Quantum Computing
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- Uses qubits: 2 dimensional quantum systems
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- Exploits entanglement
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- Requires implementing precise transformations on the qubits.
The Trouble with Qubits
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The Trouble with Qubits

- We need to be able to make exquisitely delicate manipulations of qubits
- while preserving entanglement and
- ensuring absence of decoherence.
- A tall order!
We need stability
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- Where do we find quantum braids or knots?
Quantum Statistics
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Depending on the type of particle the answer could be 1/3 (bosons) or 0 (fermions).
Symmetry
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- Symmetries can be composed, there is an identity, there is an inverse for every symmetry and composition is associative.
- Symmetries form a group.
Symmetry in QM
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$$\rho : G \to GL(H)$$
Identical particles
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The permutation group is a symmetry of a quantum system: the system looks the same if you interchange particles of the same type.
Representations of the Permutation Group
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The simplest two representations possible:
Representations of the Permutation Group

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- the trivial representation: every permutation is mapped onto the identity element of $\text{GL}(H)$,
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The simplest two representations possible:

- the trivial representation: every permutation is mapped onto the identity element of $GL(H)$,

- or the alternating representation: a permutation $P$ is mapped to $+1$ or $-1$ according to whether $P$ is odd or even.
What nature does
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- The state vector of a system either changes sign under an interchange of any pair of identical particles (fermions) or does not (bosons).
Consequences 1
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In short $v = 0$!

With fermions two particles cannot be in exactly the same state: Pauli exclusion principle. The reason for chemistry!!
Consequences 2
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Bosons can indeed be packed into the same state.
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- The fundamental reason for early quantum mechanics.
- The explanation of lasers, superconductivity and many other collective phenomena.
Spin in Quantum Mechanics
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• We have to identify a rotation of \( \theta \) and \( \pi - \theta \), so we identify antipodal points on the surface of the ball.

• The resulting group is not simply connected: there are loops that cannot be continuously deformed to a point.
A picture of $SO(3)$ showing a loop that can be shrunk to a point and one that cannot.

$SO(3)$ is not simply connected.
There is another group $SU(2)$: the group of unitary $2 \times 2$ matrices with determinant 1.

There is a homomorphism from $SU(2)$ to $SO(3)$ which is onto and 2 to 1 and which locally looks just like $SO(3)$ but globally is simply connected.

Now which is the relevant symmetry group for quantum mechanics?
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Nature has two types of particles: those for which a $2\pi$ rotation is the identity and those for which a $4\pi$ rotation is the identity.
The Spin-Statistics Theorem
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All this is true in *three* dimensions.

What happens in two dimensions?
Two dimensional physics
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Such entities are called anyons.
What happened to the Spin-Statistics theorem?

It still holds in two dimensions! The relevant group is no longer the permutation group but the braid group.

To understand why we need to think about the physics of two dimensional entities.

In the laboratory we get 2D physics with a thin gas of free electrons trapped between two semiconductor layers.

A strong magnetic field is applied in the perpendicular direction confining the “gas” to a 2D layer.

Excited states of this system are not electrons but virtual particles with strange properties.
Imagine some (5 in the picture) particles and consider what happens when some of them are exchanged.

Here $1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 3, 4 \mapsto 5$ and $5 \mapsto 2$

In 3D the strands can always be disentangled; the only thing that matters is the start and end point. So we can describe the effect just by giving a permutation.

In 2D the entangling matters. One has to distinguish between different braidings.
Here the permutations are the same but the braiding is different.
The Braid Group

Fix $n$ and consider $n$ points on a line with another $n$ points on a line below. We connect them with strands. The generators of the group are interchanges of adjacent strands.

This is an element of $B_6$.

Much richer theory than the permutation group.
For \( n \) points the generators are \( b_1 \) to \( b_{n-1} \) and their inverses. The generators obey the following equations:

\[
\begin{align*}
    b_i b_j & = b_j b_i \quad \text{for} \quad |i - j| \geq 2 \\
    b_i b_{i+1} b_i & = b_{i+1} b_i b_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1.
\end{align*}
\]

which respectively depicts as:

\[
\begin{align*}
    \cdots & \\
    i & \quad i+1 \\
    \cdots & \\
    j & \quad j+1 \\
    \cdots & = \cdots \\
    i & \quad i+1 \\
    j & \quad j+1 \\
    \cdots & \\
\end{align*}
\]

and

\[
\begin{align*}
    \cdots & \\
    i & \quad i+1 \quad i+2 \\
    \cdots & = \cdots \\
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Generalized Spin-Statistics theorem holds in dimensions 2 and 3.

