A domain of spacetime intervals for General Relativity

Keye Martin and Prakash Panangaden

Tulane University and McGill University

work done at University of Oxford

+ < >

Viewpoint

- Understand the role of order in analysing the causal structure of spacetime.

+ < >

Viewpoint

- Understand the role of order in analysing the causal structure of spacetime.
- Reconstruct spacetime topology from causal order: obvious links with domain theory.

+ < >

Viewpoint

- Understand the role of order in analysing the causal structure of spacetime.
- Reconstruct spacetime topology from causal order: obvious links with domain theory.
- Not looking at the combinatorial aspects of order: continuous posets play a vital role; Scott, Lawson and interval topologies play a vital role.

+ < >

Viewpoint

- Understand the role of order in analysing the causal structure of spacetime.
- Reconstruct spacetime topology from causal order: obvious links with domain theory.
- Not looking at the combinatorial aspects of order: continuous posets play a vital role; Scott, Lawson and interval topologies play a vital role.
- Everything is about classical spacetime: we see this as a step on Sorkin's programme to understand quantum gravity in terms of causets.

Overview

- The causal structure of globally hyperbolic spacetimes defines a *bicontinuous* poset. The topology can be recovered from the order *and from the way-below relation* but with no appeal to smoothness. The order can be taken to be fundamental.

+ < >

Overview

- The causal structure of globally hyperbolic spacetimes defines a *bicontinuous* poset. The topology can be recovered from the order *and from the way-below relation* but with no appeal to smoothness. The order can be taken to be fundamental.
- The entire spacetime manifold can be reconstructed given a countable dense subset with the induced order: no metric information need be given.

+ < >

Overview

- The causal structure of globally hyperbolic spacetimes defines a *bicontinuous* poset. The topology can be recovered from the order *and from the way-below relation* but with no appeal to smoothness. The order can be taken to be fundamental.
- The entire spacetime manifold can be reconstructed given a countable dense subset with the induced order: no metric information need be given.
- Globally hyperbolic spacetimes can be seen as the maximal elements of interval domains. There is an equivalence of categories between globally hyperbolic spacetimes and interval domains. The main theorem.

+ < >

Work in Progress

- The space of causal curves between two compact sets is itself compact in the Vietoris topology. Causal curves are maximal elements of the convex powerdomain. Paper by Keye Martin is on the ArXiv (gr-qc).

+ < >

Work in Progress

- The space of causal curves between two compact sets is itself compact in the Vietoris topology. Causal curves are maximal elements of the convex powerdomain. Paper by Keye Martin is on the ArXiv (gr-qc).
- Incorporating metric information as a measurement (in Keye Martin's sense) on top of the poset.

+ < >

Causality in Computer Science

- In distributed systems one loses synchronization and *absolute* global state just as in relativity. One works with causal structure.

+ < >

Causality in Computer Science

- In distributed systems one loses synchronization and *absolute* global state just as in relativity. One works with causal structure.
- Causal precedence in distributed systems studied by Petri (65) and Lamport (77): clever algorithms, but the mathematics was elementary and combinatorial and did not reveal the connections with general relativity.

Causality in Computer Science

- In distributed systems one loses synchronization and *absolute* global state just as in relativity. One works with causal structure.
- Causal precedence in distributed systems studied by Petri (65) and Lamport (77): clever algorithms, but the mathematics was elementary and combinatorial and did not reveal the connections with general relativity.
- Event structures studied by Winskel, Plotkin and others (80-85): more sophisticated, invoked domain theory. The mathematics comes closer to what we will see today.

The layers of spacetime structure

- Set of events: no structure

+ < >

The layers of spacetime structure

- Set of events: no structure
- Topology: 4 dimensional real manifold

+ < >

The layers of spacetime structure

- Set of events: no structure
- Topology: 4 dimensional real manifold
- Differentiable structure: tangent spaces

+ < >

The layers of spacetime structure

- Set of events: no structure
- Topology: 4 dimensional real manifold
- Differentiable structure: tangent spaces
- Causal structure: light cones (defines metric up to conformal transformations)

+ < >

The layers of spacetime structure

- Set of events: no structure
- Topology: 4 dimensional real manifold
- Differentiable structure: tangent spaces
- Causal structure: light cones (defines metric up to conformal transformations)
- Lorentzian metric: gives a length scale.

