Duality in Logic and Computation

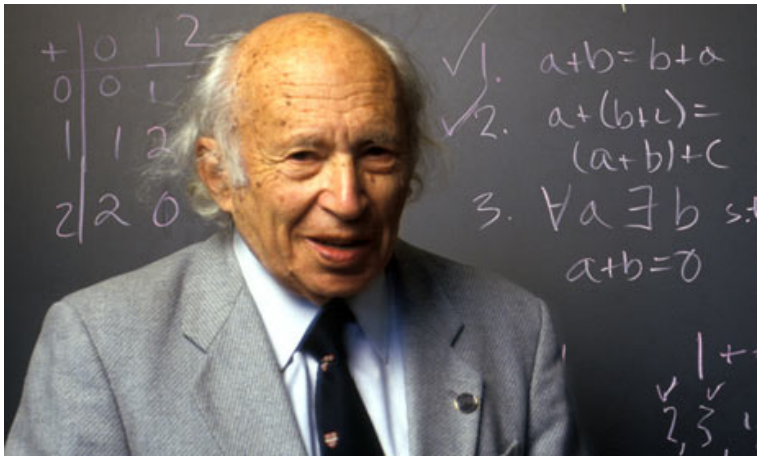
Prakash Panangaden[1]

[1]School of Computer Science
McGill University

IEEE Symposium On Logic In Computer Science, June 2013

Heros of duality

Marshall H. Stone

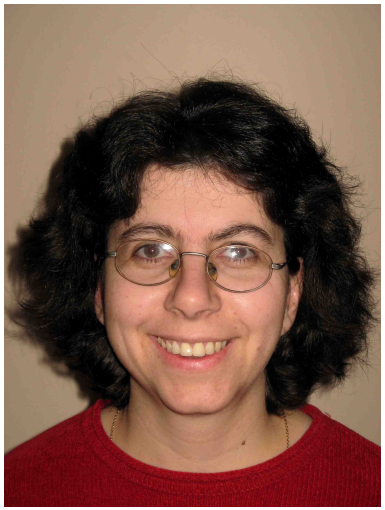


Israel M. Gelfand

Machine Learning Collaborators

Chris Hundt

Monica Dinculescu



Doina Precup

Joelle Pineau

Collaborators on today's talk

This is joint work with many people: Nick Bezahanishvili, Filippo Bonchi, Marcello Bonsangue, Helle Hvid Hansen, Dexter Kozen, Clemens Kupke, Jan Rutten and Alexandra Silva.



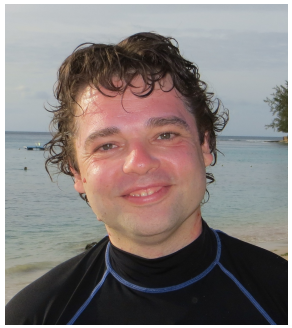



Other Stone-duality collaborators

A related paper on Stone Duality for Markov Processes with Dexter Kozen, Kim G. Larsen and Radu Mardare will be presented tomorrow.

Dexter Kozen

Kim G. Larsen



Radu Mardare

Collaborators on LMPs and duality

Previous work was with Josée Desharnais, Vincent Danos, François Laviolette (Info. Comp 2006) and Philippe Chaput, Vincent Danos and Gordon Plotkin (ICALP 2009).

Josée Desharnais

François Laviolette

Philippe Chaput

Vincent Danos

Gordon Plotkin

Examples of duality principles

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- Controllability and observability in control theory, Kalman.
- Many examples from semantics and logic.

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- The two mathematical universes are *mirror images* of each other.
- Two completely different sets of theorems that one can use.

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- Commutative unital C*-algebras and compact Hausdorff spaces. [Gelfand, Stone]
- Labelled Markov processes and C^*-algebras with operators. [Mislove, Ouaknine, Pavlovic, Worrell]

Maps matter!

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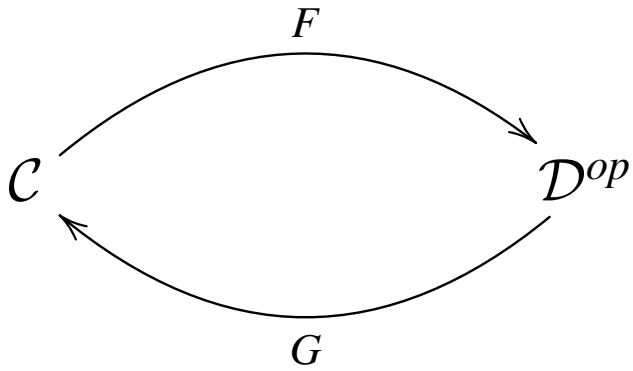
- Compositions are *preserved* or *reversed*.
- This is *functoriality*.
- From this one can often conclude *invariance properties*.

Duality categorically



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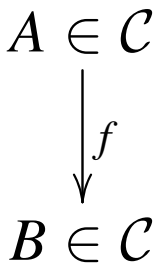
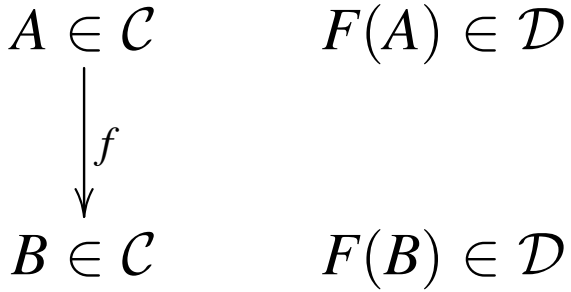


Duality categorically

We get



Duality categorically

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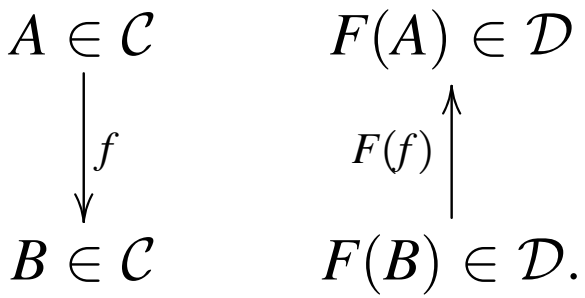

