What are Feynman diagrams?

Prakash Panangaden

Quantum Mechanics Recap

1. States are rays in a Hilbert space
2. Measurements are described by hermitian operators...
3. Evolution is given by a particular unitary operator $\exp(-iHt)$
4. The algebra of observables is non-commutative and is given by Dirac's rule

$$\{P, Q\} \longrightarrow [P, Q]$$

Wave Equations

What is the precise dynamical law?

Figure out H (and get Nobel prize) then time evolution is given by:

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

Eigenstates of H

If $H\psi = E\psi$ then $\quad \dfrac{\partial \psi}{\partial t} = iE\psi$

hence

$$\psi(t) = e^{-iEt}\psi(0)$$

$|\psi|^2$ is constant. These are the stationary states.

Harmonic Oscillator

The Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$.

The energy levels are equally spaced: $E_n = \hbar\omega(n + \frac{1}{2})$

Some marvellous operators

$$a = C(x + iC'p), \quad a^\dagger = C(x - iC'p)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \;\; a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$[a, a^\dagger] = 1, \;\; H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

Relativistic QM

Possible relativistic wave equations arise from the representation theory of the Lorentz group. Dirac guessed the right equation for the electron from physical intuition and formal arguments.

Problem: the energy spectrum was not bounded below. What stops an electron from falling into the negative energy states and radiating away an infinite amount of energy?

Dirac's hack: Fill the negative energy states. The "vacuum" is a sea of negative energy electrons and Pauli's exclusion principle will keep ordinary electrons from falling into the sea.

A negative energy electron may be kicked upstairs and become an ordinary electron leaving a "hole". The hole will behave just like a positively charged electron: a positron.

Quantum Field Theory

Hole theory was replaced by quantum field theory created by too many people to name them all but a few should be mentioned: Wigner, Weisskopf, Jordan, Heisenberg, Fermi and Dirac.

The main ideas: particles are no longer “conserved”, they can be created and destroyed. The state space is the symmetric tensor algebra or the Grassman algebra over the old Hilbert space. This is called Fock space.

The old “wave functions” become operator fields. They act on Fock space and create or annihilate particles: second quantization.

The mathematical complexity rises a whole level beyond that of ordinary quantum mechanics.

Classical Field Theory: Klein-Gordon field

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-m^2\phi=0$$

Often written $\Box\phi - m^2\phi = 0$.

Let V be the real vector space of classical solutions; it is the analogue of phase space.

The symplectic form is:

$$\Omega(\phi_1,\phi_2)=\int_\Sigma(\phi_1\vec{\nabla}\phi_2-\phi_2\vec{\nabla}\phi_1)\cdot d\vec{\sigma}$$

Traditional Quantum Field Theory

Start with $\Box\phi - m^2\phi$ Put it in a "box" to avoid hassles.

$$\phi(\vec{x},t) = \sum_{\vec{k}} \phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}; \quad \vec{k} = 2\pi(n_x, n_y, n_z).$$

Now the Hamiltonian is

$$\sum_{\vec{k}} \{\frac{1}{2}|\dot{\phi}_{\vec{k}}|^2 + \frac{1}{2}\omega_{\vec{k}}^2|\phi_{\vec{k}}|^2\} \quad \text{where } \omega_{\vec{k}}^2 = \vec{k}^2 + m^2.$$

This looks like a collection of harmonic oscillators.

Fermi

The $a, a^\dagger$ operators now destroy and create quanta of different modes:
particles have emerged from the field!

The innocent harmonic oscillator plays a foundational role in QFT.

The a and $a^\dagger$ come from the positive and negative frequencies of the field.

The vacuum is the state killed by all the a operators.

How do we know what is positive frequency and what is negative frequency?

The Fourier transform tells us:

$$\Phi(\vec{x}, t) = \sum_k f_k(\vec{x}, t)\mathbf{a}_k + \overline{f_k}(\vec{x}, t)\mathbf{a}_k^\dagger$$

Operators are in **bold face**.