Causal Structure of Spacetime I

- At every point a pair of “cones” is defined in the tangent space: future and past light cone. A vector on the cone is called **null** or **lightlike** and one inside the cone is called **timelike**.

+ < >

Causal Structure of Spacetime I

- At every point a pair of “cones” is defined in the tangent space: future and past light cone. A vector on the cone is called **null** or **lightlike** and one inside the cone is called **timelike**.
- We assume that spacetime is *time-orientable*: there is a global notion of future and past.

+ < >

Causal Structure of Spacetime I

- At every point a pair of “cones” is defined in the tangent space: future and past light cone. A vector on the cone is called **null** or **lightlike** and one inside the cone is called **timelike**.
- We assume that spacetime is *time-orientable*: there is a global notion of future and past.
- A *timelike* curve from x to y has a tangent vector that is everywhere timelike: we write $x \preceq y$. (We avoid $x \ll y$ for now.) A *causal* curve has a tangent that, at every point, is either timelike or null: we write $x \leq y$.

+ < >

Causal Structure of Spacetime I

- At every point a pair of "cones" is defined in the tangent space: future and past light cone. A vector on the cone is called **null** or **lightlike** and one inside the cone is called **timelike**.
- We assume that spacetime is *time-orientable*: there is a global notion of future and past.
- A *timelike* curve from x to y has a tangent vector that is everywhere timelike: we write $x \preceq y$. (We avoid $x \ll y$ for now.) A *causal* curve has a tangent that, at every point, is either timelike or null: we write $x \leq y$.
- A fundamental assumption is that $\leq$ is a partial order. Penrose and Kronheimer give axioms for $\leq$ and $\preceq$.

+ < >

Causal Structure of Spacetime II

- $I^+(x) := \{y \in M | x \preceq y\}$; similarly I^-

Causal Structure of Spacetime II

- $I^+(x) := \{y \in M | x \preceq y\}$; similarly I^-
- $J^+(x) := \{y \in M | x \leq y\}$; similarly J^-.

+ < >

Causal Structure of Spacetime II

- $I^+(x) := \{y \in M | x \preceq y\}$; similarly I^-
- $J^+(x) := \{y \in M | x \leq y\}$; similarly J^-.
- $I^\pm$ are always open sets in the manifold topology; $J^\pm$ are **not** always closed sets.

+ < >

Causal Structure of Spacetime II

- $I^+(x) := \{y \in M | x \preceq y\}$; similarly I^-
- $J^+(x) := \{y \in M | x \leq y\}$; similarly J^-.
- $I^\pm$ are always open sets in the manifold topology; $J^\pm$ are **not** always closed sets.
- Chronology: $x \preceq y \Rightarrow y \npreceq x$.

+ < >

Causal Structure of Spacetime II

- $I^+(x) := \{y \in M | x \preceq y\}$; similarly I^-
- $J^+(x) := \{y \in M | x \leq y\}$; similarly J^-.
- $I^\pm$ are always open sets in the manifold topology; $J^\pm$ are **not** always closed sets.
- Chronology: $x \preceq y \Rightarrow y \not\preceq x$.
- Causality: $x \leq y$ and $y \leq x$ implies $x = y$.

+ < >

Causality Conditions

- $$I^{\pm} = I^{\pm} \Rightarrow p = q.$$

+ < >

Causality Conditions

- $$I^{\pm} = I^{\pm} \Rightarrow p = q.$$
- Strong causality at p: Every neighbourhood $\mathcal{O}$ of p contains a neighbourhood $\mathcal{U} \subset \mathcal{O}$ such that no causal curve can enter $\mathcal{U}$, leave it and then re-enter it.

+ < >

Causality Conditions

- $$I^{\pm} = I^{\pm} \Rightarrow p = q.$$
- Strong causality at p: Every neighbourhood $\mathcal{O}$ of p contains a neighbourhood $\mathcal{U} \subset \mathcal{O}$ such that no causal curve can enter $\mathcal{U}$, leave it and then re-enter it.
- Stable causality: perturbations of the metric do not cause violations of causality.

