Labelled Markov Processes

Lecture 3: The logical characterization of probabilistic bisimulation

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Outline

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What are Labelled Markov Processes?

- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

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Stochastic Kernels

- A *stochastic kernel* (Markov kernel) is a function $h : S \times \Sigma \longrightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \longrightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : X \longrightarrow [0, 1]$ a measurable function.
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- and the uncountable generalization of a matrix.

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Formal Definition of LMPs

- An LMP is a tuple $(S, \Sigma, \mathsf{L}, \forall\alpha \in \mathsf{L}.\tau_\alpha)$ where
- (S, Σ) is an **analytic space**
- and $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a *transition probability* function such that
- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure and
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Larsen-Skou Bisimulation

- Let $\mathcal{S} = (S, i, \Sigma, \tau)$ be a labelled Markov process. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) = \tau_a(s', A)$. Two states are bisimilar if they are related by a bisimulation relation.
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Logical Characterization

- $$\mathcal{L} ::== \mathrm{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$
- We say $s \models \langle a \rangle_q \phi$ iff
$$\exists A \in \Sigma.(\forall s' \in A.s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

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Proof Sketch

- Show that the relation “s and s' satisfy exactly the same formulas” is a bisimulation.
- Can easily show that $\tau_a(s, A) = \tau_a(s', A)$ for A of the form $[\![\phi]\!]$.
- Use Dynkin’ $\lambda - \pi$ theorem to show that we get a well defined measure on the σ-algebra generated by such sets and the above equality holds.
- Use special properties of analytic spaces to show that this σ-algebra is the same as the original σ-algebra.

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The Easy Direction

- Let R be a bisimulation relation on an LMP (S, Σ, τ_a). We prove by induction on ϕ that $\forall \phi \in \mathcal{L}$

$$\forall s, s' \in S.sRs' \Rightarrow s \models \phi \Leftrightarrow s' \models \phi.$$

- Base case trivial.
- $\wedge$ is obvious from Inductive Hypothesis.
- For $\phi = \langle a \rangle_q \psi$ we have that $[\![\psi]\!]$ is R-closed from inductive hypothesis. Thus

$$\tau_a(s, [\![\psi]\!]) = \tau_a(s', [\![\psi]\!])$$

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What is measure theory?

- We want to assign a “size” to sets so that we can use it for quantitative purposes, like integration or probability.
- We could count the number of points but this is useless for the continuum.
- We want to generalize the notion of “length” or “area.”
- What is the “length” of the rational numbers between 0 and 1?
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What are measurable sets anyway?

- Alas! Not all sets can be given a sensible notion of size that generalizes the notion of length of an interval.
- We take a family of sets satisfying “reasonable” axioms and deem them to be “measurable.”
- Countable unions are the key.

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Measurable spaces

A **measurable space** (X, Σ) is a set X together with a family Σ of subsets of X, called a σ**-algebra** or σ**-field**, satisfying the following axioms:

1. $\emptyset \in \Sigma$,
2. $A \in \Sigma$ implies that $A^c \in \Sigma$, and
3. if $\{A_i \in \Sigma | i \in I\}$ is a *countable* family then $\cup_{i \in I} A_i \in \Sigma$.

If we require only finite additivity rather than countable additivity we get a **field** or **algebra**.

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Basic facts

- The intersection of any collection of σ-algebras on a set is another σ-algebra.
- Thus given any family of sets $\mathcal{B}$ there is a least σ-algebra containing $\mathcal{B}$: the σ-algebra *generated* by $\mathcal{B}$.
- Measurable sets are complicated beasts, we often want to work with the sets of family of simpler sets that generate the σ-algebra.
- The σ-algebra generated by the intervals in $\mathbf{R}$ is called the *Borel* algebra.
- There is a larger σ-algebra containing the Borel sets called the Lebesgue σ-algebra; we will not use it.

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Functions

- What are the “right” functions between measurable spaces?
- Let $f : X \longrightarrow Y$ be a function and let Σ be a σ-algebra on Y. The sets of the form $\{f^{-1}(A) | A \in \Sigma\}$ form a σ-algebra on X.
- σ-algebras behave well under inverse image.
- A function f from a σ-algebra (X, Σ_X) to a σ-algebra (Y, Σ_Y) is said to be **measurable** if $f^{-1}(B) \in \Sigma_X$ whenever $B \in \Sigma_Y$.

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Measures

A **measure (probability measure)** μ on a measurable space (X, Σ) is a function from Σ (a set function) to $[0, \infty]$ ($[0, 1]$), such that if $\{A_i | i \in I\}$ is a countable family of pairwise disjoint sets then

$$\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i).$$

In particular if I is empty we have

$$\mu(\emptyset) = 0.$$

A set equipped with a σ-algebra and a measure defined on it is called a **measure space**.

An example

Fix a set X and a point x of X. We define a measure, in fact a probability measure, on the σ-algebra of all subsets of X as follows. We use the slightly peculiar notation $\delta(x, A)$ to emphasize that x is a parameter in the definition.

$$\delta(x, A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This measure is called the *Dirac delta measure*. Note that we can fix the set A and view this as the definition of a (measurable) function on X. What we get is the characteristic function of the set A, χ_A.