The f_k are classical positive energy solutions:

$$f_k = (\cdot\cdot)\exp(i\vec{k}\cdot\vec{x} - i\omega t)$$

One needs the canonical Fourier transform that one has in a flat spacetime.

Fock Space

A Hilbert space that accomodates multiple particles.

Suppose that $\mathcal{H}$ is the ordinary (1 particle) Hilbert space.

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \bigoplus \mathcal{H} \bigoplus (\mathcal{H} \bigotimes_S \mathcal{H}) \bigoplus (\mathcal{H} \bigotimes_S \mathcal{H} \bigotimes_S \mathcal{H}) \ldots$$

$$C(\sigma)\Psi = (0, \sigma^\alpha \xi, \sqrt{2}\sigma^{(\alpha}\xi^{\beta)}, \sqrt{3}\sigma^{(\alpha}\xi^{\beta\gamma)}, \ldots)$$

$$A(\bar{\tau})\Psi = (\xi^\mu \bar{\tau}_\mu, \sqrt{2}\xi^{\mu\alpha}\bar{\tau}_\mu, \sqrt{3}\xi^{\mu\alpha\beta}\bar{\tau}_\mu, \ldots)$$

The "harmonic oscillators" give the creation and annihilation operators of QFT.

Summary

In QFT particles may be created and destroyed.

The space of quantum states has a representation: the Fock representation, to support reasoning about multi-particle states with varying numbers of particles.

The classical fields become operator fields: $\phi = C + A$.

Interacting Quantum Fields

$$\nabla_\nu F^{\mu\nu} = ej^\mu$$

Maxwell's equation with a current.

Here F describes the electric and magnetic fields and is given
$F^{\mu\nu} = \nabla^{[\mu} A^{\nu]}$

$$(\gamma^\mu \nabla_\mu + m)\psi = -e\gamma_\mu \psi A^\mu$$

The Dirac equation for an electron in an electromagnetic field

Each equation mentions the other field.

We do now know how to solve equations like this at all:
they are non-linear and coupled.

Perturbation Theory

Pretend that the coupling is "small."

For QED the coupling constant is e.

First solve the free equations exactly.

Then use these solutions to compute j and A
and plug these back into the equations.

Then recompute the solutions and recompute j and A.

You get an approximate solution
in powers of the coupling constant.

Keep going until you have the desired accuracy.

Problems

- This approximate expansion does not reveal all the interesting properties of the exact solutions.
- The power series does not converge!
- Even the individual terms (after first order) are infinite!!

Feynman's Brilliant Intuition

A simple visualization of the terms in the perturbation series.

Think in terms of particles and their trajectories.

Particles coast freely until they interact. For a given type of theory the interaction is always the same.

Coasting particles are represented by straight lines; interactions by vertices.

The pictures define integrals that express the probability (amplitude) for the process shown.

An electron (fermion) propagator

A photon (boson) propagator

We need a mathematical function to describe how a particle moves from x to y:
this is called a Feynman propagator.