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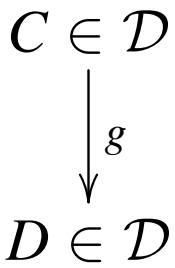
Similarly, given



Duality categorically

We get

$$G(C) \in \mathcal{C} \qquad C \in \mathcal{D}$$

$$\Big\downarrow g$$

$$G(D) \in \mathcal{C} \qquad D \in \mathcal{D}$$

Duality categorically

and

$$\begin{array}{ccc} G(C) \in \mathcal{C} & \qquad & C \in \mathcal{D} \\ \uparrow{\scriptstyle G(g)} & & \downarrow{\scriptstyle g} \\ G(D) \in \mathcal{C} & & D \in \mathcal{D}. \end{array}$$

Isomorphisms

We have isomorphisms

$$A \simeq G(F(A)) \text{ and } C \simeq F(G(C)).$$

Duality categorically

Stone-type Duality

We have a (contravariant) adjunction between categories $\mathcal{C}$ and $\mathcal{D}$, which is an *equivalence* of categories.

Often obtained by looking at maps into an object living in both categories: a schizophrenic object.

Duality for high school students I

- Finite-dimensional vector space V over, say $\mathbb{R}$,

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- Finite-dimensional vector space V over, say $\mathbb{R}$,
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- V^* has the same dimension as V and a (basis-dependent) isomorphism between V and V^*.
- The double dual V^{**} is also isomorphic to V
- with a "nice" canonical isomorphism: $v \in V \mapsto \lambda\sigma \in V^*.\sigma(v)$.

Duality for high school students II

$$U \xrightarrow{\theta} V$$

$$U^* \xleftarrow[\theta^*]{} V^*$$

Given a linear maps θ between vector spaces U and V we get a map θ^* *in the opposite direction* between the dual spaces:

$$\theta^*(\sigma \in V^*)(u \in U) = \sigma(\theta(u)).$$

Boolean algebras

A Boolean algebra is a set A equipped with two constants, $0, 1$, a unary operation $(\cdot)'$ and two binary operations $\vee, \wedge$ which obey the following axioms, p, q, r are arbitrary members of A:

$$\begin{aligned} 0' &= 1 & 1' &= 0 \\ p \wedge 0 &= 0 & p \vee 1 &= 1 \\ p \wedge 1 &= p & p \vee 0 &= p \\ p \wedge p' &= 0 & p \vee p' &= 1 \\ p \wedge p &= p & p \vee p &= p \end{aligned}$$

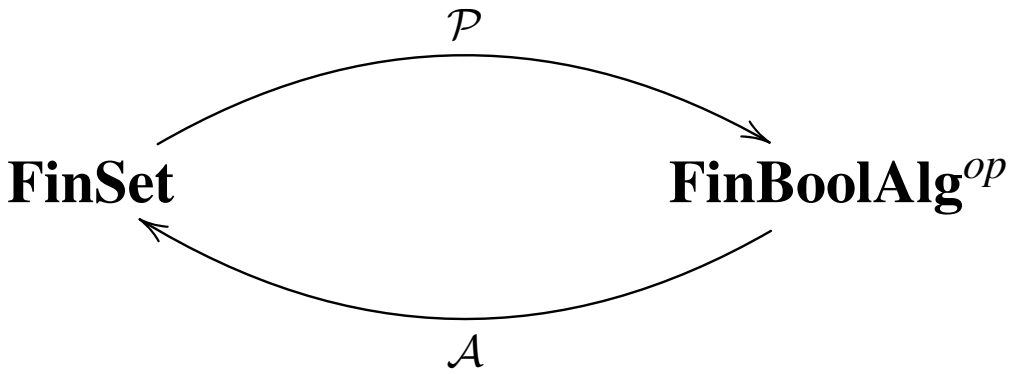
Boolean algebras II

$$
\begin{aligned}
p'' &= p \\
(p \wedge q)' &= p' \vee q' \\
(p \vee q)' &= p' \wedge q' \\
p \wedge q &= q \wedge p \\
p \vee q &= q \vee p \\
p \wedge (q \wedge r) &= (p \wedge q) \wedge r \\
p \vee (q \vee r) &= (p \vee q) \vee r \\
p \wedge (q \vee r) &= (p \wedge q) \vee (p \wedge r) \\
p \vee (q \wedge r) &= (p \vee q) \wedge (p \vee r)
\end{aligned}
$$

The operation $\vee$ is called *join*, $\wedge$ is called *meet* and $(\cdot)'$ is called *complement*.

Maps are Boolean algebra homomorphisms.

(Toy) Stone duality



Here $\mathcal{P}$ is power-set and $\mathcal{A}$ takes the *atoms* of a Boolean algebra.

Stone spaces

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- Many, but not all, Stone spaces are Polish.

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- Everything that can, and should be, an isomorphism is an isomorphism.

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- Denotational semantics: compositional, equivalent to operational semantics.

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- Relate predicate-transformer semantics to state-transformer semantics: Jaco De Bakker (1978).
- Duality: The category of state transformers is equivalent to the (opposite of) the category of predicate transformers: Plotkin (1979).

Duality for probabilistic programs: Kozen

Probabilistic programs and *expectation transformers*: Kozen (1981)

Logic	Probability
States s	Distributions μ
Formulas P	Random variables f
Satisfaction $s \models P$	Integration $\int f \mathrm{d}\mu$

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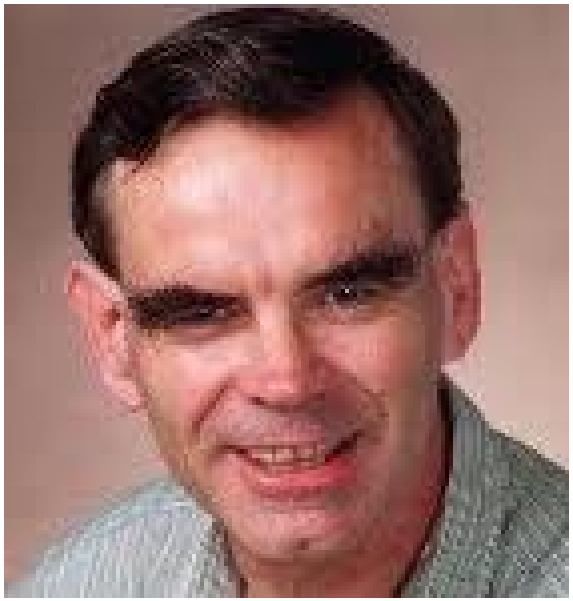
- The two interpretations are Stone duals.
- Ties together semantics, logic and verification.