Causality Conditions

- $$I^{\pm} = I^{\pm} \Rightarrow p = q.$$
- Strong causality at p: Every neighbourhood $\mathcal{O}$ of p contains a neighbourhood $\mathcal{U} \subset \mathcal{O}$ such that no causal curve can enter $\mathcal{U}$, leave it and then re-enter it.
- Stable causality: perturbations of the metric do not cause violations of causality.
- Causal simplicity: for all $x \in M$, $J^{\pm}(x)$ are closed.

+ < >

Causality Conditions

- $$I^{\pm} = I^{\pm} \Rightarrow p = q.$$
- Strong causality at p: Every neighbourhood $\mathcal{O}$ of p contains a neighbourhood $\mathcal{U} \subset \mathcal{O}$ such that no causal curve can enter $\mathcal{U}$, leave it and then re-enter it.
- Stable causality: perturbations of the metric do not cause violations of causality.
- Causal simplicity: for all $x \in M$, $J^{\pm}(x)$ are closed.
- Global hyperbolicity: M is strongly causal and for each p, q in M, $[p, q] := J^{+}(p) \cap J^{-}(q)$ is compact.

+ < >

The Alexandrov Topology

Define

$$\langle x, y \rangle := I^{+}(x) \cap I^{-}(y).$$

The sets of the form $\langle x, y \rangle$ form a base for a topology on M called the Alexandrov topology.
Theorem (Penrose): TFAE:

1. (M, g) is strongly causal.
2. The Alexandrov topology agrees with the manifold topology.
3. The Alexandrov topology is Hausdorff.

The proof is geometric in nature.

+ < >

The Way-below relation

- In domain theory, in addition to $\leq$ there is an additional, (often) irreflexive, transitive relation written $\ll$: $x \ll y$ means that x has a “finite” piece of information about y or x is a “finite approximation” to y. If $x \ll x$ we say that x is *finite*.

+ < >

The Way-below relation

- In domain theory, in addition to $\leq$ there is an additional, (often) irreflexive, transitive relation written $\ll$: $x \ll y$ means that x has a "finite" piece of information about y or x is a "finite approximation" to y. If $x \ll x$ we say that x is *finite*.
- The relation $x \ll y$ - pronounced x is "way below" y - is directly defined from $\leq$.

+ < >

The Way-below relation

- In domain theory, in addition to $\leq$ there is an additional, (often) irreflexive, transitive relation written $\ll$: $x \ll y$ means that x has a “finite” piece of information about y or x is a “finite approximation” to y. If $x \ll x$ we say that x is *finite*.
- The relation $x \ll y$ - pronounced x is “way below” y - is directly defined from $\leq$.
- Official definition of $x \ll y$: If $X \subset D$ is directed and $y \leq (\sqcup X)$ then there exists $u \in X$ such that $x \leq u$. If a limit gets past y then, at some finite stage of the limiting process it already got past x.

+ < >

The role of way below in spacetime structure

- **Theorem:** Let (M, g) be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If (M, g) is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.

\+ < >

The role of way below in spacetime structure

- **Theorem:** Let (M, g) be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If (M, g) is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.
- One can recover I from J without knowing what smooth or timelike means.

+ < >

The role of way below in spacetime structure

- **Theorem:** Let (M, g) be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If (M, g) is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.
- One can recover I from J without knowing what smooth or timelike means.
- Intuition: any way of approaching y must involve getting into the timelike future of x.

+ < >

The role of way below in spacetime structure

- **Theorem:** Let (M,g) be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If (M,g) is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.
- One can recover I from J without knowing what smooth or timelike means.
- Intuition: any way of approaching y must involve getting into the timelike future of x.
- We can stop being coy about notational clashes: henceforth $\ll$ is way-below *and* the timelike order.

+ < >

Continuous Domains and Topology

- A continuous domain D has a basis of elements $B \subset D$ such that for every x in D the set $x\!\downarrow\!\!\downarrow := \{u \in B | u \ll x\}$ is directed and $\sqcup(x\!\downarrow\!\!\downarrow) = x$.

+ < >

Continuous Domains and Topology

- A continuous domain D has a basis of elements $B \subset D$ such that for every x in D the set $x{\downarrow} := \{u \in B | u \ll x\}$ is directed and $\sqcup(x{\downarrow}) = x$.
- The Scott topology: the open sets of D are upwards closed and if $\mathcal{O}$ is open, then if $X \subset D$, directed and $\sqcup X \in \mathcal{O}$ it must be the case that some $x \in X$ is in $\mathcal{O}$.