Propagators

The free wave equation is: $D_x \psi(x) = 0.$

Consider the equation: $D_x G(x, y) = \delta(x, y).$

If we solve for $G(x, y)$, which is a distribution,

$$\phi(x) = \int G(x, y) h(y) \mathrm{d}^4 y$$

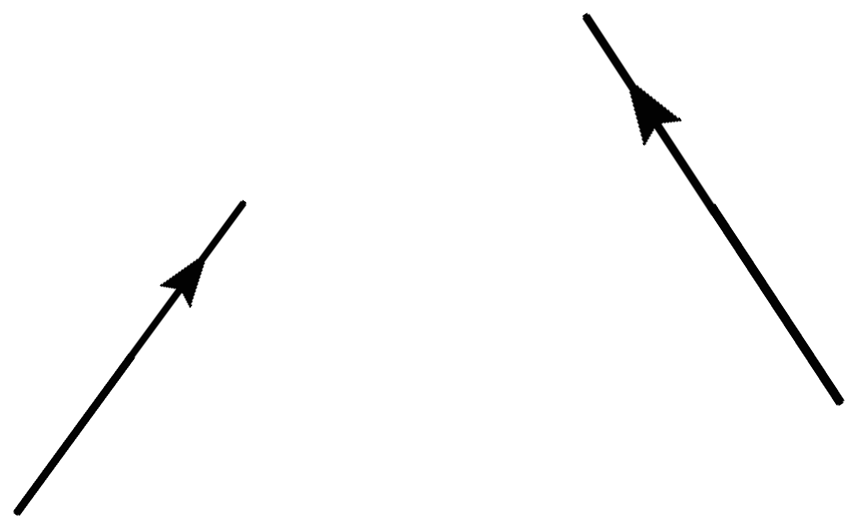
solves the equation $D_x \phi(x) = h(x)$.

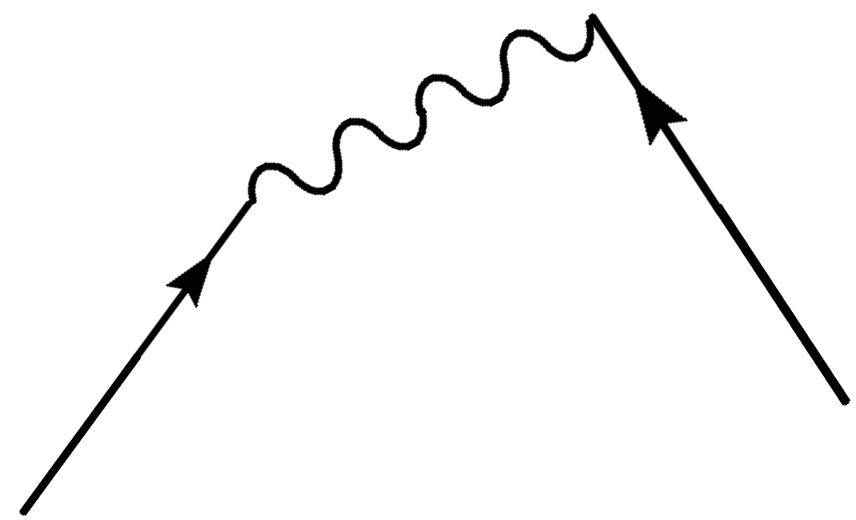
Thus G tells us how to propagate a solution;
we call it a propagator.

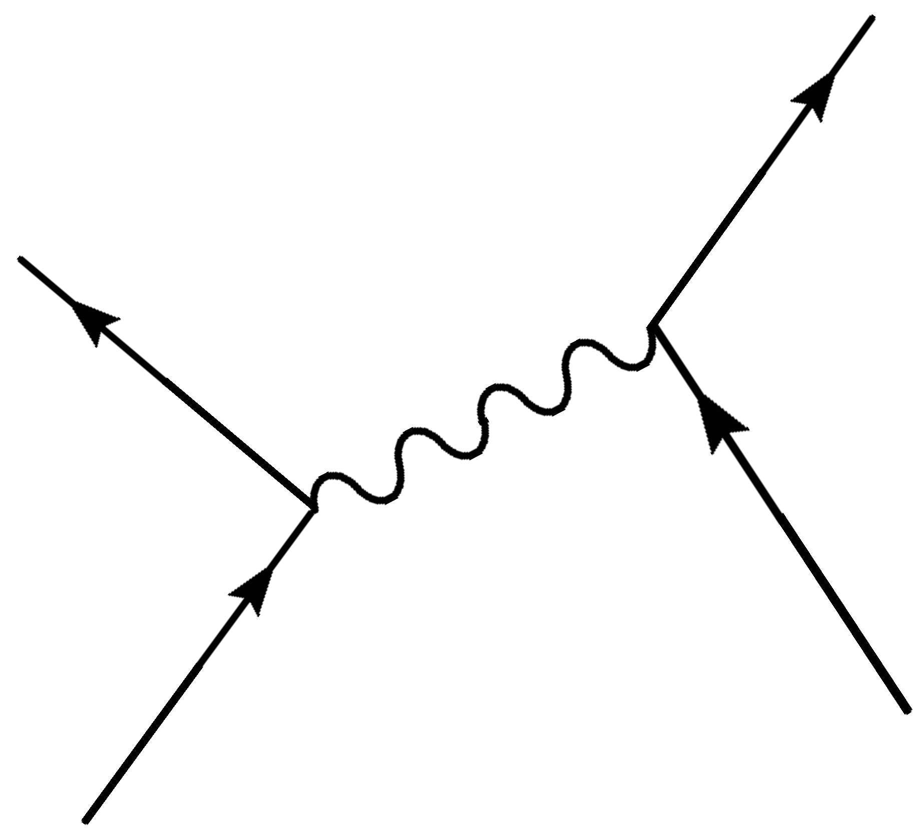
It is the "matrix inverse" of D_x.

