Markov Processes as Function Transformers

Prakash Panangaden[1]

[1]School of Computer Science
McGill University

Dagstuhl Oct 2012

Outline

1. Introduction

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- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

Formal Definition of LMPs

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- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure
  and
  $\forall A : \Sigma.\lambda s : S.\tau_\alpha(s, A)$ is a measurable function.

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$$\exists A \in \Sigma.(\forall s' \in A.s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

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- Here one is thinking of a "predicate" as simply a subset of S, but such a subset can be described by a logical formula.

Logic and Probability

Classical logic	Generalization
Truth values $\{0, 1\}$	Probabilities $[0, 1]$
Predicate	Random variable
State	Distribution
The satisfaction relation $\models$	Integration $\int$

The “predicate transformer” view of Markov processes

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- $F_\tau(\mu)(D \in \Lambda) = \int_X \tau(x, D)\mathrm{d}\mu(x)$.
- If μ is the measure representing the “current” distribution on X then after a τ-step, $F_\tau(\mu)$ is the distribution on Y.

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- Because of our backward view, bisimulation becomes a cospan instead of a span. But this actually makes everything easier!
- We can develop a theory of bisimulation, logical characterization, approximation and minimal realization in this framework.

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- These are all Banach spaces.

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- We will switch to a cone view and the situation will be much improved.

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- Unfortunately for us, many of the structures that we want to look at are cones but are not part of any obvious vector space: *e.g.* the measures on a space.
- We could artificially embed them in a vector space, for example, by introducing signed measures.

Abstract cones d'après Selinger

Definition of Cones

A **cone** is a commutative monoid $(V, +, 0)$ with an action of $\mathbb{R}^{\geq 0}$. Multiplication by reals distributes over addition and the following cancellation law holds:

$$\forall u, v, w \in V, v + u = w + u \Rightarrow v = w.$$

The following strictness property also holds:

$$v + w = 0 \Rightarrow v = w = 0.$$

Note that every cone comes with a natural order.

An order on a cone

If $u, v \in V$, a cone, one says $u \leq v$ if and only if there is an element $w \in V$ such that $u + w = v$.

Normed cones

Definition of a normed cone

A **normed cone** C is a cone with a function $||\cdot|| : C \longrightarrow \mathbb{R}^{\geq 0}$ satisfying the usual conditions:
$||v|| = 0$ if and only if $v = 0$
$\forall r \in \mathbb{R}^{\geq 0}, v \in C, ||r \cdot v|| = r||v||$
$||u + v|| \leq ||u|| + ||v||$
$u \leq v \Rightarrow ||u|| \leq ||v||$.

Normally one uses norms to talk about convergence of Cauchy sequences. But without negation how can we talk about Cuchy sequences?

Completeness

However, order-theoretic concepts can be used instead.

Complete normed cones

An ω-**complete normed cone** is a normed cone such that if $\{a_i \mid i \in I\}$ is an increasing sequence with $\{||a_i||\}$ bounded then the lub $\bigvee_{i \in I} a_i$ exists and $\bigvee_{i \in I} ||a_i|| = ||\bigvee_{i \in I} a_i||$.

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Suppose that u_i is an ω-chain with a l.u.b. in an ω-complete normed cone and u is an upper bound of the u_i. Suppose furthermore that $\lim_{i \longrightarrow \infty} \|u - u_i\| = 0$. Then $u = \bigvee_i u_i$.

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Here we are writing $u - u_i$ informally

We really mean w_i where $u_i + w_i = u$.

Maps between cones

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Bounded maps

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Norm of a bounded map

The norm of a bounded linear map $f : C \longrightarrow D$ is defined as $||f|| = \sup\{||f(u)|| \, : \, u \in C, \, ||u|| \leq 1\}$.

A category of normed cones

The ambient category

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The subcategory of interest

we define the subcategory $\omega\mathbf{CC}_1$: the norms of the maps are all bounded by 1. Isomorphisms in this category are always isometries.

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Basic facts

C^* is an ω-complete normed cone as well, and the cone order corresponds to the point wise order.

The duality functor

In ω**CC**, the dual operation becomes a contravariant functor.
If $f : C \longrightarrow D$ is a map of cones, we define $f^* : D^* \longrightarrow C^*$ as follows: given a map L in D^*, we define a map f^*L in C^* as $f^*L(u) = L(f(u))$.

