version of the process; hence with the same behaviour,
- such that any other process with the same behaviour contains this one.
- This is a classic couniversality property.

Bisimulation-minimal realization

Definition of minimal realization

Given an AMP (X, Σ, p, τ_a), a **bisimulation-minimal realization** of this abstract Markov process is an AMP $(\tilde{X}, \Gamma, r, \pi_a)$ and a zigzag in $\mathbf{AMP}$ $\eta : X \longrightarrow \tilde{X}$ such that for every zigzag β from X to another AMP (Y, Λ, q, ρ_a), there is a zigzag γ from (Y, Λ, q, ρ_a) to $(\tilde{X}, \Gamma, r, \pi_a)$ with $\eta = \gamma \circ \beta$.

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If we think of a zigzag as defining a quotient of the original space then $\tilde{X}$ is the "most collapsed" version of X.

Existence theorem

Given any AMP (X, Σ, p, τ_a) there exists another AMP $(\tilde{X}, \Gamma, r, \pi_a)$ and a zigzag η in $\mathbf{AMP}$, $\eta : X \longrightarrow \tilde{X}$ such that $(\tilde{X}, \Gamma, r, \pi_a)$ and η define a bisimulation-minimal realization of (X, Σ, p, τ_a).

Proof idea: Intersect all event bisimulations to get a smallest (fewest sets in the σ-algebra) event bisimulation. Define the associated equivalence relation and form the quotient.

Two AMPs and are bisimilar if and only if their minimal realizations are isomorphic.

A modal logic

We define a logic $\mathcal{L}$ as follows, with $a \in \mathcal{A}$:

$$\mathcal{L} ::= \mathbf{T} | \phi \wedge \psi | \langle a \rangle_q \psi$$

Given a labelled AMP (X, Σ, p, τ_a), we associate to each formula ϕ a measurable set $[\![\phi]\!]$, defined recursively as follows:

$$\begin{aligned} [\![\mathbf{T}]\!] &= X \\ [\![\phi \wedge \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\ \left[\!\!\left[\langle a \rangle_q \psi \right]\!\!\right] &= \{ s : \tau_a(\mathbf{1}_{[\![\psi]\!]})(s) > q \} \end{aligned}$$

We let $[\![\mathcal{L}]\!]$ denotes the measurable sets obtained by all formulas of $\mathcal{L}$.

Logical characterization of bisimulation

Main theorem

Given a LAMP (X, Σ, p, τ_a), the σ-field $\sigma([\![\mathcal{L}]\!])$ generated by the logic $\mathcal{L}$ is the smallest event-bisimulation on X. That is, the map $i : (X, \Sigma, p, \tau_a) \longrightarrow (X, \sigma([\![\mathcal{L}]\!]), p, \tau_a)$ is a zigzag; furthermore, given any zigzag $\alpha : (X, \Sigma, p, \tau_a) \longrightarrow (Y, \Lambda, q, \rho_a)$, we have that $\sigma([\![\mathcal{L}]\!]) \subseteq \alpha^{-1}(\Lambda)$.

Hence, the σ-field obtained on X by the smallest event bisimulation is precisely the σ-field we obtain from the logic.

Finite approximations

Let (X, Σ, p, τ_a) be a LAMP. Let $\mathcal{P} = 0 < q_1 < q_2 < \ldots < q_n < 1$ be a finite partition of the unit interval with each q_i a rational number. We call these *rational partitions*. We define a family of finite π-systems, subsets of Σ, as follows:

$$
\begin{aligned}
\Phi_{\mathcal{P},0} &= \{X, \emptyset\} \\
\Phi_{\mathcal{P},n} &= \pi\left(\{\tau_a(\mathbf{1}_A)^{-1}(q_i, 1] : q_i \in \mathcal{P}, A \in \Phi_{\mathcal{P},n-1}, a \in \mathcal{A}\} \cup \Phi_{\mathcal{P},n-1}\right) \\
&= \pi\left(\left\{\left[\!\left[\langle a \rangle_{q_i} \mathbf{1}_A\right]\!\right] : q_i \in \mathcal{P}, A \in \Phi_{\mathcal{P},n-1}, a \in \mathcal{A}\right\} \cup \Phi_{\mathcal{P},n-1}\right)
\end{aligned}
$$

where $\pi(\Omega)$ means the π-system generated by the family of sets Ω.

Approximation pairs

For each pair $(\mathcal{P}, M)$ consisting of a rational partition and a natural number, we define a σ-algebra $\Lambda_{\mathcal{P},M}$ on X as $\Lambda_{\mathcal{P},M} = \sigma\left(\Phi_{\mathcal{P},M}\right)$, the σ-algebra generated by $\Phi_{\mathcal{P},M}$. We call each pair $(\mathcal{P}, M)$ consisting of a rational partition and a natural number an *approximation pair*.

Approximation and Logic

The following result links the finite approximation with the formulas of the logic used in the characterization of bisimulation.

Crucial fact

Given any labelled AMP (X, Σ, p, τ_a), the σ-algebra $\sigma\left(\bigcup \Phi_{\mathcal{P},M}\right)$, where the union is taken over all approximation pairs, is precisely the σ-algebra $\sigma\,[\![\mathcal{L}]\!]$ obtained from the logic.