Chu spaces: Pratt



Vaughan Pratt

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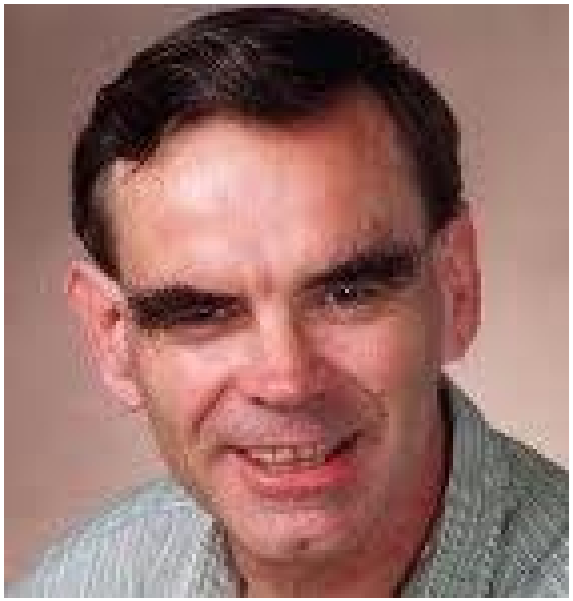

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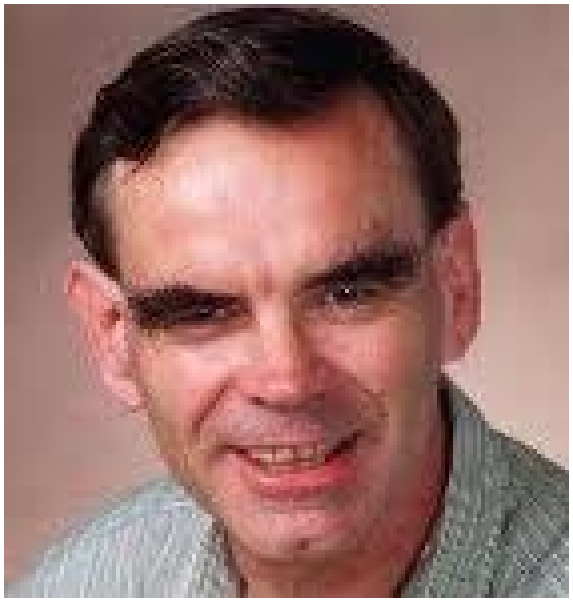

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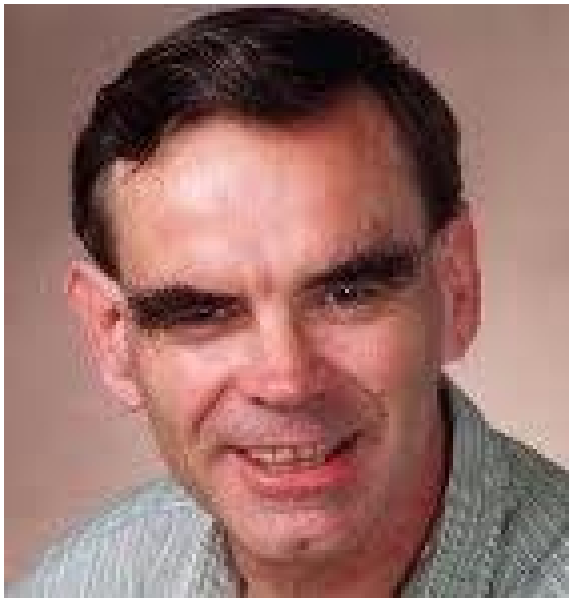

Vaughan Pratt

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- Generalized matrices.
- Pratt's observation: many interesting categories embed in Chu categories.
- Stone duality is "transposition" of matrices.

Many other contributors

- Bart Jacobs, ...
- Achim Jung and Drew Moshier
- Mai Gehrke, Jean-Eric Pin, ...
- Bezhanishvilis

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- The intermediate step can blow up the size of the automaton exponentially before minimizing it.

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- Take the reachable states.
- Repeat.
- This gives the minimal DFA recognizing the same language!
- The intermediate step can blow up the size of the automaton exponentially before minimizing it.
- But experimental results seem to indicate that it often works well in practice.

Deterministic Automata

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$: a deterministic finite (Moore) automaton. S is the set of **states**, $\mathcal{A}$ an **input alphabet** (actions), $\mathcal{O}$ is a set of **observations**.
- $\delta : S \times \mathcal{A} \to S$ is the **state transition function**.
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- *except* that we do not have a start state,
- which means that reachability makes no sense.
- We will worry about that in a minute.

A Simple Modal Logic

- View $\mathcal{O}$ as propositions, define a simple modal logic. A *formula* φ is:

$$\varphi ::== \omega \in \mathcal{O} \mid (a)\varphi$$

  where $a \in \mathcal{A}$.

- We say $s \models \omega$, if $\omega \in \gamma(s)$ (or $\gamma(s,\omega) = T$).
  We say $s \models (a)\varphi$ if $\delta(s,a) \models \varphi$.

- Now we define $[\![\varphi]\!]_{\mathcal{M}} = \{s \in S | s \models \varphi\}$.

An Equivalence Relation on Formulas

- We write sa as shorthand for $\delta(s, a)$.
- Define $\sim_{\mathcal{M}}$ between *formulas* as $\varphi \sim_{\mathcal{M}} \psi$ if $[\![\varphi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}}$.

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- Equivalence class for φ same as of states $[\![\varphi]\!]_{\mathcal{M}}$ that satisfy φ.