How does this compare with Banach spaces?

This dual is stronger than the dual in usual Banach spaces, where we only require the maps to be bounded. For instance, it turns out that the dual to $L_\infty^+(X)$ (to be defined later) is isomorphic to $L_1^+(X)$, which is not the case with the Banach space $L_\infty(X)$.

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- The cones $\mathcal{M}^p_{\mathsf{UB}}(X)$ and $L_\infty^+(X, \Sigma, p)$ are isomorphic.

Duality for cones

A Reisz-like theorem

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Corollary

Since $\mathcal{M}^{\ll p}(X)$ is isometrically isomorphic to $L_1^+(X)$, an immediate corollary is that $L_\infty^{+,*}(X)$ is isometrically isomorphic to $L_1^+(X)$, which is of course false in general in the context of Banach spaces.

Duality for cones II

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Corollary

$\mathcal{M}^p_{\mathrm{UB}}(X)$ is isometrically isomorphic to $L_\infty^+(X)$, hence immediate corollary is that $L_1^{+,*}(X)$ is isometrically isomorphic to $L_\infty^+(X)$.

The pairing

Pairing function

There is a map from the product of the cones $L_\infty^+(X,p)$ and $L_1^+(X,p)$ to $\mathbb{R}^+$ defined as follows:

$$\forall f \in L_\infty^+(X,p), g \in L_1^+(X,p) \quad \langle f,\ g\rangle = \int fg\mathrm{d}p.$$

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$$\forall f \in L^+_\infty(X,p), g \in L^+_1(X,p) \quad \langle f,\ g\rangle = \int fg\mathrm{d}p.$$

This map is bilinear and is continuous and ω-continuous in both arguments; we refer to it as the pairing.

Duality expressed via pairing

This pairing allows one to express the dualities in a very convenient way. For example, the isomorphism between $L_\infty^+(X, p)$ and $L_1^+(X, p)$ sends $f \in L_\infty^+(X, p)$ to $\lambda g.\langle f,\ g\rangle = \lambda g. \int fg\mathrm{d}p$.

Summary of cones

We fix a probability triple (X, Σ, p) and focus on six spaces of cones that are based on them. They break into two natural groups of three isomorphic spaces. The first three spaces are:

A1 $\mathcal{M}^{\ll p}(X)$ - the cone of all measures on (X, Σ, p) that are absolutely continuous with respect to p,

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- A3 $L_\infty^{+,*}(X, p)$ - the dual cone of the the cone of almost-everywhere positive bounded measurable functions.

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B3 $L^{+,*}_1(X,p)$ - the dual of the cone of almost-everywhere positive functions in the normed vector space $L_1(X,p)$.

Summary of dualities and isos

The spaces defined in A1, A2 and A3 are dual to the spaces defined in B1, B2 and B3 respectively. The situation may be depicted in the diagram

$$\begin{array}{ccccc}
\mathcal{M}^{\ll p}(X) & \xleftrightarrow{\sim} & L_1^+(X,p) & \xleftrightarrow{\sim} & L_\infty^{+,*}(X,p) \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathcal{M}^p_{\mathsf{UB}} & \xleftrightarrow{\sim} & L_\infty^+(X,p) & \xleftrightarrow{\sim} & L_1^{+,*}(X,p)
\end{array} \qquad (1)$$

where the vertical arrows represent dualities and the horizontal arrows represent isomorphisms.

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Duality is the Key

$$\begin{array}{ccccc} \mathcal{M}^{\ll p}(X) & \xleftrightarrow{\sim} & L_1^+(X,p) & \xleftrightarrow{\sim} & L_\infty^{+,*}(X,p) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{M}_{\text{UB}}^p & \xleftrightarrow{\sim} & L_\infty^+(X,p) & \xleftrightarrow{\sim} & L_1^{+,*}(X,p) \end{array} \quad (2)$$

where the vertical arrows represent dualities and the horizontal arrows represent isomorphisms.

Pairing function

There is a map from the product of the cones $L_\infty^+(X,p)$ and $L_1^+(X,p)$ to $\mathbb{R}^+$ defined as follows:

$$\forall f \in L_\infty^+(X,p), g \in L_1^+(X,p) \quad \langle f,\ g\rangle = \int fg\mathrm{d}p.$$

Some notation

1. Given (X, Σ, p) and (Y, Λ) and a measurable function $f : X \longrightarrow Y$ we obtain a measure q on Y by $q(B) = p(f^{-1}(B))$. This is written $M_f(p)$ and is called the *image measure* of p under f.