At last!! A Feynman diagram.

Electron-electron scattering







What are the propagators?

If we take Fourier transforms we get for the Klein-Gordon field:

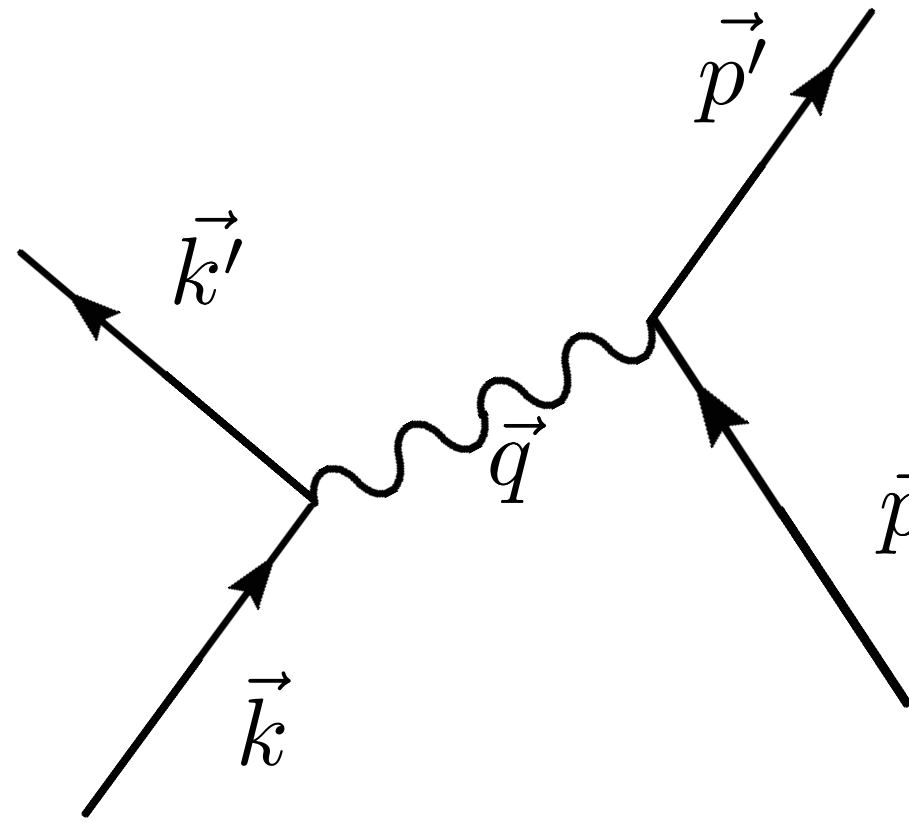
$$\frac{1}{k^2 - m^2}$$

This becomes singular at $k = \pm m$.

One has to have a contour of integration that avoids the poles.

