Program Verification:
An overview of the series

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School of Computer Science
McGill University
The lecturers
The lecturers
The lecturers

- Joel Ouaknine; University of Oxford
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- Joel Ouaknine; University of Oxford
- James Worrell; University of Oxford
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- Joel Ouaknine; University of Oxford
- James Worrell; University of Oxford
- Amy Felty; University of Ottawa
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- Joel Ouaknine; University of Oxford
- James Worrell; University of Oxford
- Amy Felty; University of Ottawa
- Stephen Brookes; Carnegie-Mellon University
The topics
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- Real-time systems (Ouaknine)
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- Probabilistic systems (Worrell)
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- Theorem proving techniques (Felty)
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- Probabilistic systems (Worrell)
- Theorem proving techniques (Felty)
- Concurrent systems (Brookes)
Early history: formalisms
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- Scott, deBakker 1969: fixed-point induction
- Hoare 1969: axiomatic semantics
- Early 1970s: predicate transformers (Dijkstra)
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* Abstract interpretation (1976) Cousot and Cousot
* Model checking (Clarke, Emerson, Sifakis)
Disasters and Opportunities
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- The Ariane-5 disaster
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- The attack on the Needham-Schroder protocol
Some areas of current research
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- Abstraction techniques
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- Probabilistic verification
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- Probabilistic verification
- Real-time systems
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- Security
The Main Point

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Semantics and Axioms
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- Ideally it should be compositional.
- One should be able to extract the relevant aspects of the program through axioms that capture the semantics.
Hoare Logic
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- {P} S {Q}: If P holds before execution of S then Q will hold after S terminates, if S does indeed terminate.
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- \{P\} S \{Q\}: If \(P\) holds before execution of \(S\) then \(Q\) will hold after \(S\) terminates, if \(S\) does indeed terminate.

- Compositionality: From \(\{P\} S \{R\}\) and \(\{R\} S' \{Q\}\) deduce \(\{P\} S;S' \{Q\}\).
Dynamic Logic
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- A modal logic with programs and formulas defined by mutual induction.
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- Every program defines a modality.
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- Every program defines a modality.
- Challenging from the point of view of basic theory: the canonical model construction does not work.
Recursion

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Recursion

- How to make sense of recursion compositionally?
- Fixed-point theory (Kleene).
- \( \text{rec } f. \ F[f] \) is the solution of \( f = F[f] \).
- Fixed-point induction for programs: Scott and deBakker, Park.
Denotational Semantics
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Continuous functions from \( D \) to itself have *least fixed points*. 
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The function spaces are themselves data types.

Continuous functions from $D$ to itself have *least fixed points*.

The meaning of a recursively defined function from $D$ to $D$ is given by the least fixed point of a functional from $D \to D$ to $D \to D$. 
Fixed-point Induction
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Let $D$ be a dcpo. A subset $S \subseteq D$ is called \textit{chain-closed} if for all chains

$$d_0 \leq d_1 \leq d_2 \leq \ldots$$

in $D$, we have

$$\forall n.d_n \in S \Rightarrow \bigvee_n d_n \in S.$$
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Similarly a property $\Phi$, may be admissible.
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If \( S \) contains \( \bot \) and is chain closed, we call it \textit{admissible}.

Similarly a property \( \Phi \), may be admissible.

If \( f : D \to D \) is continuous, \( \Phi \) is admissible and
\[
 \forall d \in S, f(d) \in S
\]
then, \( \text{fix}(f) \in S \).
Fixed-point Induction

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d_0 \leq d_1 \leq d_2 \leq \ldots
in $D$, we have
\[ \forall n. d_n \in S \Rightarrow \bigvee_n d_n \in S. \]

If $S$ contains $\perp$ and is chain closed, we call it **admissible**.

Similarly a property $\Phi$, may be admissible.

If $f : D \to D$ is continuous, $\Phi$ is admissible and
\[ \forall d \in S, f(d) \in S \]
then, $fix(f) \in S$.

With this many properties of recursively defined functions can be proved.
Abstract Interpretation

How can we use denotational semantics to prove properties without computing the detailed behaviour of the program?
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How can we use denotational semantics to prove properties without computing the detailed behaviour of the program?

Use abstracted data types!
Abstract Interpretation

How can we use denotational semantics to prove properties without computing the detailed behaviour of the program?

Use abstracted data types!

\[
\begin{align*}
D & \xrightarrow{f} E \\
\alpha_1 & \iff \gamma_1 \iff \gamma_2 \iff \alpha_2 \\
A & \xrightarrow{g} B
\end{align*}
\]
Model Checking
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- Describe the system (program) as a transition system of some kind.
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- Give the specification in a suitable (dynamic) logic.
Model Checking

- Describe the system (program) as a transition system of some kind.
- Give the specification in a suitable (dynamic) logic.
- Show automatically that the system is a model of the specification.
Theorem proving
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- Describe the (relevant parts of or behaviour of) the system using formulas. [Beh]
Theorem proving

- Describe the (relevant parts of or behaviour of) the system using formulas. [Beh]
- Define the specification as another formula. [Spec]
Theorem proving

- Describe the (relevant parts of or behaviour of) the system using formulas. [Beh]
- Define the specification as another formula. [Spec]
- Prove, using semi-automatic tools if possible, that Beh implies Spec.
Which is better?
Which is better?