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2. We say that a measure ν is **absolutely continuous** with respect to another measure μ if for any measurable set A, $\mu(A) = 0$ implies that $\nu(A) = 0$. We write $\nu \ll \mu$.

The Radon-Nikodym Theorem

The Radon-Nikodym theorem is a central result in measure theory allowing one to define a “derivative” of a measure with respect to another measure.

Radon-Nikodym

If $\nu \ll \mu$, where ν, μ are finite measures on a measurable space (X, Σ) there is a positive measurable function h on X such that for every measurable set B

$$\nu(B) = \int_B h \, \mathrm{d}\mu.$$

The function h is defined uniquely up to a set of μ-measure 0. The function h is called the Radon-Nikodym derivative of ν with respect to μ; we denote it by $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Since ν is finite, $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L_1^+(X, \mu)$.

Notation for Radon-Nikodym

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   - given $f \in L_1^+(X,p)$, $\frac{\mathrm{d}f \cdot p}{\mathrm{d}p} = f$
3. These two identities just say that the operations $(-) \cdot p$ and $\frac{\mathrm{d}(-)}{\mathrm{d}p}$ are inverses of each other as maps between $L_1^+(X,p)$ and $\mathcal{M}^{\ll p}(X)$ the space of finite measures on X that are absolutely continuous with respect to p.

Expectation and conditional expectation

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5. Now she can recompute the expectation values given this information.

Where the action happens

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- This will allow for L_∞ and L_1 versions of the theory.
- Going between these versions by duality will be very useful.

The "infinity" category

$\mathbf{Rad}_\infty$

The category $\mathbf{Rad}_\infty$ has as objects probability spaces, and as arrows $\alpha : (X,p) \longrightarrow (Y,q)$, measurable maps such that $M_\alpha(p) \leq Kq$ for some real number K.

The reason for choosing the name $\mathbf{Rad}_\infty$ is that $\alpha \in \mathbf{Rad}_\infty$ maps to $d/dqM_\alpha(p) \in L^+_\infty(Y,q)$.

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2. The fact that the category $\mathbf{Rad}_\infty$ embeds in $\mathbf{Rad}_1$ reflects the fact that L_∞^+ embeds in L_1^+.

Pairing function revisited

Recall the isomorphism between $L_\infty^+(X,p)$ and $L_1^{+,*}(X,p)$ mediated by the pairing function:

$$f \in L_\infty^+(X,p) \mapsto \lambda g : L_1^+(X,p).\langle f,\ g\rangle = \int fg\mathrm{d}p.$$

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Expectation value functor

- The **functor** $\mathbb{E}_\infty(\cdot)$ is a functor from $\mathbf{Rad}_\infty$ to ω**CC** which, on objects, maps (X, p) to $L^+_\infty(X, p)$ and on maps is given as follows:

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- Given $\alpha : (X, p) \longrightarrow (Y, q)$ in $\mathbf{Rad}_\infty$ the action of the functor is to produce the map $\mathbb{E}_\infty(\alpha) : L_\infty^+(X, p) \longrightarrow L_\infty^+(Y, q)$ obtained by composing $(P_1(\alpha))^*$ with the isomorphisms between $L_1^{+,*}$ and L_∞^+

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$$\begin{array}{ccc} L_1^{+,*}(X,p) & \longleftarrow & L_\infty^+(X,p) \\ {\scriptstyle (P_1(\alpha))^*}\downarrow & & \downarrow{\scriptstyle \mathbb{E}_\infty(\alpha)} \\ L_1^{+,*}(Y,q) & \longrightarrow & L_\infty^+(Y,q) \end{array}$$

Markov kernels as linear maps

1. Given τ a Markov kernel from (X, Σ) to (Y, Λ), we define $T_\tau : \mathcal{L}^+(Y) \longrightarrow \mathcal{L}^+(X)$, for $f \in \mathcal{L}^+(Y)$, $x \in X$, as $T_\tau(f)(x) = \int_Y f(z)\tau(x, dz)$.