See the paper by Froelich and Gabbiani: Local Quantum Theory and Braid Group Statistics.

There is a lot more to be said about knots, braids, physics and related things but we need to get on with the main story.
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Consider the exchange process. If we exchange two clusters of \( n \) anyons (of type \( \theta \)) each, we get a phase change of \( n^2 \theta \). Thus we have a particle of type \( n^2 \theta \).
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Thus if we have a cluster of $n$ anyons and another cluster of $m$ anyons (all the basic anyons are type $\theta$) when we combine them we get a cluster of type $(n + m)^2\theta$. 
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\textbf{Not all anyons are so simple!}
Representations of the braid group.
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Physical systems in 2D have to carry representations of the braid group. What do they look like?
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Let us consider 1D representations. A 1D vector space is just a copy of $\mathbb{C}$. So every linear map on $\mathbb{C}$ is just a complex number. So every generator $b_j$ of the braid group looks like $e^{i\theta_j}$ in a 1D rep.
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One of the basic equations in the braid group is:

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b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}
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The Yang-Baxter equation.
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Applying this we get that $e^{i\theta_j + i\theta_{j+1} + i\theta_j} = e^{i\theta_{j+1} + i\theta_j + i\theta_{j+1}}$
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or $\theta_j = \theta_{j+1}$. All the generators of the group produce the same phase shift.
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However, there are more interesting representations.
Non-abelian anyons
Non-abelian anyons

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There are candidates but there are no definite laboratory demonstrations of non-abelian anyons.
Fusing non-abelian anyons

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What happens when we combine anyons of different types? Write \([a, b]\) for the combination of a type-\(a\) anyon and a type-\(b\) anyon.
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What happens when we combine anyons of different types? Write $[a, b]$ for the combination of a type-$a$ anyon and a type-$b$ anyon.

We get general fusion rules of the form $[a, b] = \sum_c N_{ac}^b$; where the $N$s are just natural numbers.
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Thus a rule like \([a, b] = 2a + b + 3c\) means that fusing an \(a\) and a \(b\) produces either an \(a\) – and this can happen in two ways – or a \(b\) or a \(c\), which last can happen in 3 ways.
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It is the space of fusion possibilities that describes the qubits! If \([a, b] = 2c\) we use the 2D fusion space of the resulting \(c\) anyon to encode a qubit.
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How do we describe all this complicated algebra? There are different types of things that combine in non-trivial ways. We have essentially an exotic type theory.
What do we need?
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Friday, July 10, 2009
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To accomodate everything we use what are called *modular tensor categories*. 
An example: Fibonacci anyons

Two basic types: 1 and $\tau$.

$1 \otimes 1 \simeq 1$

$1 \otimes \tau \simeq \tau \otimes 1 \simeq \tau$

$\tau \otimes \tau \simeq 1 \oplus \tau$

Here are the fusion rules.
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$$(\tau \otimes \tau) \otimes \tau \simeq (1 \oplus \tau) \otimes \tau$$
$$\simeq (1 \otimes \tau) \oplus (\tau \otimes \tau)$$
$$\simeq \tau \oplus (1 \oplus \tau)$$
$$\simeq 1 \oplus 2 \cdot \tau.$$
We now pass to the context of finite-dimensional complex vector spaces via the splitting spaces whose basis vectors are dual to the fusion states described above. Consider $\text{Hom}(\tau, (\tau \otimes \tau) \otimes \tau) \cong \text{Hom}(\tau, \mathbf{1} \oplus 2 \cdot \tau) \cong \text{Hom}(\tau, \mathbf{1}) \oplus 2 \cdot \text{Hom}(\tau, \tau)$. As $2 \cdot \tau := \tau \oplus \tau$ this is $\cong \text{Hom}(\tau, \mathbf{1}) \oplus 2 \cdot \text{Hom}(\tau, \tau)$. Now, using Lemma 1 in conjunction with the property that for any $b \in \{1, \tau\}$, $\text{End}(b) \cong \mathbb{C}$; if we set $b = 1$ the last expression is isomorphic to $\mathbb{C} \oplus 2 \cdot 0$. Conversely if $b = \tau$, then it is isomorphic to $0 \oplus 2 \cdot \mathbb{C}$. From this, we conclude that considering the space of states with global charge $b \in \{1, \tau\}$ is the same as considering $\text{Hom}(\tau, (\tau \otimes \tau) \otimes \tau)$. In its turn, such a consideration fixes either of the splitting spaces $\mathbb{C}$ or $2 \cdot \mathbb{C} = \mathbb{C}^2$ as orthogonal subspaces of $\mathbb{C}^3$, the topological space representing our triple of anyons. It is within this two-dimensional complex vector space that we will simulate our qubit. Indeed, if $b = \tau$, we are left with two degrees of freedom which are the two possible outputs of the second splitting.

