Semantics of Probabilistic Languages

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Estonia Winter School March 2015

Outline

1. Introduction
2. Semantics of a language with while loops
3. Partially additive categories
4. Back to semantics

Syntax

Kozen's Language

$$S ::== x_i := f(\vec{x}) | S_1; S_2 | \textit{if } \mathbf{B} \textit{ then } S_1 \textit{ else } S_2 | \textit{while } \mathbf{B} \textit{ do } S.$$

- There are a fixed set of variables $\vec{x}$ taking values in a measurable space (X, Σ_X).
- f is a measurable function.
- B is a measurable subset.

Outline of the semantics

- State transformer semantics: distribution (measure) transformer semantics.
- Meaning of statements: Markov kernels *i.e.* **SRel** morphisms.
- The only subtle part: how to give fixed-point semantics to the while loop?

Partially additive monoids

- Back to $\mathbf{SRel}$ structure.
- Can we "add" $\mathbf{SRel}$ morphisms?
- Not always, the sum may exceed 1, but we can define *summable families* which may even be countaby infinite.
- The homsets of $\mathbf{SRel}$ form *partially additive monoids*.

Partially additive monoids

A **partially additive monoid** is a pair $(M, \sum)$ where M is a nonempty set and $\sum$ is a partial function which maps *some* countable subsets of M to M. We say that $\{x_i | i \in I\}$ is **summable** if $\sum_{i \in I} x_i$ is defined.

Axioms for partially-additive monoids

1. The sums can be rearranged at will.
2. **Partition-Associativity:** Suppose that $\{x_i | i \in I\}$ is a countable family and $\{I_j | j \in J\}$ is a countable partition of I. Then $\{x_i | i \in I\}$ is summable iff for every $j \in J$ $\{x_i | i \in I_j\}$ is summable and $\{\sum_{i \in I_j} x_i | j \in J\}$ is summable. In this case we require

$$\sum_{i \in I} x_i = \sum_{j \in J} \sum_{i \in I_j} x_i.$$

3. **Unary-sum:** A singleton family is always summable.
4. **Limit:** If $\{x_i | i \in I\}$ is countable and *every finite subfamily* is summable then the whole family is summable.

Zero morphisms

The sum of the empty family exists, call it 0. It is the identity for $\sum$.

Partially additive structure in a category

Let $\mathcal{C}$ be a category. A **partially additive structure** on $\mathcal{C}$ is a partially additive monoid structure on the homsets of $\mathcal{C}$ such that if $\{f_i : X \longrightarrow Y | i \in I\}$ is summable, then $\forall W, Z, g : W \longrightarrow X, h : Y \longrightarrow Z$, we have that $\{h \circ f_i | i \in I\}$ and $\{f_i \circ g | i \in I\}$ are summable and, furthermore, the equations

$$h \circ \sum_{i \in I} f_i = \sum_{i \in I} h \circ f_i, (\sum_{i \in I} f_i) \circ g = \sum_{i \in I} f_i \circ g$$

hold.

A category has **zero morphisms** if there is a distinguished morphism in every homset – we write 0_{XY} for the distinguished member of $hom(X, Y)$ – such that $\forall W, X, Y, Z, f : W \longrightarrow X, g : Z \longrightarrow Y$ we have $g \circ 0_{WZ} = 0_{XY} \circ f$.

If a category has a partially additive structure it has zero morphisms.

SRel has partially additive structure

- A family $\{h_i : X \longrightarrow Y | i \in I\}$ in **SRel** is summable if
$$\forall x \in X. \sum h_i(x, Y) \leq 1.$$
We define the sum by the evident pointwise formula.
- Partition associativity follows immediately from the fact that we are dealing with absolute convergence since all the values are nonnegative.
- The unary sum axiom is immediate.
- The limit axiom follows from the fact that the finite partial sums are bounded by 1.
- Countable additivity follows from the fact that each h_i is countably additive and the sums in question can be rearranged since we have only nonnegative terms.
- The verification of the two distributivity equations is by the monotone convergence theorem

Quasi-projections

Let $\mathcal{C}$ be a category with countable coproducts and zero morphisms and let $\{X_i | i \in I\}$ be a countable family of objects of $\mathcal{C}$.