A Dual Automaton

- Given a finite automaton $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$.
  Let T be the set of $\sim_{\mathcal{M}}$-equivalence classes of formulas on $\mathcal{M}$.
- We define $\mathcal{M}' = (S', \mathcal{A}, \mathcal{O}', \delta', \gamma')$ as follows:
- $S' = T = \{[\![\varphi]\!]_{\mathcal{M}}\}$
- $\mathcal{O}' = S$
- $\delta'([\![\varphi]\!]_{\mathcal{M}}, a) = [\![(a)\varphi]\!]_{\mathcal{M}}$
- $\gamma'([\![\varphi]\!]_{\mathcal{M}}) = [\![\varphi]\!]_{\mathcal{M}}$ or $\gamma'([\![\varphi]\!]_{\mathcal{A}}, s) = (s \models \varphi)$.

The intuition

Interchange states and observations.

Minimality Properties

- In general, the double dual is the minimal machine with the same behaviour!

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- In general, the double dual is the minimal machine with the same behaviour!
- For deterministic machines bisimulation is the same as trace equivalence.
- This gives an intuition for why Brzozowski's algorithm works,
- but it does not really address the role of reachability properly.

Probabilistic systems

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- because the double-dual serves as a substitute for the original machine.

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- There should be no such thing as absolute state!
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- Double dual: state can be regarded as the summary of the outcomes of experiments.

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- why is the dual another automaton?

Automata as Coalgebras

- Our automata are coalgebras of the following functor:

$$F(S) = S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}, \quad F(f : S \to S') = \lambda\langle \alpha : \mathcal{A} \to S,\ O \subseteq \mathcal{O}\rangle.\langle f \circ \alpha,\ O\rangle.$$

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- with S: states, $\mathcal{A}$: actions, $\mathcal{O}$: observations, δ is a transition function and γ is an observation function.
- They are well known as Moore machines.

Homomorphisms

A homomorphism for these coalgebras is a function $f : S \to S'$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & S' \\ {\scriptstyle \langle \delta,\ \gamma \rangle}\downarrow & & \downarrow{\scriptstyle \langle \delta',\ \gamma' \rangle} \\ S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} & \xrightarrow[f^{\mathcal{A}} \times \mathrm{id}]{} & S'^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} \end{array}$$

where $f^{\mathcal{A}}(\alpha) = f \circ \alpha$.

This translates to the following conditions:

$$\forall s \in S, \omega \in \mathcal{O},\ \omega \in \gamma(s) \iff \omega \in \gamma'(f(s)) \tag{1}$$

and

$$\forall s \in S, a \in \mathcal{A},\ f(\delta(s,a)) = \delta'(f(s),a). \tag{2}$$

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- the usual operations $\wedge$, $\neg$ and constants T and $\bot$ and, in addition,
- together with unary operators (a) and constants $\underline{\omega}$.
- We denote an object by $\mathcal{B} = (B, \{(a)|a \in \mathcal{A}\}, \{\underline{\omega}|\omega \in \mathcal{O}\}, \mathsf{T}, \wedge, \neg)$.

Equations

The following three equations hold:

$$(a)(b_1 \wedge b_2) = (a)b_1 \wedge (a)b_2, \tag{3a}$$

$$(a)\mathsf{T} = \mathsf{T}, \tag{3b}$$

$$\neg(a)\neg b = (a)b. \tag{3c}$$

Morphisms

The morphisms are the usual boolean homomorphisms preserving, in addition, the constants and the unary operators.

Duality Theorem

There is a dual equivalence of categories

$$\mathbf{PODFA}^{op} \cong \mathbf{FBAO}.$$

One functor $\mathcal{P}$ is just the contravariant power set functor and the other one $\mathcal{H}$ maps a boolean algebra to its set of atoms.

Minimization?

- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.

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- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.
- So how do we explain the minimization?

Definable Subsets

Define a logic $\mathcal{L}$ by

$$\phi ::== \mathsf{T} | \bot | \phi_1 \wedge \phi_2 | \neg\phi | (a)\phi | \underline{\omega}$$

and define the **definable subsets** $\mathcal{D}(S)$ of a machine $\mathcal{M} = (S, \delta, \gamma)$ as sets of the form $[\![\phi]\!]$.

- $\mathcal{D}(S)$ is a subobject of $\mathcal{P}(\mathcal{M})$

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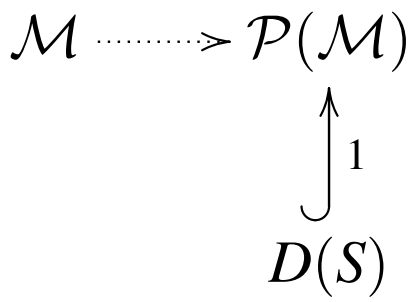
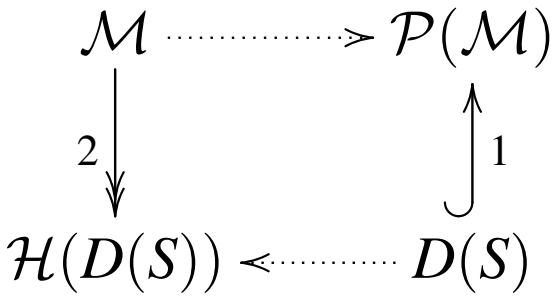
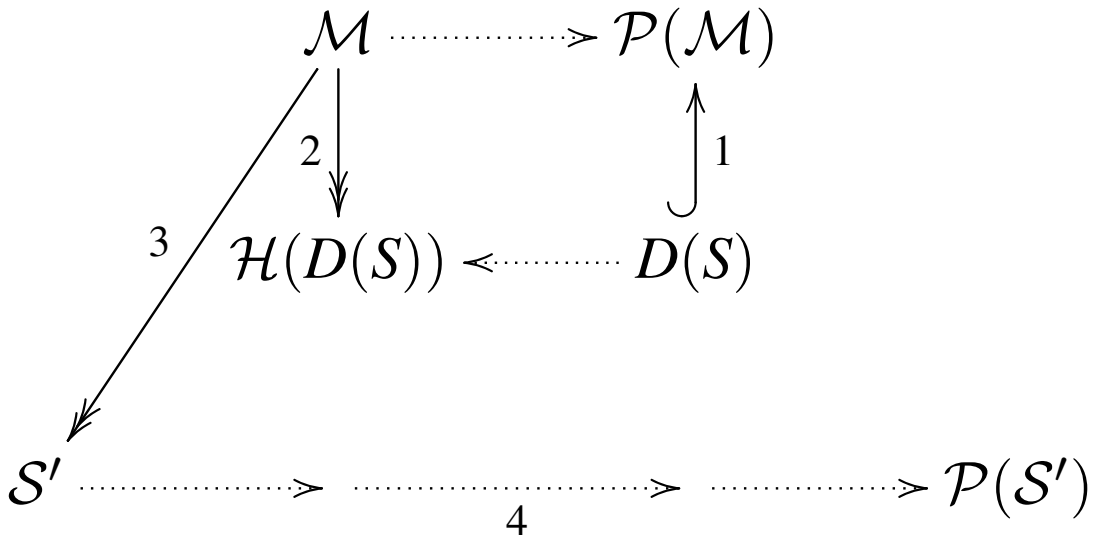
- any other subalgebra must contain $\mathcal{D}(S)$.