+ < >

Continuous Domains and Topology

- A continuous domain D has a basis of elements $B \subset D$ such that for every x in D the set $x\Downarrow := \{u \in B | u \ll x\}$ is directed and $\sqcup(x\Downarrow) = x$.
- The Scott topology: the open sets of D are upwards closed and if $\mathcal{O}$ is open, then if $X \subset D$, directed and $\sqcup X \in \mathcal{O}$ it must be the case that some $x \in X$ is in $\mathcal{O}$.
- The Lawson topology: basis of the form

$$\mathcal{O} \setminus [\cup_i (x_i \uparrow)]$$

where $\mathcal{O}$ is Scott open. This topology is metrizable if the domain is ω-continuous.

+ < >

Continuous Domains and Topology

- A continuous domain D has a basis of elements $B \subset D$ such that for every x in D the set $x{\downarrow} := \{u \in B | u \ll x\}$ is directed and $\sqcup(x{\downarrow}) = x$.
- The Scott topology: the open sets of D are upwards closed and if $\mathcal{O}$ is open, then if $X \subset D$, directed and $\sqcup X \in \mathcal{O}$ it must be the case that some $x \in X$ is in $\mathcal{O}$.
- The Lawson topology: basis of the form

$$\mathcal{O} \setminus [\cup_i (x_i \uparrow)]$$

  where $\mathcal{O}$ is Scott open. This topology is metrizable if the domain is ω-continuous.
- The interval topology: basis sets of the form $(x, y) := \{u | x \ll u \ll y\}$.

+ < >

Bicontinuity and Global Hyperbolicity

- The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).

+ < >

Bicontinuity and Global Hyperbolicity

- The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).
- Theorem: If (M, g) is globally hyperbolic then $(M, \leq)$ is a bicontinuous poset. In this case the interval topology is the manifold topology.

Bicontinuity and Global Hyperbolicity

- The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).
- Theorem: If (M, g) is globally hyperbolic then $(M, \leq)$ is a bicontinuous poset. In this case the interval topology is the manifold topology.
- We feel that bicontinuity is a significant causality condition in its own right; perhaps it sits between globally hyperbolic and causally simple.

Bicontinuity and Global Hyperbolicity

- The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).
- Theorem: If (M, g) is globally hyperbolic then $(M, \leq)$ is a bicontinuous poset. In this case the interval topology is the manifold topology.
- We feel that bicontinuity is a significant causality condition in its own right; perhaps it sits between globally hyperbolic and causally simple.
- Topological property of causally simple spacetimes: If (M, g) is causally simple then the Lawson topology is contained in the interval topology.

An “abstract” version of globally hyperbolic

We *define* a globally hyperbolic poset $(X, \leq)$ to be

1. bicontinuous and,
2. all segments $[a, b] := \{x : a \leq x \leq b\}$ are compact in the interval topology on X.

+ < >

Properties of globally hyperbolic posets

- A globally hyperbolic poset is locally compact and Hausdorff.

+ < >

Properties of globally hyperbolic posets

- A globally hyperbolic poset is locally compact and Hausdorff.
- The Lawson topology is contained in the interval topology.

+ < >

Properties of globally hyperbolic posets

- A globally hyperbolic poset is locally compact and Hausdorff.
- The Lawson topology is contained in the interval topology.
- Its partial order $\leq$ is a closed subset of X^2.

+ < >

Properties of globally hyperbolic posets

- A globally hyperbolic poset is locally compact and Hausdorff.
- The Lawson topology is contained in the interval topology.
- Its partial order $\leq$ is a closed subset of X^2.
- Each directed set with an upper bound has a supremum.

+ < >

Properties of globally hyperbolic posets

- A globally hyperbolic poset is locally compact and Hausdorff.
- The Lawson topology is contained in the interval topology.
- Its partial order $\leq$ is a closed subset of X^2.
- Each directed set with an upper bound has a supremum.
- Each filtered set with a lower bound has an infimum.

+ < >

Second countability

- Globally hyperbolic posets share a remarkable property with metric spaces, that separability (countable dense subset) and second countability (countable base of opens) are equivalent.