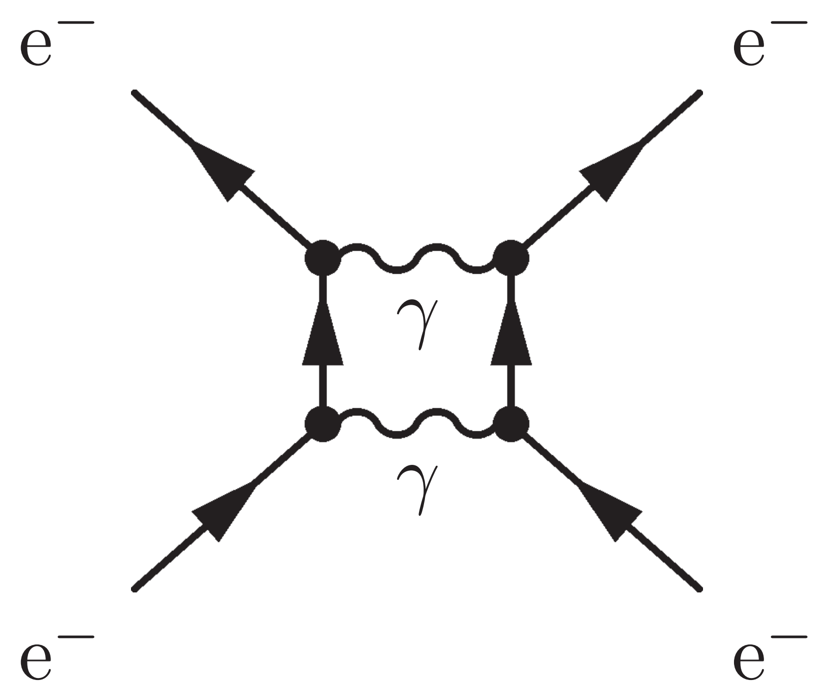
Feynman's idea: Positive frequency propagates to the future but negative frequency propagates to the past.

Feynman diagrams are usually drawn in momentum space (the Fourier transformed versions of the propagators are used.)



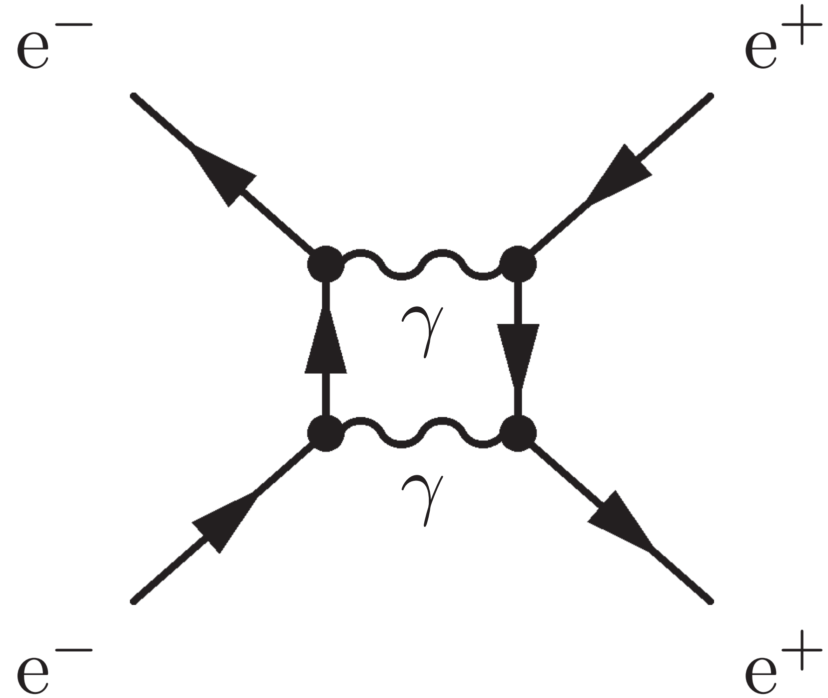
$$\vec{k'} = \vec{k} - \vec{q} \qquad\qquad \vec{p'} = \vec{p} + \vec{q}$$

Electron-electron scattering: fourth order



Every vertex has exactly two electrons and one photon.