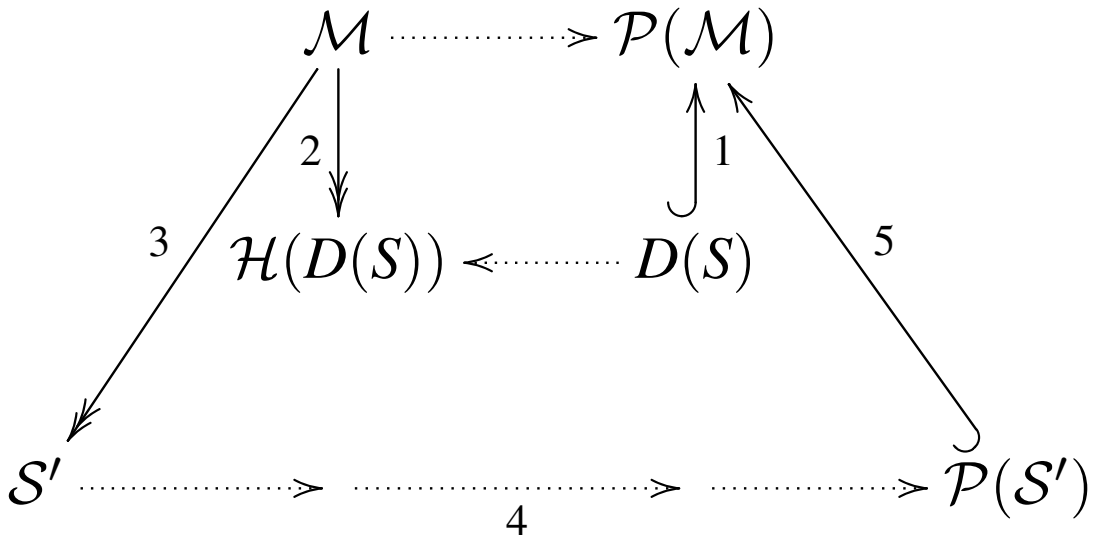
In Pictures

$$\mathcal{M} \dashrightarrow \mathcal{P}(\mathcal{M})$$


$\mathcal{M}$
$\mathcal{P}(\mathcal{M})$
1
$D(S)$


$\mathcal{M}$
$\mathcal{P}(\mathcal{M})$
2
1
$\mathcal{H}(D(S))$
$D(S)$


$\mathcal{M}$
$\mathcal{P}(\mathcal{M})$
2
1
3
$\mathcal{H}(D(S))$
$D(S)$
$\mathcal{S}'$
$\mathcal{P}(\mathcal{S}')$
4


$\mathcal{M}$
$\mathcal{P}(\mathcal{M})$
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1
3
$\mathcal{H}(D(S))$
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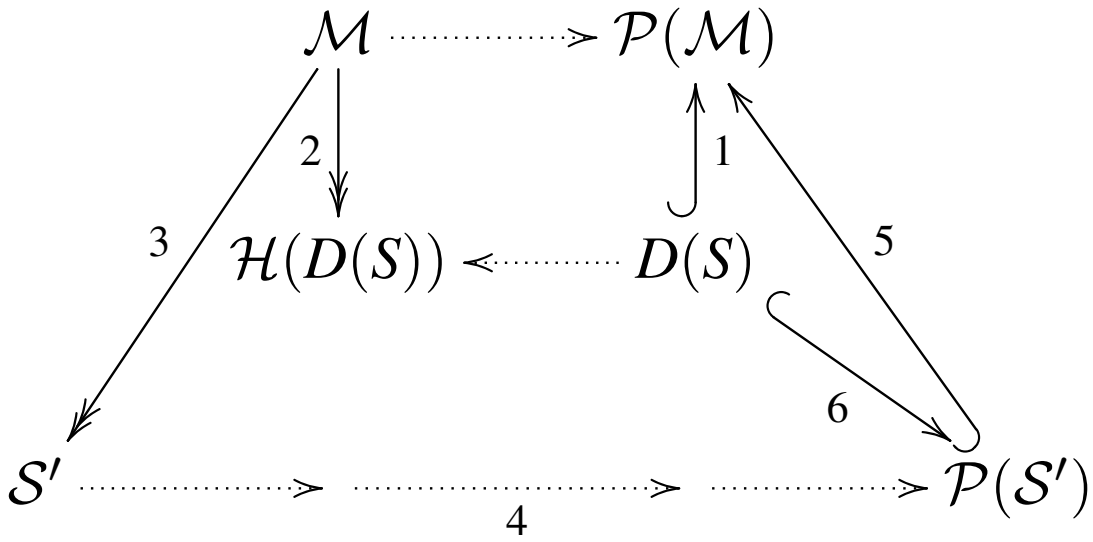
5
S'
4
$\mathcal{P}(S')$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6
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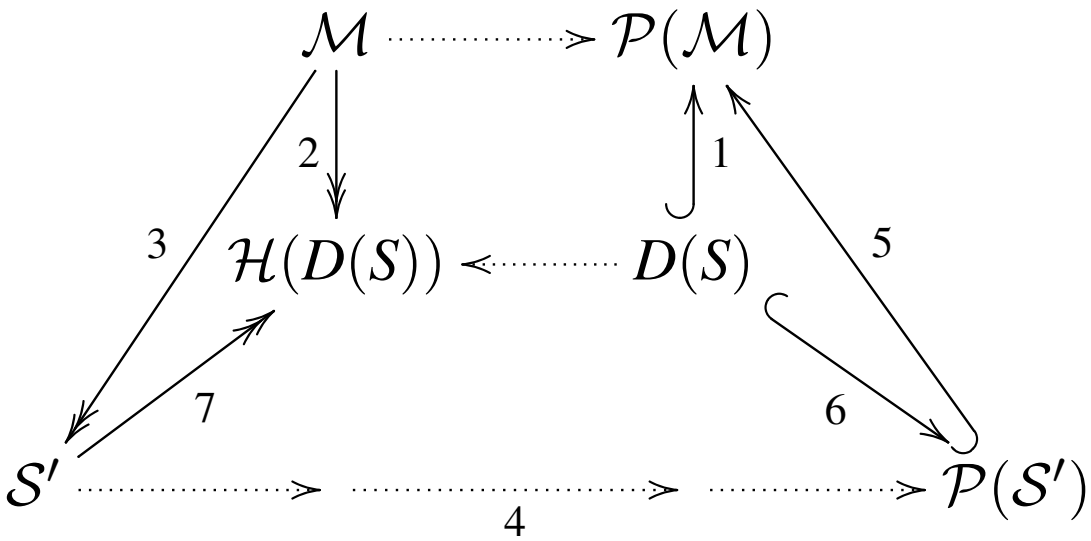
4