Relating finite approximations

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$$i_{(\mathcal{Q},N),(\mathcal{P},M)} : (X, \Lambda_{\mathcal{Q},N}, \Lambda_{\mathcal{Q},N}(\tau_a)) \longrightarrow (X, \Lambda_{\mathcal{P},M}, \Lambda_{\mathcal{P},M}(\tau_a))$$

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- Furthermore if $(\mathcal{P}, M) \leq (\mathcal{Q}, N) \leq (\mathcal{R}, K)$ the maps compose to give

$$i_{(\mathcal{R},K),(\mathcal{P},M)} = i_{(\mathcal{R},K),(\mathcal{Q},N)} \circ i_{(\mathcal{Q},N),(\mathcal{P},M)}.$$

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- Furthermore if $(\mathcal{P}, M) \leq (\mathcal{Q}, N) \leq (\mathcal{R}, K)$ the maps compose to give
$$i_{(\mathcal{R},K),(\mathcal{P},M)} = i_{(\mathcal{R},K),(\mathcal{Q},N)} \circ i_{(\mathcal{Q},N),(\mathcal{P},M)}.$$
- In short we have a projective system of such maps indexed by our poset of approximation pairs.

Finite spaces

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- When the approximations are refined the quotients compose so we can define maps between quotient spaces.

Projective diagram fragment

We get the following commuting diagram:

$$\begin{array}{ccc}
(X, \Lambda_{\mathcal{Q},N}, \Lambda_{\mathcal{Q},N}(\tau_a)) & \xrightarrow{i_{(\mathcal{Q},N),(\mathcal{P},M)}} & (X, \Lambda_{\mathcal{P},M}, \Lambda_{\mathcal{P},M}(\tau_a)) \\
\downarrow {\scriptstyle \pi_{\mathcal{Q},N}} & & \downarrow {\scriptstyle \pi_{\mathcal{P},M}} \\
(\hat{X}_{\mathcal{Q},N}, \phi_{\mathcal{Q},N}(\tau_a)) & \xrightarrow[j_{(\mathcal{Q},N),(\mathcal{P},M)}]{} & (\hat{X}_{\mathcal{P},M}, \phi_{\mathcal{P},M}(\tau_a))
\end{array} \quad (7)$$

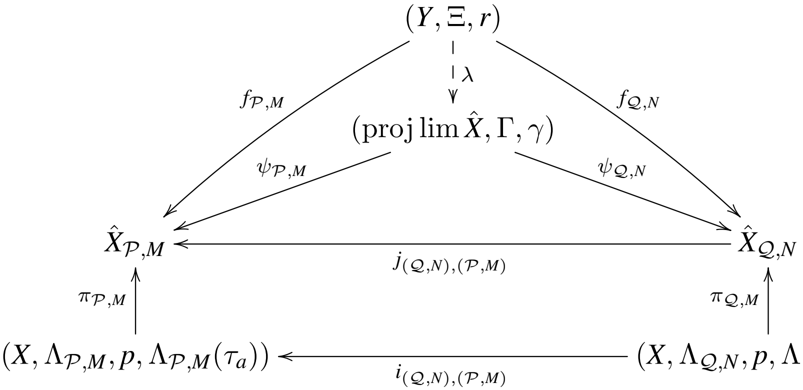
Existence of a projective limit

Main theorem

The probability spaces of finite approximants $\hat{X}_{\mathcal{P},M}$ of a measure space (X, Σ, p, τ_a) each equipped with the discrete σ-algebra (i.e. the σ-algebra of all subsets) indexed by the approximation pairs, form a projective system in the category $\mathbf{Rad}_=$. This system of finite approximants to the LAMP (X, Σ, p, τ_a) has a projective limit in the category $\mathbf{Rad}_=$.

This uses a theorem of Choksi from 1958. In typical analysis style, he constructs the required limit but does not prove any universal property. It was a non-trivial extension to show this.

Picture of the situation



What can we say about the LAMP?

We can now consider the LAMP structure. We do not get a universal property in the category $\mathbf{AMP}$, however, the universality of the construction in $\mathbf{Rad}_{=}$ almost forces the structure of a LAMP on the projective limit constructed in $\mathbf{Rad}_{=}$.

LAMP on the projective limit

A LAMP can be defined on the projective limit constructed in $\mathbf{Rad}_{=}$ so that the cone formed by this limit object and the maps to the finite approximants yields a commuting diagram in the category $\mathbf{AMP}$.

Approximation and minimal realization

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- This gives a very pleasing connection between the approximation process and the minimal realization.

Two routes to the minimal realization

Given an AMP (X, Σ, p, τ_a), the projective limit of its finite approximants $(\mathrm{proj}\lim \hat{X}, \Gamma, \gamma, \zeta_a)$ is isomorphic to its minimal realization $(\tilde{X}, \Xi, r, \xi_a)$.

Summary

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- Approximation via averaging is done through conditional expectation.
- There is a modal logic characterizing bisimulation which naturally defines finite approximants.
- The limit of these finite approximants reconstructs a minimal realization of the original process.

Other related results

- A general theory with all L_p spaces done and lost in a JACM black hole.

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- We have developed a Stone-type duality for Markov processes.
- Projective limit in $\mathbf{AMP}$?
- My student Florence Clerc is using these ideas for continuous-time processes (Feller-Dynkin).