+ < >

Second countability

- Globally hyperbolic posets share a remarkable property with metric spaces, that separability (countable dense subset) and second countability (countable base of opens) are equivalent.
- Let $(X, \leq)$ be a bicontinuous poset. If $C \subseteq X$ is a countable dense subset in the interval topology, then:
(i) The collection

$$\{(a_i, b_i) : a_i, b_i \in C, a_i \ll b_i\}$$

is a countable basis for the interval topology.
(ii) For all $x \in X$, $\Downarrow x \cap C$ contains a directed set with supremum x, and $\Uparrow x \cap C$ contains a filtered set with infimum x.

+ < >

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a,b] : a,b \in \mathbb{R} \;\&\; a \leq b\}$$

ordered under reverse inclusion

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

is an ω-continuous dcpo.

+ < >

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a,b] : a,b \in \mathbb{R} \;\&\; a \leq b\}$$

ordered under reverse inclusion

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

is an ω-continuous dcpo.

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,

+ < >

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \ \& \ a \leq b\}$$

ordered under reverse inclusion

$$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

is an ω-continuous dcpo.

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $I \ll J \Leftrightarrow J \subseteq \text{int}(I)$, and

+ < >

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a,b] : a,b \in \mathbb{R} \;\&\; a \leq b\}$$

ordered under reverse inclusion

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

is an ω-continuous dcpo.

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $I \ll J \Leftrightarrow J \subseteq \text{int}(I)$, and
- $\{[p,q] : p,q \in \mathbb{Q} \;\&\; p \leq q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$.

+ < >

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a,b] : a,b \in \mathbb{R} \,\&\, a \leq b\}$$

ordered under reverse inclusion

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

is an ω-continuous dcpo.

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $I \ll J \Leftrightarrow J \subseteq \mathrm{int}(I)$, and
- $\{[p,q] : p,q \in \mathbb{Q} \,\&\, p \leq q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$.
- The domain $\mathbf{I}\mathbb{R}$ is called the interval domain.

Generalizing $\mathbf{I}\mathbb{R}$

- The closed segments of a globally hyperbolic poset X
$$\mathbf{I}X := \{[a, b] : a \leq b \,\&\, a, b \in X\}$$
ordered by reverse inclusion form a continuous domain with

\+ < >

Generalizing $\mathbf{I}\mathbb{R}$

- The closed segments of a globally hyperbolic poset X

$$\mathbf{I}X := \{[a,b] : a \leq b \;\&\; a,b \in X\}$$

ordered by reverse inclusion form a continuous domain with

- $$[a,b] \ll [c,d] \equiv a \ll c \;\&\; d \ll b.$$

+ < >

Generalizing $\mathbf{I}\mathbb{R}$

- The closed segments of a globally hyperbolic poset X

$$\mathbf{I}X := \{[a,b] : a \leq b \;\&\; a, b \in X\}$$

ordered by reverse inclusion form a continuous domain with

-

$$[a,b] \ll [c,d] \equiv a \ll c \;\&\; d \ll b.$$

- X has a countable basis iff $\mathbf{I}X$ is ω-continuous.

+ < >

Generalizing $\mathbf{I}\mathbb{R}$

- The closed segments of a globally hyperbolic poset X

$$\mathbf{I}X := \{[a,b] : a \le b \;\&\; a, b \in X\}$$

  ordered by reverse inclusion form a continuous domain with

- $$[a,b] \ll [c,d] \equiv a \ll c \;\&\; d \ll b.$$

- X has a countable basis iff $\mathbf{I}X$ is ω-continuous.

- $$\max(\mathbf{I}X) \simeq X$$

  where the set of maximal elements has the relative Scott topology from $\mathbf{I}X$.

+ < >

Spacetime from a discrete ordered set

- If we have a countable dense subset $\mathcal{C}$ of $\mathcal{M}$, a globally hyperbolic spacetime, then we can view the induced causal order on C as defining a discrete poset. An ideal completion construction in domain theory, applied to a poset constructed from $\mathcal{C}$ yields a domain $\mathbf{I}C$ with

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements have the Scott topology. Thus from a countable subset of the manifold we can reconstruct the whole manifold.