The Secret of Minimization



A Simpler Logic

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- With this logic the definable subsets $E(S)$ do not form a boolean algebra,
- it is just a “set with operations”
- in other words, it can be viewed as an automaton!

Why the simpler logic works

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- Thus for **deterministic** automata the boolean algebra generated by $E(S)$ is just the same as $D(S)$ so the minimization picture works with boolean algebra generated by $E(S)$.
- For nondeterministic automata the story is different.

Minimality = fewest states?

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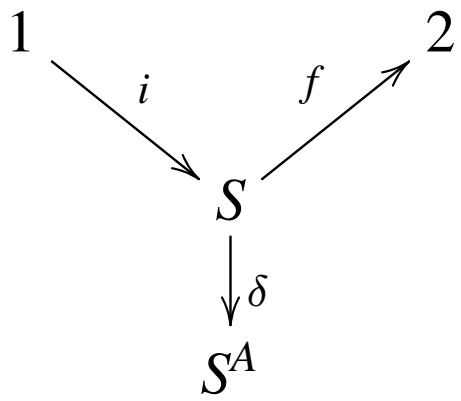
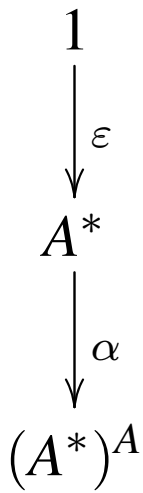
- To really get the system with the fewest states one needs to deal with reachability.
- The following discussion is a rapid version of what Jan Rutten discussed in his beautiful MFPS talk on Monday.

An automaton in diagrams



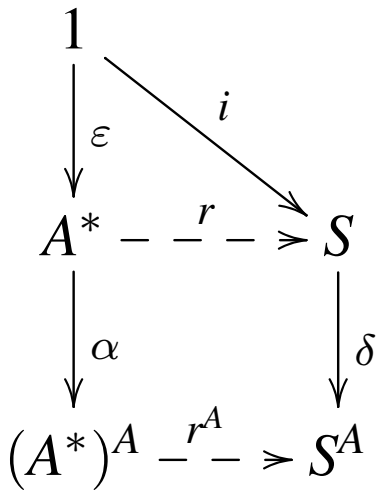
- Here S is the state space, A is the set of actions, 1 is the one-element set and 2 is a two-element set.
- The map i defines an initial state and f defines a set of final states. I will write i for the map and for the initial state itself.
- the transition function $\delta : S \times A \to S$ has been written as $\delta : S \to S^A$.
- There is a natural extension $\delta^* : S \to S^{A^*}$.

A very special (infinite) automaton



- This automaton has all words as its state space.
- The initial state is the empty word ε.
- The transition function α acts by $\alpha(w) = \lambda a : A.w \cdot a$.
- We do not bother to define “final” states in this machine.

Reachability



- Given any function between sets $f : V \to W$, we have a map $f^A : V^A \to W^A$, given by $f^A(\phi) = \lambda a : A.f(\phi(a)) = f \circ \phi$.

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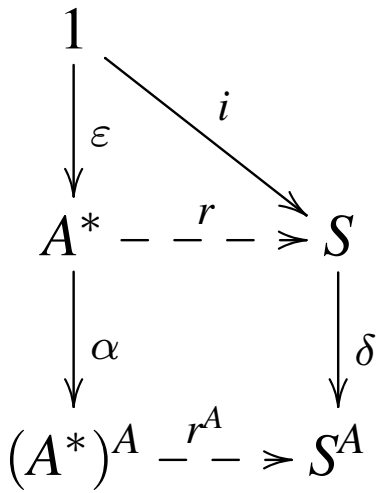

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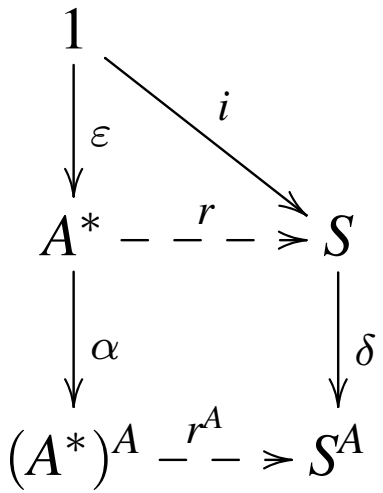
Reachability



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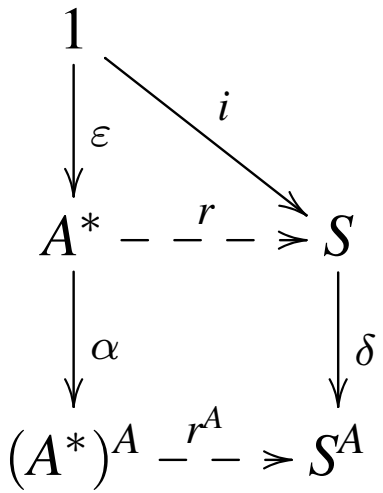