Lebesgue measure on **R**

- For *any* subset of **R** we define *outer measure* as the infimum of the total length of the intervals of any covering family of intervals.
- The rationals have outer measure zero.
- This is not additive so it does not give a measure defined on all sets.
- It does however give a measure on the Borel sets.

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Digression on Analytic Spaces

- An analytic set A is the image of a Polish space X (or a Borel subset of X) under a continuous (or measurable) function $f : X \longrightarrow Y$, where Y is Polish. If (S, Σ) is a measurable space where S is an analytic set in some ambient topological space and Σ is the Borel σ-algebra on S.
- Analytic sets do not form a σ-algebra but they are in the completion of the Borel algebra under **any** measure. [Universally measurable.]
- Regular conditional probability densities can be defined on analytic spaces.

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Amazing Facts about Analytic Spaces

- Given A an analytic space and $\sim$ an equivalence relation such that there is a *countable* family of real-valued measurable functions $f_i : S \longrightarrow \mathbf{R}$ such that

$$\forall s, s' \in S . s \sim s' \iff \forall f_i . f_i(s) = f_i(s')$$

then the quotient space (Q, Ω) - where $Q = S/\sim$ and Ω is the finest σ-algebra making the canonical surjection $q : S \longrightarrow Q$ measurable - is also analytic.

- If an analytic space (S, Σ) has a sub-σ-algebra Σ_0 of Σ which separates points and is countably generated then Σ_0 is Σ! The Unique Structure Theorem (UST).

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The Quotient

- Given (S, Σ, τ_a) an LMP, we define $s \simeq s'$ if s and s' obey exactly the same formulas of $\mathcal{L}_0$.
- The functions $I_{[\phi]} : S \longrightarrow \mathbf{R}$ defined by $I_{[\phi]}(s) = 1$ if $s \models \phi$ and 0 otherwise are a countable family of measurable functions such that $s \simeq s'$ if and only if all the functions agree on s and s'. Thus the quotient space (Q, Ω) is analytic.
- We define an LMP (Q, Ω, ρ_a) where $\rho_a(t, U) := \tau_a(s, q^{-1}(U)); s \in q^{-1}(\{t\})$.

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ρ is well defined - I

- Easy to check that $q^{-1}(q([\![\phi]\!])) = [\![\phi]\!]$:
  $s \in q^{-1}(q([\![\phi]\!]))$ implies that $q(s) \in q([\![\phi]\!])$, i.e. $\exists s' \in [\![\phi]\!].s \simeq s'$, so $s \models \phi$ so $s \in [\![\phi]\!]$.
- Thus $q([\![\phi]\!])$ is measurable.
- Thus the σ-algebra generated -say, Λ - by $q([\![\phi]\!])$ is a sub-σ-algebra of Ω.
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- The collection $q([\![\phi]\!])$ is a π-system (because $\mathcal{L}_0$ has conjunction) and it generates Ω; thus if we can show that two measures agree on these sets they agree on all of Ω.
- If $q(s) = q(s') = t$ then $\tau_a(s, [\![\phi]\!]) = \tau_a(s', [\![\phi]\!])$ (simple interpolation).
- Thus $\tau_a(s, q^{-1}(q([\![\phi]\!]))) = \tau_a(s', q^{-1}(q([\![\phi]\!])))$ and hence ρ is well defined. We have $\rho_a(q(s), B) = \tau_a(s, q^{-1}(B))$.

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Finishing the Argument

- Let X be any $\simeq$-closed subset of S.
- Then $q^{-1}(q(X)) = X$ and $q(X) \in \Omega$.
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Other Logics

$$
\begin{array}{rcl}
\mathcal{L}_{\mathrm{Can}} & := & \mathcal{L}_0 \mid \mathrm{Can}(a) \\
\mathcal{L}_{\Delta} & := & \mathcal{L}_0 \mid \Delta_a \\
\mathcal{L}_{\neg} & := & \mathcal{L}_0 \mid \neg\phi \\
\mathcal{L}_{\vee} & := & \mathcal{L}_0 \mid \phi_1 \vee \phi_2 \\
\mathcal{L}_{\wedge} & := & \mathcal{L}_{\neg} \mid \bigwedge_{i \in \mathbf{N}} \phi_i
\end{array}
$$

where

$$
\begin{array}{ll}
s \models \mathrm{Can}(a) & \text{to mean that } \tau_a(s, S) > 0; \\
s \models \Delta_a & \text{to mean that } \tau_a(s, S) = 0.
\end{array}
$$

We need $\mathcal{L}_{\vee}$ to characterise simulation.

Conclusions

- Strong probabilistic bisimulation is characterised by a very simple modal logic with no negative constructs.
- There is a logical characterisation of simulation.
- There is a “metric” on LMPs which is based on this logic.
- Why did the proof require so many subtle properties of analytic spaces? There is a more general definition of bisimulation for which the logical characterisation proof is “easy” but to prove that that definition coincides with this one in analytic spaces requires roughly the same proof as that given here.

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