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3. If we write $\mathbf{1}_B$ for the indicator function of the measurable set B we have that $T_\tau(\mathbf{1}_B)(x) = \tau(x, B)$.
4. It encodes all the transition probability information

From linear maps to markov kernels

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2. The interpretation of L is that $L(\mathbf{1}_B)$ is a measurable function on X such that $L(\mathbf{1}_B)(x)$ is the probability of jumping from x to B.

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3. It is easy to show that this map is linear and ω-continuous.

What do they mean?

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2. The operator T_τ may be interpreted as a likelihood transformer which propagates information "backwards", just as we expect from predicate transformers.
3. $T_\tau(f)(x)$ is just the expected value of f after one τ-step given that one is at x.

Labelled abstract Markov processes

The definition

An **abstract Markov kernel** from (X, Σ, p) to (Y, Λ, q) is an ω-continuous linear map $\tau : L^+_\infty(Y) \longrightarrow L^+_\infty(X)$ with $\|\tau\| \leq 1$.

Labelled abstract Markov processes

The definition

An **abstract Markov kernel** from (X, Σ, p) to (Y, Λ, q) is an ω-continuous linear map $\tau : L_\infty^+(Y) \longrightarrow L_\infty^+(X)$ with $\|\tau\| \leq 1$.

LAMPS

A **labelled abstract Markov process** on a probability space (X, Σ, p) with a set of labels (or actions) $\mathcal{A}$ is a family of abstract Markov kernels $\tau_a : L_\infty^+(X, p) \longrightarrow L_\infty^+(X, p)$ indexed by elements a of $\mathcal{A}$.

The approximation map

The expectation value functors project a probability space onto another one with a possibly coarser σ-algebra.
Given an AMP on (X, p) and a map $\alpha : (X, p) \longrightarrow (Y, q)$ in $\mathbf{Rad}_\infty$, we have the following approximation scheme:

Approximation scheme

$$
\begin{array}{ccc}
L^+_\infty(X,p) & \xrightarrow{\tau_a} & L^+_\infty(X,p) \\
P_\infty(\alpha) \big\uparrow & & \big\downarrow \mathbb{E}_\infty(\alpha) \\
L^+_\infty(Y,q) & \overset{\alpha(\tau_a)}{\dashrightarrow} & L^+_\infty(Y,q)
\end{array}
$$

A special case

- Take (X, Σ) and (X, Λ) with $\lambda \subset \Sigma$ and use the measurable function $id : (X, \Sigma) \longrightarrow (X, \Lambda)$ as α.

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Coarsening the σ-algebra

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- Thus $id(\tau_a)$ is the approximation of τ_a obtained by averaging over the sets of the coarser σ-algebra Λ.
- We now have the machinery to consider approximating along arbitrary maps α.

Bisimulation traditionally

Larsen-Skou definition

Given an $\mathbf{LMP}$ (S, Σ, τ_a) an equivalence relation R on S is called a *probabilistic bisimulation* if sRt then for every *measurable* R-closed set C we have for every a

$$\tau_a(s, C) = \tau_a(t, C).$$

This variation to the continuous case is due to Josée Desharnais and her Indian friends.

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Given a LMP (X, Σ, τ_a), an **event-bisimulation** is a sub-σ-algebra Λ of Σ such that (X, Λ, τ_a) is still an LMP.

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- This means τ_a sends the subspace $L^+_\infty(X, \Lambda, p)$ to itself; where we are now viewing τ_a as a map on $L^+_\infty(X, \Lambda, p)$.

The bisimulation diagram

$$
\begin{array}{ccc}
L^+_\infty(X,\Sigma,p) & \xrightarrow{\tau_a} & L^+_\infty(X,\Sigma,p) \\
\big\uparrow & & \big\uparrow \\
L^+_\infty(X,\Lambda,p) & \xrightarrow{\tau_a} & L^+_\infty(X,\Lambda,p)
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$$