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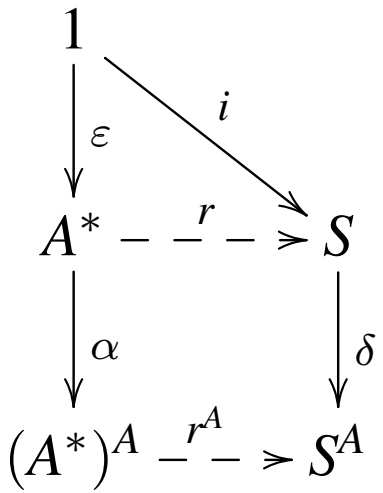

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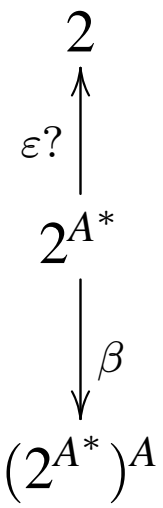
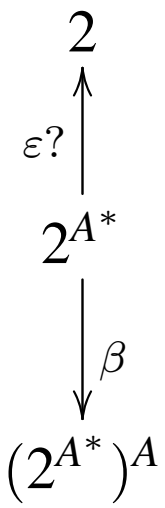
- The image of A^* under r is exactly the reachable subset of S.
- The entire state space is *reachable* exactly when r is a surjection.
- Note, final states play no role.

Another very special infinite automaton



- This automaton has all *languages* as its state space.

Another very special infinite automaton



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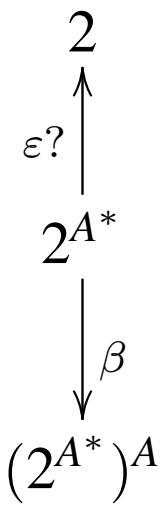

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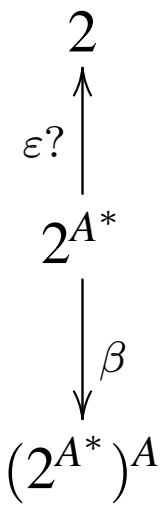

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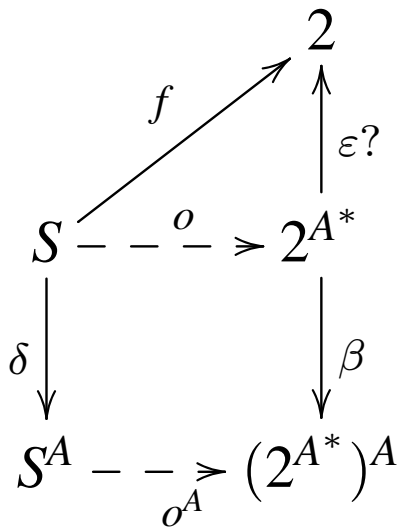
- We do not bother to define an "initial" state in this machine.

Observability



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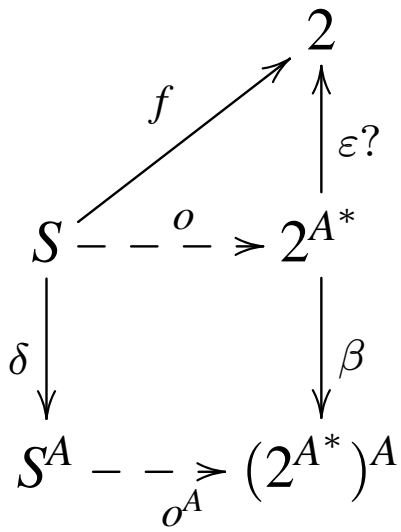

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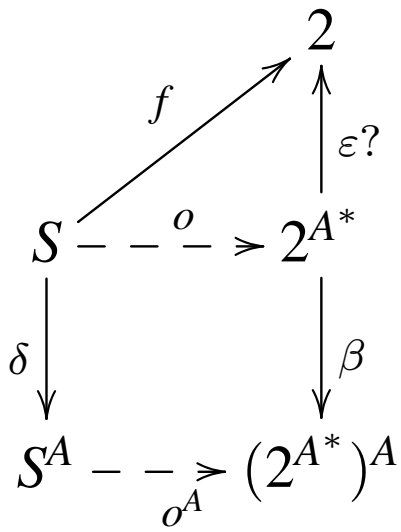

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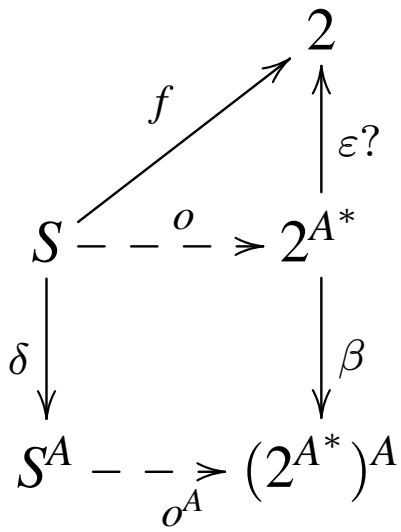
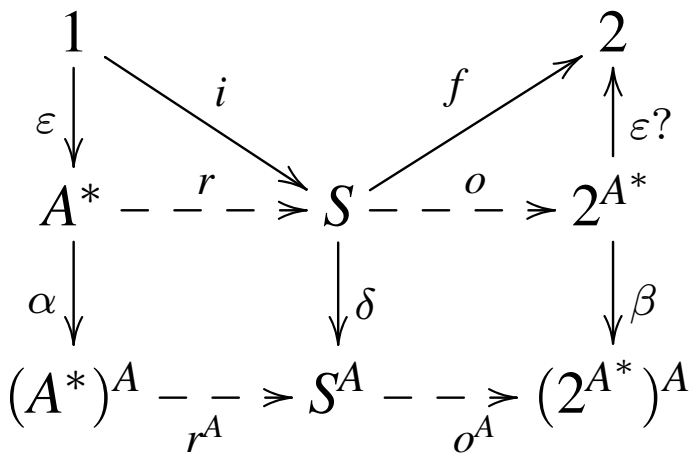


- Here o is the map that takes a state to the language recognized starting from that state.
- It is the unique map making the upper triangle and the lower square commute.
- Think of o as giving the observable behaviour of a state.
- A machine is *observable* exactly when distinct states recognize different languages, i.e. when o is an injection.