- Theorem proving is good when one doesn’t have a complete picture of the model and one can capture some of their properties using axioms.
Which is better?

- Theorem proving is good when one doesn’t have a complete picture of the model and one can capture some of their properties using axioms.

- Model checking allows a different formalism for describing the model and writing the specification. This allows one to use a rather restricted language for the specifications which has a better chance of being decidable.
Peace, flowers and love
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- Of course, the two approaches should co-exist.
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Theorem proving can settle properties that would require induction proofs and is much more powerful.
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- Theorem proving can settle properties that would require induction proofs and is much more powerful.
- Model checking can be a powerful tactic within a theorem proving environment.
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Theorem proving can settle properties that would require induction proofs and is much more powerful.

Model checking can be a powerful tactic within a theorem proving environment.

Abstraction is a vital tool in both cases.
Model Checking

The system is a transition system

$S$: States

$P$: Propositions

$\rightarrow \subseteq S \times S$: Transition relation

$L : S \rightarrow 2^P$: Labelling function.
Model Checking

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$S$: States
$P$: Propositions
$\rightarrow \subset S \times S$: Transition relation
$L : S \rightarrow 2^P$: Labelling function.
The Logic

○ : Next
◊ : Eventually
□ : Always
∪ : Until
The Logic

Usually temporal logic: Linear Temporal Logic (LTL) or Computation Tree Logic (CTL).

\[ \bigcirc \quad : \quad \text{Next} \]
\[ \diamond \quad : \quad \text{Eventually} \]
\[ \Box \quad : \quad \text{Always} \]
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For transition systems of the type shown CTL is more natural.

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The Logic

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For transition systems of the type shown CTL is more natural.

State formulas:
\[ \phi ::= true \mid p \mid \phi_1 \land \phi_2 \mid \neg \phi \mid \exists \psi \mid \forall \psi \]

\[ \bigcirc : \text{Next} \]
\[ \Diamond : \text{Eventually} \]
\[ \square : \text{Always} \]
\[ \bigcup : \text{Until} \]
The Logic

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For transition systems of the type shown CTL is more natural.

State formulas:
\( \phi ::= \text{true} | p | \phi_1 \land \phi_2 | \neg \phi | \exists \psi | \forall \psi \)

Path formulas:
\( \psi ::= \bigcirc \phi | \diamond \phi | \Box \phi | \phi_1 \bigcup \phi_2 \)

\( \bigcirc \) : Next
\( \diamond \) : Eventually
\( \Box \) : Always
\( \bigcup \) : Until
start state $\not\models \forall \Diamond q$
\[
\text{start state } \not\models \forall \Diamond q
\]
\[
\text{start state } \not\models \forall \Box \Diamond p
\]
This is LTL not CTL
start state \( \not\models \forall \lozenge q \)

start state \( \not\models \forall \blacksquare \Diamond p \) \hspace{1em} \text{This is LTL not CTL}

start state \( \models \exists \Diamond \blacksquare q \)
start state $\not\models \forall \Box q$

start state $\not\models \forall \Box p$  
This is LTL not CTL

start state $\models \exists \Diamond \Box q$  
This is CTL* not LTL
start state $\not\models \forall \lozenge q$

start state $\not\models \forall \Box \lozenge p$    This is LTL not CTL

start state $\models \exists \lozenge \Box q$    This is CTL* not LTL

start state $\models \exists \text{“every second state satisfies } q\text{.”}$
start state  \( \not\models \forall \Box q \)

start state  \( \not\models \forall \Diamond p \)  \( \) This is LTL not CTL

start state  \( \models \exists \Diamond \Box q \)  \( \) This is CTL* not LTL

start state  \( \models \exists \Diamond \Box q \)  \( \) “every second state satisfies \( q \).”

But “every second state satisfies \( q \)” cannot be expressed with these temporal formulas.
start state $\not\models \forall \Diamond q$

start state $\not\models \forall \square \Diamond p$  This is LTL not CTL

start state $\models \exists \Diamond \square q$  This is CTL* not LTL

start state $\models \exists \text{“every second state satisfies } q\text{.”}$

But “every second state satisfies $q$” cannot be expressed with these temporal formulas.