Remark 8. It is worth stressing that it takes three anyons of charge $\tau$ to simulate a single qubit. Moreover, we shall see later that braiding these anyons together simulates a unitary transformation on such a simulated qubit.

Remark 9. Since $\text{Fib}$ is rigid, we can apply Proposition 1. We have $\text{Hom}(\tau, (\tau \otimes \tau) \otimes \tau) \cong \text{Hom}(\mathbf{1} \otimes \tau, (\tau \otimes \tau) \otimes \tau) \cong \text{Hom}(\mathbf{1}, ((\tau \otimes \tau) \otimes \tau) \otimes \tau)$. Comparing this fact with what we got in Example [example], we see that the two different encodings are essentially the same. It is also because of this, some authors, for instance J. Preskill in [31], prefer to encode their qubit within a quadruple of anyons of individual charge $\tau$ with global charge $1$ instead. We choose the former to align with the work of Bonesteel et al. [7] that we will explain in section 6.
The basic idea to simulate quantum computation with anyons is given by the following steps:

1. Consider a compound system of anyons. We initialise a state in the splitting space by fixing the charges the subsets of anyons according to the way they will fuse. This determines the basis state in which the computation starts.
2. We braid the anyons together, it will induce a unitary action on the chosen splitting space.
3. Finally, we let the anyon fuse together and the way they fuse determines which state is measured and this constitutes the output of our computation.
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In fact it is possible to show that the Fibonacci anyons are universal for quantum computation.
Simulating qubits
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If we fuse 3 $\tau$s together we get one two-dimensional space of possible $\tau$ results and we can label the basis vectors as:

$|((\tau \otimes \tau) \otimes \tau; \tau, 1)\rangle$ and $|((\tau \otimes \tau) \otimes \tau; \tau, 2)\rangle$. 

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The two-dimensional space of fusion outcomes is our qubit.
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The one-dimensional space represents possible “leakage.”
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We are almost there, but we need at least one *two*-qubit gate.
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How do they come up with this? By being clever!
A categorical presentation of quantum computation with anyons

As an action on the fusion space of the three anyons involved, this is:

\[
B_3 R^{-2} B^{-4} R_2 B_4 R_2 B_2 R^{-2} B^{-4} R^{-4} B_2 R_2 B_2 R_3 \sim \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & i & 0 & 0 \\
i & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

This tells us how the given combination of braid insert an anyon within a triplet without disturbing it. In fact, this stresses the distinction between the dynamics of the anyons and the consequences on the fusion space. Indeed, even if we disturbed the initial configuration of anyons via multiple braidings, the effect on the splitting space is approximately the identity.

b) Now, we implement an \textit{i·NOT} as the following braid:

The unitary acting on the splitting space of the initial triple is given by:

\[
R^{-2} B^{-4} R_4 B_2 R_2 B_2 R_2 R_2 B^{-2} R^{-2} B_4 R_2 B_2 R_3 \sim \begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

This combination of braids tells us how to implement a \textit{i·NOT} gate on the two dimensional fusion space of our triple of anyons. Again, this gate is approximated.

c) Finally, the \textit{i·CNOT} gate acting on two topological qubits is realised as follows:

First, instead of inserting 1 anyons, we insert a couple that will be used as a test couple and that in the very same manner as described in a) – as these two will fuse together yielding either 1 or \(\tau\), this is exactly what we want.

Secondly, we apply the \textit{i·NOT}-gate computed in b). Finally, we extract the control pair returning it to its original position by applying the insertion procedure in reverse order. This is done, again, without disturbing the triple at stance here.

The above shows the general scheme.

A \textbf{NOT} can be implemented as a one-qubit unitary. We insert a \textit{pair} of test anyons. They fuse to produce a \(\tau\) or a 1.

If the fusion produces a 1 then any tensoring with the other anyons has no effect. If it produces a \(\tau\) the \textbf{NOT} will have an effect. At the end we restore the state of the control triplet.

Details are admittedly hairy and formalizing all this is daunting.
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Tremendously exciting synergy between the three communities.
Some references

J. Preskill, Lectures notes in quantum computation, chapter 9. Available at http://www.theory.caltech.edu/people/preskill/ph229