The butterfly



A deterministic automaton (S, δ, i, f) is minimal if it is both reachable and observable.

The power-set functor

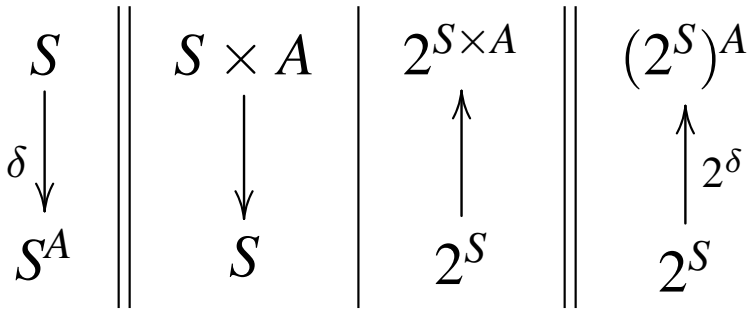
Given sets U, V and a function $f : U \to V$ we define

$$\mathcal{P}(f) : \mathcal{P}(V) \to \mathcal{P}(U)$$

by

$$\mathcal{P}(f)(P \subseteq V) = f^{-1}(P).$$

Reverse functorially



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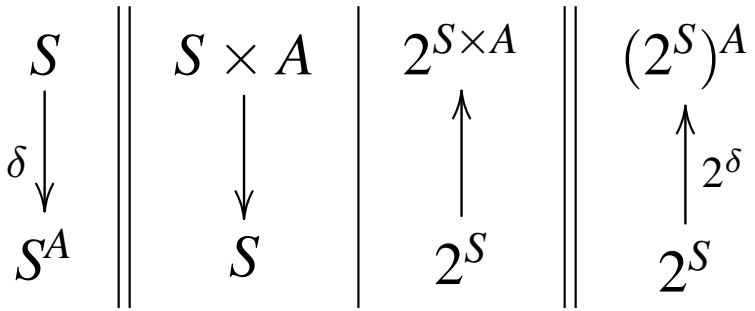

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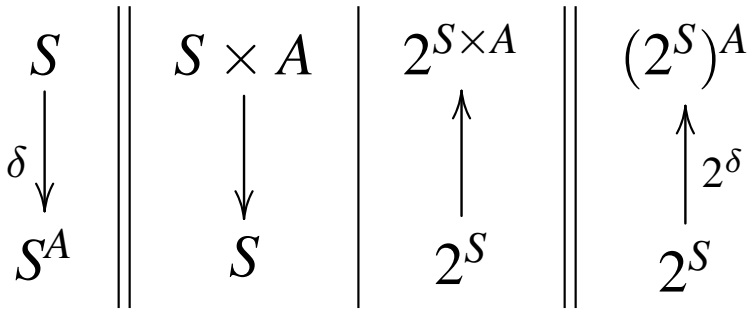

- The power-set functor produces the reversed determinized automaton.
- Initial becomes final under powerset. The final state $S \to \mathbf{2}$ becomes the new initial state by observing that such a function is the same thing as a subset.
- It makes reachable into observable, *but not vice versa*.

Why Brzozowski's algorithm works

Theorem

If (S, δ, i, f) is a reachable deterministic automaton accepting L, then $(2^S, 2^\delta, f, 2^i)$ is an observable deterministic automaton accepting $rev(L)$.

If, we take its reachable part again and reverse it again we again get an observable automaton this time recognizing L. If we take the reachable part we get a minimal automaton recognizing L.

Abstract nonsense?

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- “Surely, this is categorical mumbo-jumbo for something that can be explained simply!”

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- Weighted automata (i.e. automata over vector spaces) can be minimized by using the same idea with the self duality of vector spaces.
- Belief automata can be minimized using Gelfand duality.

What is Gelfand duality?

- Problem in undergrad algebra courses: Given the ring of continuous real-valued functions defined on a compact Hausdorff space X, call this $C(X)$, show that for every maximal ideal M there is an $x \in X$ such that

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- In fact, one can even get the topology.
- $C(X)$ is more than a ring it is a *commutative, unital* C^*-algebra.
- Incidentally, this works just as well with complex-valued functions.

C^*-algebras

- A (complex) C^*-algebra is a (complex) vector space with
- an associative multiplication (satisfying obvious laws)
- and a norm $\|\cdot\|$ with respect to which it is complete (hence a Banach space).
- The norm satisfies: $\|a \cdot b\| \leq \|a\| \cdot \|b\|$.
- There is also an involution $*$ satisfying $(ab)^* = b^*a^*$ and $(\alpha a)^* = \overline{\alpha} a^*$.
- The crucial property is:
- $\|a^*a\| = \|a\|^2$.
- Morphisms are homomorphisms preserving the $*$.
- We say that A is *unital* if there is a unit element for the multiplication.

Gelfand duality

The category of commutative unital C^*-algebras is dually equivalent to the category of compact Hausdorff spaces.

It does not matter if the C^* algebras are complex (Gelfand) or real (Stone); though the proofs are very different.

Belief automata

- A **probabilistic automaton with observations** is

$$\mathcal{F} = (S, \mathcal{A}, \mathcal{O}, \delta : S \times \mathcal{A} \times S \to [0,1], \gamma : S \times \mathcal{O} \to [0,1]).$$

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- Minimization via Stone duality $\longrightarrow$ minimization via Gelfand duality.

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- Completeness theorems, which typically work by constructing transition systems from consistent sets of formulas embody a key aspect of duality results *but*,
- the arrow part of the duality is crucial for proving our minimization results.

Ongoing and Future Work

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- Metric analogue of Stone duality: Mardare and Kozen.
- Pressing research topic of great interest in quantum information theory: what is the duality theory for *non-commutative* C^*-algebras?: Tobias Fritz.

Thank you!