+ < >

Spacetime from a discrete ordered set

- If we have a countable dense subset $\mathcal{C}$ of $\mathcal{M}$, a globally hyperbolic spacetime, then we can view the induced causal order on C as defining a discrete poset. An ideal completion construction in domain theory, applied to a poset constructed from $\mathcal{C}$ yields a domain $\mathbf{I}C$ with

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

  where the set of maximal elements have the Scott topology. Thus from a countable subset of the manifold we can reconstruct the whole manifold.
- We do not know any conditions that allow us to look at a given poset and say that it arises as a dense subset of a manifold, globally hyperbolic or otherwise.

Compactness of the space of causal curves

- A fundamental result in relativity is that the space of causal curves between points is compact on a globally hyperbolic spacetime. We use domains as an aid in proving this fact for any globally hyperbolic poset. This is the analogue of a theorem of Sorkin and Woolgar: they proved it for K-causal spacetimes; we did it for globally hyperbolic posets.

+ < >

Compactness of the space of causal curves

- A fundamental result in relativity is that the space of causal curves between points is compact on a globally hyperbolic spacetime. We use domains as an aid in proving this fact for any globally hyperbolic poset. This is the analogue of a theorem of Sorkin and Woolgar: they proved it for K-causal spacetimes; we did it for globally hyperbolic posets.
- The Vietoris topology on causal curves arises as the natural counterpart to the manifold topology on events, so we can understand that its use by Sorkin and Woolgar is very natural.

+ < >

Compactness of the space of causal curves

- A fundamental result in relativity is that the space of causal curves between points is compact on a globally hyperbolic spacetime. We use domains as an aid in proving this fact for any globally hyperbolic poset. This is the analogue of a theorem of Sorkin and Woolgar: they proved it for K-causal spacetimes; we did it for globally hyperbolic posets.
- The Vietoris topology on causal curves arises as the natural counterpart to the manifold topology on events, so we can understand that its use by Sorkin and Woolgar is very natural.
- The causal curves emerge as the maximal elements of a natural domain; in fact a "powerdomain": a domain-theoretic analogue of a powerset.

+ < >

Globally Hyperbolic Posets and Interval Domains

- One can define categories of globally hyperbolic posets and an abstract notion of “interval domain”: these can also be organized into a category.

+ < >

Globally Hyperbolic Posets and Interval Domains

- One can define categories of globally hyperbolic posets and an abstract notion of “interval domain”: these can also be organized into a category.
- These two categories are equivalent.

+ < >

Globally Hyperbolic Posets and Interval Domains

- One can define categories of globally hyperbolic posets and an abstract notion of “interval domain”: these can also be organized into a category.
- These two categories are equivalent.
- Thus globally hyperbolic spacetimes *are* domains - not just posets - but

+ < >

Globally Hyperbolic Posets and Interval Domains

- One can define categories of globally hyperbolic posets and an abstract notion of “interval domain”: these can also be organized into a category.
- These two categories are equivalent.
- Thus globally hyperbolic spacetimes *are* domains - not just posets - but
- not with the causal order but, rather, with the order coming from the notion of intervals; i.e. from notions of approximation.

Interval Posets

- An *interval* poset D has two functions left $: D \to \max(D)$ and right $: D \to \max(D)$ such that

$$(\forall x \in D)\, x = \text{left}(x) \sqcap \text{right}(x).$$

+ < >

Interval Posets

- An *interval* poset D has two functions left $: D \to \max(D)$ and right $: D \to \max(D)$ such that

$$(\forall x \in D)\, x = \text{left}(x) \sqcap \text{right}(x).$$

- The union of two intervals with a common endpoint is another interval and

+ < >

Interval Posets

- An *interval* poset D has two functions left $: D \longrightarrow \max(D)$ and right $: D \longrightarrow \max(D)$ such that

$$(\forall x \in D)\, x = \text{left}(x) \sqcap \text{right}(x).$$

- The union of two intervals with a common endpoint is another interval and
- each point $p \in \max(D)$ above x determines two subintervals left$(x) \sqcap p$ and $p \sqcap$ right(x) with evident endpoints.