Electron-positron scattering

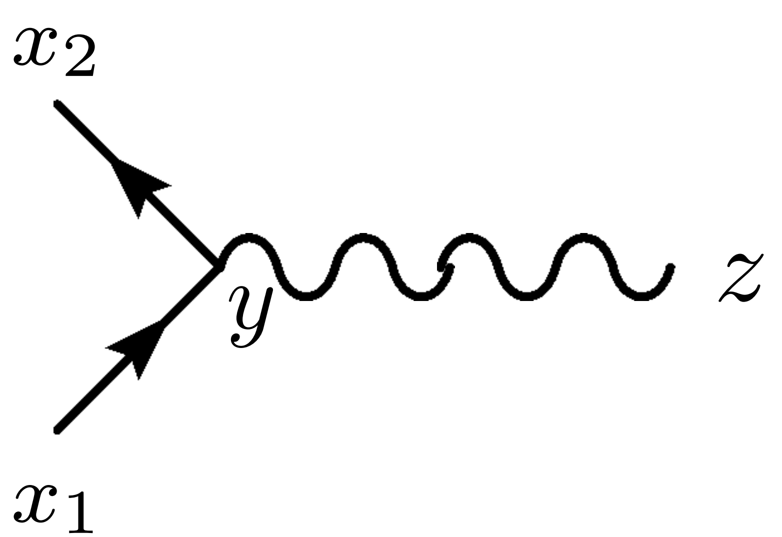


A positron is an electron traveling "backwards" in time.

From pictures to integrals

$= G(x,y)$: electron propagator

$= D(u,v)$: photon propagator



$= e \int G(x_1, y) D(y, z) G(y, x_2) \mathrm{d}^4 y$

We integrate over the internal vertices.
They are like "bound" variables.

Fron QFT to Feynman Diagrams

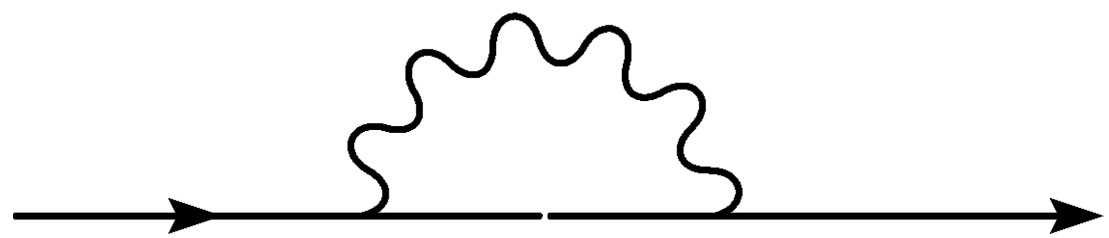
Schwinger showed how to calculate quantities of interest from QFT using perturbation theory in a scary algebraic formalism.

Feynman wrote rules for calculating terms in the perturbation expansion using the diagrammatic formalism.
QFT for the masses!

Dyson *derived* the Feynman rules from QFT.
Then people could trust the Feynman rules.

Some other time, I will derive these rules.

Loops

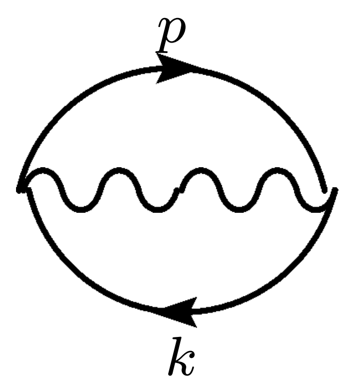


Diagrams like this yield infinity when you calculate the integral.

It models a *self interaction*. The theory is trying to correct the electron mass by adding in the energy of the electron interacting with itself.