For any $J \subset I$ we define the **quasi-projection** $PR_J : \coprod_{i \in I} X_i \longrightarrow \coprod_{j \in J} X_j$ by

$$PR_J \circ \iota_i = \begin{cases} \iota_i & i \in J \\ 0 & i \notin J \end{cases}$$

Diagonal-injection

We write $I \cdot X$ for the coproduct of $|I|$ copies of X. We define the **diagonal-injection** Δ by couniversality:

$$
\begin{array}{ccc}
\coprod(X_i|i \in I) & \xrightarrow{\Delta} & I \cdot \coprod(X_i|i \in I) \\
\uparrow{\scriptstyle in_j} & & \uparrow{\scriptstyle in_j} \\
X_j & \xrightarrow[in_j]{} & \coprod(X_i|i \in I)
\end{array}
$$

We have a morphism σ from $I \cdot X$ to X given by:

$$
\begin{array}{ccc}
I \cdot X & \xrightarrow{\sigma} & X \\
\uparrow{\scriptstyle in_j} & \nearrow{\scriptstyle id_X} & \\
X & &
\end{array}
$$

These maps in **SRel**

- $$PR_J((x,k),\uplus_{j\in J}) = \begin{cases} \delta(x,A_k) & k\in J \\ 0 & k\notin J \end{cases}.$$
- The Δ morphism in **SRel** is
$$\Delta((x,k),\uplus_{i\in I}(\uplus_{j\in I}A^i_j)) = \delta(x,A^k_k).$$
The analogous map in **Set** is $\Delta((x,k)) = ((x,k),k)$.
- Finally
$$\sigma((x,k),A) = \delta(x,A)$$
in **SRel** while in **Set** we have $\sigma((x,k)) = x$.

Partially additive category

A **partially additive category**, $\mathcal{C}$, is a category with countable coproducts and a partially additive structure satisfying the following two axioms.

1. **Compatible sum axiom**: If $\{f_i | i \in I\}$ is a countable set of morphisms in $\mathcal{C}(X, Y)$ and there is a morphism $f : X \longrightarrow I \cdot Y$ with $PR_i \circ f = f_i$ then $\{f_i | i \in I\}$ is summable.
2. **Untying axiom**: If $f, g : X \longrightarrow Y$ are summable then $\iota_1 \circ f$ and $\iota_2 \circ g$ are summable as morphisms from X to $Y + Y$.

SRel is a PAC

The category **SRel** is a partially additive category.

All verifications are routine.

Iteration in a PAC

Arbib-Manes

Given $f : X \longrightarrow X + Y$ in a partially additive category, we can find a unique $f_1 : X \longrightarrow X$ and $f_2 : X \longrightarrow Y$ such that $f = \iota_1 \circ f_1 + \iota_2 \circ f_2$. Furthermore there is a morphism $\dagger f =_{df} \sum_{n=0}^{\infty} f_2 \circ f_1^n : X \longrightarrow Y$. The morphism $\dagger f$ is called the **iterate** of f.

- First claim is trivial.
- The second is about the summability of a specific family.
- Can prove easily by induction that the finite subfamilies are summable.
- The limit axiom then guarantees that the whole family is summable.

Semantics of Kozen's Language I

- Statements are **SRel** morphisms of type $(X^n, \Sigma^n) \longrightarrow (X^n, \Sigma^n)$.
- **Assignment**: $x := f(\vec{x})$

$$[\![x_i := f(\vec{x})]\!](\vec{x}, \vec{A}) = \delta(x_1, A_1) \ldots \delta(x_{i-1}, A_{i-1}) \delta(f(\vec{x}), A_i) \delta(x_{i+1}, A_{i+1}) \ldots$$

- **Sequential Composition**: $S_1; S_2$

$$[\![S_1; S_2]\!] = [\![S_2]\!] \circ [\![S_1]\!]$$

where the composition on the right hand side is the composition in **SRel**.

- **Conditionals**: *if* $\mathbf{B}$ *then* S_1 *else* S_2

$$[\![\mathit{if}\ \mathbf{B}\ \mathit{then}\ S_1\ \mathit{else}\ S_2]\!](\vec{x}, \vec{A}) = \delta(\vec{x}, \mathbf{B})[\![S_1]\!](\vec{x}, \vec{A}) + \delta(\vec{x}, \mathbf{B}^c)[\![S_2]\!](\vec{x}, \vec{A})$$

Semantics of Kozen's Language II

While Loops: *while* **B** *do* *S*

$$[\![while\ \mathbf{B}\ do\ S]\!] = h^*$$

where we are using the $*$ in **SRel** and the morphism

$$h : (X^n, \Sigma^n) \longrightarrow (X^n, \Sigma^n) + (X^n, \Sigma^n)$$

is given by

$$h(\vec{x}, \vec{A_1} \uplus \vec{A_2}) = \delta(\vec{x}, \mathbf{B})[\![S]\!](\vec{x}, \vec{A_1}) + \delta(\vec{x}, \mathbf{B}^c)\delta(\vec{x}, \vec{A_2}).$$

Weakest precondition semantics

- We can construct a category of probabilistic predicate transformers: $\mathbf{SPT}$.
- Objects are measurable spaces.
- Given (X, Σ_X) we can construct the (Banach) space of bounded measurable functions on X (the "predicates") $\mathcal{F}(X)$.
- A morphism $X \longrightarrow Y$ in $\mathbf{SPT}$ is a bounded (continuous) linear map from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$.
- $$\mathbf{SPT} \simeq \mathbf{SRel}^{op}.$$
- This gives us the structure needed for a $\mathbf{wp}$ semantics.