+ < >

Interval Domains

- $(D, \mathrm{left}, \mathrm{right})$ with D a continuous dcpo

+ < >

Interval Domains

- $(D, \mathrm{left}, \mathrm{right})$ with D a continuous dcpo
- satisfying some reasonable conditions about how left and right interact with sups and with $\ll$ and

+ < >

Interval Domains

- $(D, \mathrm{left}, \mathrm{right})$ with D a continuous dcpo
- satisfying some reasonable conditions about how left and right interact with sups and with $\ll$ and
- intervals are compact: $\uparrow x \cap \max(D)$ is Scott compact.

+ < >

Globally Hyperbolic Posets are an Example

- For a globally hyperbolic $(X, \leq)$, we define $\mathrm{left} : \mathbf{I}X \to \mathbf{I}X :: [a,b] \mapsto [a]$ and $\mathrm{right} : \mathbf{I}X \to \mathbf{I}X :: [a,b] \mapsto [b]$.

+ < >

Globally Hyperbolic Posets are an Example

- For a globally hyperbolic $(X, \leq)$, we define $\mathrm{left} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [a]$ and $\mathrm{right} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [b]$.
- Lemma: If $(X, \leq)$ is a globally hyperbolic poset, then $(\mathbf{I}X, \mathrm{left}, \mathrm{right})$ is an interval domain.

+ < >

Globally Hyperbolic Posets are an Example

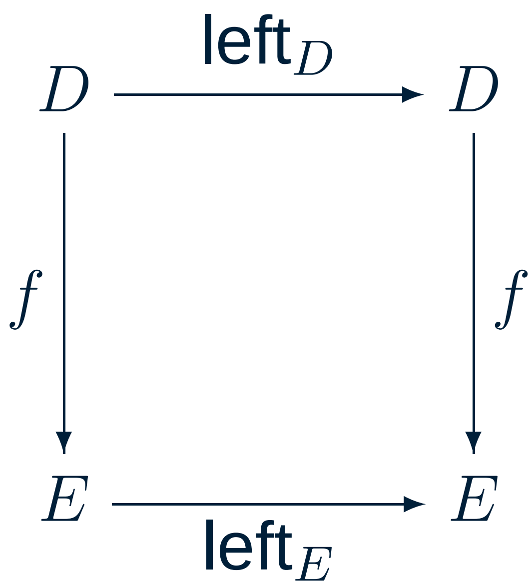
- For a globally hyperbolic $(X, \leq)$, we define $\text{left} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [a]$ and $\text{right} : \mathbf{I}X \to \mathbf{I}X :: [a, b] \mapsto [b]$.
- Lemma: If $(X, \leq)$ is a globally hyperbolic poset, then $(\mathbf{I}X, \text{left}, \text{right})$ is an interval domain.
- In essence, we now prove that this is the only example.

+ < >

The category of Interval Domains

The category $\mathbf{IN}$ of interval domains and commutative maps is given by

- **objects** Interval domains $(D, \text{left}, \text{right})$.
- **arrows** Scott continuous $f : D \to E$ that commute with left and right, i.e., such that both



and

The category of Interval Domains cont.

-
$$\begin{array}{ccc} D & \xrightarrow{\text{right}_D} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ E & \xrightarrow[\text{right}_E]{} & E \end{array}$$

  commute.
- **identity** $1 : D \to D$.
- **composition** $f \circ g$.

The Category GlobHyP

The category $\mathbf{GlobHyP}$ is given by

- **objects** Globally hyperbolic posets $(X, \leq)$.
- **arrows** Continuous in the interval topology, monotone.
- **identity** $1 : X \to X$.
- **composition** $f \circ g$.

+ < >

From GlobHyP to IN

The correspondence $\mathbf{I} : \mathbf{GlobHyP} \to \mathbf{IN}$ given by

$$(X, \leq) \mapsto (\mathbf{I}X, \mathrm{left}, \mathrm{right})$$

$$(f : X \to Y) \mapsto (\bar{f} : \mathbf{I}X \to \mathbf{I}Y)$$

is a functor between categories.

+ < >

From IN to GlobHyP

- Given $(D, \text{left}, \text{right})$ we have a poset $(\max(D), \leq)$ where the order on the maximal elements is given by:

$$a \leq b \equiv (\exists\, x \in D)\, a = \text{left}(x)\ \&\ b = \text{right}(x).$$

+ < >

From IN to GlobHyP

- Given $(D, \text{left}, \text{right})$ we have a poset $(\max(D), \leq)$ where the order on the maximal elements is given by:

$$a \leq b \equiv (\exists\, x \in D)\, a = \text{left}(x)\ \&\ b = \text{right}(x).$$

- After a five page long proof (due entirely to Keye!) it can be shown that $(\max(D), \leq)$ is always a globally hyperbolic poset.