It can be expressed with fixed-point operators in the logic.
Semantics of the Logic

$ s \models p \quad \text{iff } p \in L(s) $

$ s \models \phi_1 \land \phi_2 \quad \text{iff } s \models \phi_1 \text{ and } s \models \phi_2 $ 

$ s \models \forall \psi \quad \text{iff } \forall \text{ paths } \pi = ss_1s_2\ldots, \pi \models \psi $ 

$ s \models \exists \psi \quad \text{iff } \exists \text{ a path } \pi = ss_1s_2\ldots, \pi \models \psi $ 

A path is a sequence of states: $ \pi = s_0s_1s_2\ldots $ 

$ \pi \models \Box \phi \quad \text{iff } s_1 \models \phi $ 

$ \pi \models \Diamond \phi \quad \text{iff } \exists j \text{ such that } s_j \models \phi $ 

$ \pi \models \square \phi \quad \text{iff } \forall j \ s_j \models \phi $ 

$ \pi \models \phi_1 \bigcup \phi_2 \quad \text{iff } \exists j \text{ such that } s_j \models \phi_2 \text{ and } \forall i < j \ s_i \models \phi_1 $
The Model-Checking Algorithm
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\[ Sat(\phi) = \{ s \mid s \models \phi \} \]
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\[ Post(s) = \{ s' \mid s \rightarrow s' \}, Pre(s) = \{ s' \mid s' \rightarrow s \} \]
The Model-Checking Algorithm

\[ Sat(\phi) = \{ s | s \models \phi \} \]

\[ Post(s) = \{ s' | s \rightarrow s' \}, \ Pre(s) = \{ s' | s' \rightarrow s \} \]

Input: TS with states \( S \), CTL state formula \( \Phi \)
Output: \( T(\subseteq S) = \{ s | s \models \Phi \} = Sat(\Phi) \).
The Model-Checking Algorithm

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Output: \( T(\subset S) = \{ s \mid s \models \Phi \} = Sat(\Phi) \).

\( p : \)
\[ T = \{ s \mid p \in L(s) \} \]

\( \phi_1 \land \phi_2 : \)
\[ T = Sat(\phi_1) \bigcap Sat(\phi_2) \]

\( \neg \phi : \)
\[ T = S \setminus Sat(\phi) \]

\( \exists \bigcirc \phi : \)
\[ T = \{ s \mid Post(s) \bigcap Sat(\phi) \neq \emptyset \} \]

\( \forall \bigcirc \phi : \)
\[ T = \{ s \mid Post(s) \subseteq Sat(\phi) \} \]
Suppose the formula is $\phi = \exists(\phi_1 \cup \phi_2))$.
Note that $\phi = \phi_2 \lor \exists \Diamond \phi$; a fixed-point formula!
Suppose the formula is $\phi = \exists(\phi_1 \cup \phi_2)$.
Note that $\phi = \phi_2 \lor \exists \Box \phi$; a fixed-point formula!

Iterative algorithm to compute this (least) fixed point:
Suppose the formula is $\phi = \exists (\phi_1 \cup \phi_2))$.
Note that $\phi = \phi_2 \lor \exists \bigcirc \phi$; a fixed-point formula!

Iterative algorithm to compute this (least) fixed point:

$$T := \text{Sat}(\phi_2)$$
for all
$$s \in \text{Sat}(\phi_1) \setminus T$$
do
$$\text{if } \text{Post}(s) \cap T \neq \emptyset$$
then $T := T \cup \{s\}$. 
Similarly,
\[ \exists \Box \phi = \phi \land \exists \bigcirc \exists \Box \phi, \]
so we have a *greatest* fixed point.
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An iterative algorithm for computing the greatest fixed point.
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\[ \exists \Box \phi = \phi \land \exists \bigcirc \exists \Box \phi, \]
so we have a greatest fixed point.

An iterative algorithm for computing the greatest fixed point.

\[
T := \text{Sat}(\phi)
\]
repeat
\[
\text{choose } s \in T;
\]
\[
\text{if } \text{Post}(s) \cap T = \emptyset
\]
\[
\text{then } T := T \setminus \{s\}
\]
until
\[
\forall s \in T, \text{Post}(s) \cap T \neq \emptyset.
\]
For a transition system with $n$ states and $t$ transitions and a CTL formula $\phi$ of size $k$, the model-checking problem can be solved in time

$$O((n + t).k).$$
Further directions
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- Model checking with fairness assumptions
Further directions

- Model checking with fairness assumptions
- Finding counterexamples and witnesses
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- Symbolic model checking: dealing with large systems by working with sets of states symbolically
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- Model checking with fairness assumptions
- Finding counterexamples and witnesses
- Symbolic model checking: dealing with large systems by working with sets of states symbolically
- Using BDDs to represent sets and set operations efficiently
LTL, CTL*, mu-calculus
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- LTL uses a single outermost universal path quantifier.
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- Very good for dealing with systems specified as sets of possible runs.
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- Both are fragments of CTL*
LTL, CTL*, mu-calculus

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- Very good for dealing with systems specified as sets of possible runs.
- LTL and CTL have different expressive power: neither subsumes the other.
- Both are fragments of CTL*
- mu-calculus, allows general fixed-point operators.
LTL model checking
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* Based on automata-theoretic techniques.
LTL model checking

- Based on automata-theoretic techniques.
- PSPACE hard.
LTL model checking

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LTL model checking

- Based on automata-theoretic techniques.
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- Handles fairness nicely.
LTL model checking

- Based on automata-theoretic techniques.
- PSPACE hard.
- So what? Still very useful!
- Handles fairness nicely.
- CTL* not significantly harder.
Extensions

- Timed automata
- Probabilistic transition systems
THE END