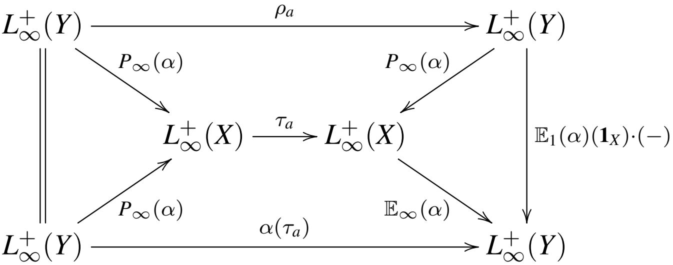
Zigzag maps

We can generalize the notion of event bisimulation by using maps other than the identity map on the underlying sets. This would be a map α from (X, Σ, p) to (Y, Λ, q), equipped with LMPs τ_a and ρ_a respectively, such that the following commutes:

$$\begin{array}{ccc} L^+_\infty(X,\Sigma,p) & \xrightarrow{\tau_a} & L^+_\infty(X,\Sigma,p) \\ {\scriptstyle P_\infty(\alpha)}\uparrow & & \uparrow{\scriptstyle P_\infty(\alpha)} \\ L^+_\infty(Y,\Lambda,q) & \xrightarrow{\rho_a} & L^+_\infty(Y,\Lambda,q) \end{array} \tag{3}$$

A key diagram

When we have a zigzag the following diagram commutes:



(4)

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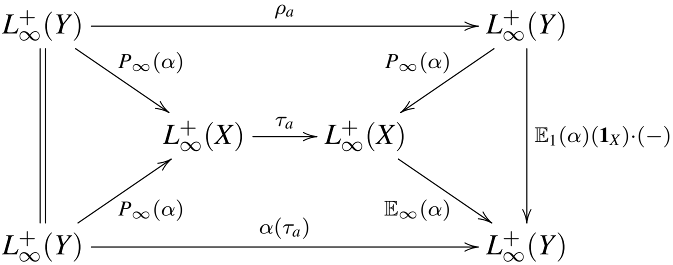

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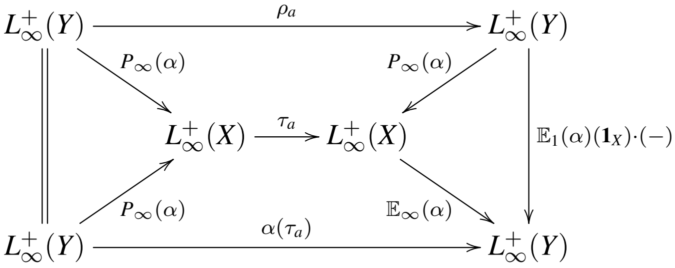

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- The upper trapezium says we have a zigzag. The lower trapezium says that we have an “approximation” and the triangle on the right is an earlier lemma.
- If we “approximate” along a zigzag we actually get the exact result.
- Approximations are approximate bisimulations.

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- With spans one can prove logical characterization of bisimulation on analytic spaces.
- With the cospan definition we get logical characterization on *all* measurable spaces.
- On analytic spaces the two concepts co-incide.
- Recent results show that the theory cannot be made to work with spans on general measurable spaces.

The official definition of bisimulation

Bisimulation

We say that two objects of $\mathbf{AMP}$, (X, Σ, p, τ) and (Y, Λ, q, ρ), are *bisimilar* if there is a third object (Z, Γ, r, π) with a pair of zigzags

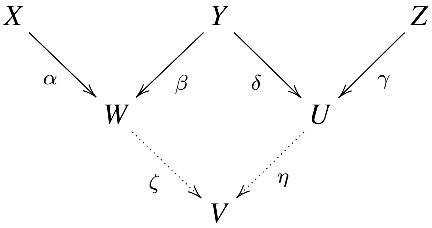
$$\alpha : (X, \Sigma, p, \tau) \longrightarrow (Z, \Gamma, r, \pi)$$
$$\beta : (Y, \Lambda, q, \rho) \longrightarrow (Z, \Gamma, r, \pi)$$

giving a cospan diagram

$$\begin{array}{ccc} (X, \Sigma, p, \tau) & & (Y, \Lambda, q, \rho) \\ & \searrow^{\alpha} \quad \swarrow_{\beta} & \\ & (Z, \Gamma, r, \pi) & \end{array} \tag{5}$$

Note that the identity function on an AMP is a zigzag, and thus that any zigzag between two AMPs X and Y implies that they are bisimilar.

Bisimulation is an equivalence



(6)

The pushouts of the zigzags β and δ yield two more zigzags ζ and η (and the pushout object V). As the composition of two zigzags is a zigzag, X and Z are bisimilar. Thus bisimulation is transitive.

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- We can construct a unique minimal (couniversal property) version of any LAMP.
- We can construct approximations using expectation values to project a LAMP onto a *finite* sub-σ-algebra.
- We can show that the projective limit of the finite approximations give the minimal representation.