+ < >

From IN to GlobHyP

- Given $(D, \text{left}, \text{right})$ we have a poset $(\max(D), \leq)$ where the order on the maximal elements is given by:

$$a \leq b \equiv (\exists\, x \in D)\, a = \text{left}(x)\ \&\ b = \text{right}(x).$$

- After a five page long proof (due entirely to Keye!) it can be shown that $(\max(D), \leq)$ is always a globally hyperbolic poset.
- Showing that this gives an equivalence of categories is easy.

+ < >

Summary

- We can recover the topology from the order.

+ < >

Summary

- We can recover the topology from the order.
- We can reconstruct the spacetime from a countable dense subset.

+ < >

Summary

- We can recover the topology from the order.
- We can reconstruct the spacetime from a countable dense subset.
- We can characterise causal simplicity order theoretically.

+ < >

Summary

- We can recover the topology from the order.
- We can reconstruct the spacetime from a countable dense subset.
- We can characterise causal simplicity order theoretically.
- We can prove the Sorkin-woolgar theorem on compactness of the space of causal curves.

+ < >

Summary

- We can recover the topology from the order.
- We can reconstruct the spacetime from a countable dense subset.
- We can characterise causal simplicity order theoretically.
- We can prove the Sorkin-woolgar theorem on compactness of the space of causal curves.
- We have shown that globally hyperbolic posets are essentially a certain kind of domain: generalizing one of the earliest and most-loved example of a continuous dcpo.

+ < >

Summary

- We can recover the topology from the order.
- We can reconstruct the spacetime from a countable dense subset.
- We can characterise causal simplicity order theoretically.
- We can prove the Sorkin-woolgar theorem on compactness of the space of causal curves.
- We have shown that globally hyperbolic posets are essentially a certain kind of domain: generalizing one of the earliest and most-loved example of a continuous dcpo.

+ < >

Conclusions

- Domain theoretic methods are fruitful in this setting.

+ < >

Conclusions

- Domain theoretic methods are fruitful in this setting.
- The fact that globally hyperbolic posets are interval domains gives a sensible way of thinking of "approximations" to spacetime points in terms of intervals. Gives us a way to understand coarse graining.

+ < >

What is to be done?

- There is a notion of *measurement* on a domain; a way of adding quantitative information. This was invented by Keye Martin. We are trying to see if there is a natural measurement on a domain that corresponds to spacetime volume of an interval or maximal geodesic length in an interval from which the rest of the geometry may reappear.

+ < >

What is to be done?

- There is a notion of *measurement* on a domain; a way of adding quantitative information. This was invented by Keye Martin. We are trying to see if there is a natural measurement on a domain that corresponds to spacetime volume of an interval or maximal geodesic length in an interval from which the rest of the geometry may reappear.
- We would like to understand conditions that allow us to tell if a given poset came from a manifold. Can we look at a poset and discern a “dimension”? Perhaps this will be a fusion of topology and combinatorics.

What is to be done?

- There is a notion of *measurement* on a domain; a way of adding quantitative information. This was invented by Keye Martin. We are trying to see if there is a natural measurement on a domain that corresponds to spacetime volume of an interval or maximal geodesic length in an interval from which the rest of the geometry may reappear.
- We would like to understand conditions that allow us to tell if a given poset came from a manifold. Can we look at a poset and discern a “dimension”? Perhaps this will be a fusion of topology and combinatorics.
- Understand the quantum theory of causal sets.

+ < >

What is to be done?

- There is a notion of *measurement* on a domain; a way of adding quantitative information. This was invented by Keye Martin. We are trying to see if there is a natural measurement on a domain that corresponds to spacetime volume of an interval or maximal geodesic length in an interval from which the rest of the geometry may reappear.
- We would like to understand conditions that allow us to tell if a given poset came from a manifold. Can we look at a poset and discern a “dimension”? Perhaps this will be a fusion of topology and combinatorics.
- Understand the quantum theory of causal sets.
- Destroy string theory!