But QFT is not designed to compute fundamental constants and definitely not in perturbation theory.

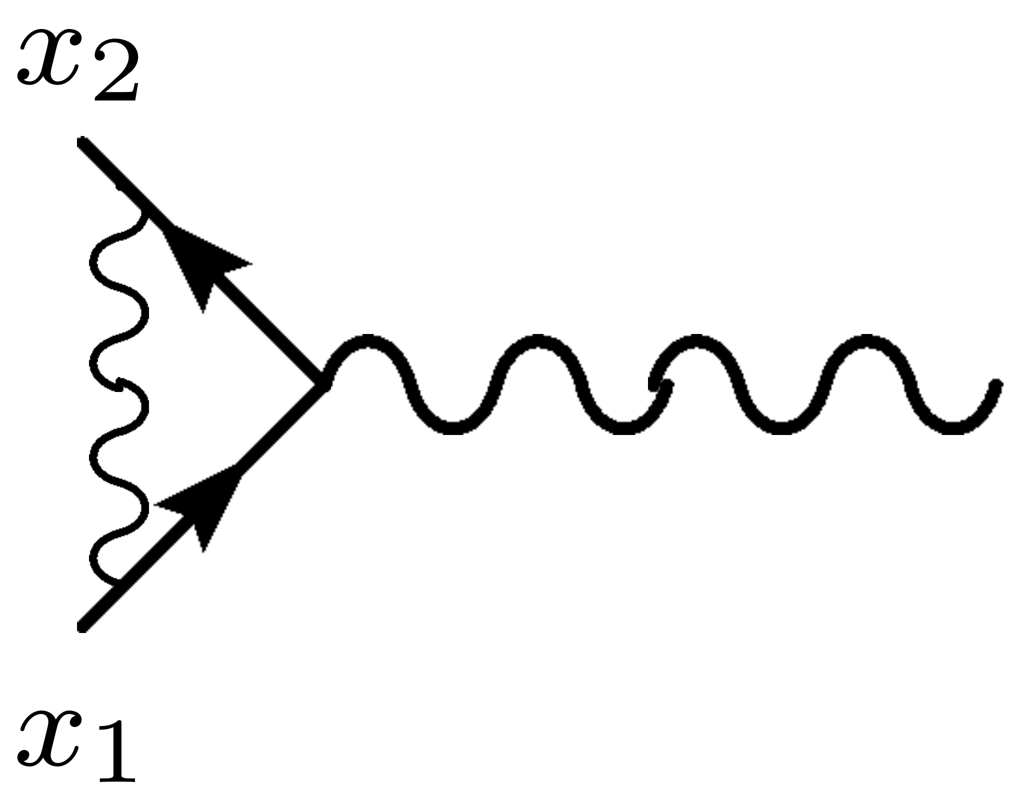
Vacuum Bubbles



These are also divergent.

They can appear free-floating as parts of other diagrams
and have the effect of multiplying by an (infinite) constant.

Another divergent diagram.



This time the theory is trying to correct the electric charge.

Loops cause divergences

All the divergences of QED can be identified as the attempt to compute 3 parameters: mass, charge and vacuum energy. We factor out these terms and use the experimental values.

This is renormalization.

Whether one can do it depends on the combinatorics of the graphs. It works for QED (Feynman, Schwinger, Tomanaga, Dyson, Abdus Salam), it works for (most) gauge theories (t'Hooft, Veltmann) but

it does not work for quantum gravity!

Conclusions

Beautiful *and useful* diagrammatic formalism.

Seems to have interesting connections with logic:
the diagrams have an eerie resemblance to proof nets.

The nature of loops reminds me of the scalars in the Abramsky-Coecke formalism.

But don't take the particle imagery too seriously. In curved spacetime, the notion of particle is not absolute but diagrammology still works.

Bunch, P, Parker 1980; Birrell and Davies 1980,
QED: P 1981, Bunch 1982.