A Logical Characterization of Probabilistic Bisimulation

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- for probabilistic systems.
- In the last few weeks: Logical characterization for simulation in systems with countably many transitions; game characterization of bisimulation.

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- their logic had some negative constructs.

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- Stone duality for Markov processes [LICS 2013]
- Dualized view of Markov processes [JACM 2014]
- Simulation, games and continuous action spaces. [in preparation]

Collaborators

Josée Desharnais, Abbas Edalat, Vineet Gupta, Radha Jagadeesan, Vincent Danos, Philippe Chaput, Gordon Plotkin, Françis Laviolette, Norm Ferns, Doina Precup, Chris Hundt, Sherry Ruan, Gheorghe Comanici, Radu Mardare, Dexter Kozen, Kim Larsen, Bartek Klin, Nathanaël Fijalkow.

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- a set of *labels* or *actions*, L or $\mathcal{A}$ and
- a transition relation $\subseteq S \times \mathcal{A} \times S$, usually written

$$\rightarrow_a \subseteq S \times S.$$

The transitions could be indeterminate (nondeterministic).

Markov Chains

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- This is what allows the probabilistic data to be given as a single matrix T.

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- $$(S, \mathsf{L}, \forall a \in \mathsf{L}\ T_a : S \times S \longrightarrow [0,1])$$
- The model is *reactive*: All probabilistic data is *internal* - no probabilities associated with environment behaviour.

Examples of PTSs

$$
\begin{tikzcd}
 & s_0 \arrow[dl, "a[\frac{1}{4}]"'] \arrow[dr, "a[\frac{3}{4}]"] & \\
s_1 & & s_2 \arrow[d, "a[1]"] \\
 & & s_3
\end{tikzcd}
$$

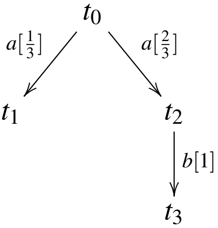
A_1

$$
\begin{tikzcd}
 & s_0 \arrow[dl, "a[1]"'] \arrow[dr, "b[1]"] & \\
s_1 \arrow[d, "c[\frac{1}{2}]"'] \arrow[drr, "c[\frac{1}{2}]"] & & s_2 \arrow[d, "a[1]"] \\
s_4 & & s_3
\end{tikzcd}
$$

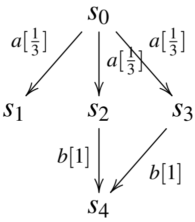
A_2

Bisimulation for PTS: Larsen and Skou

- Consider



P_1



P_2

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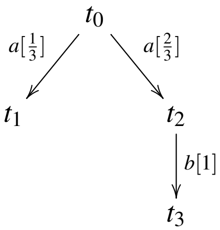

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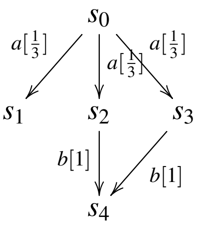

P_2

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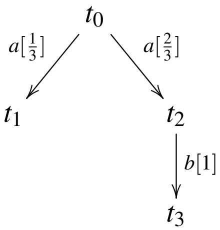

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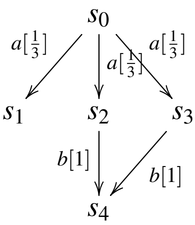

P_2

- Should s_0 and t_0 be bisimilar?
- Yes, but we need to add the probabilities.

The Official Definition

- Let $\mathcal{S} = (S, \mathsf{L}, T_a)$ be a PTS. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-equivalence class, A, $T_a(s, A) = T_a(s', A)$.

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- The notation $T_a(s, A)$ means "the probability of starting from s and jumping to a state in the set A."
- Two states are bisimilar if there is some bisimulation relation R relating them.

What are labelled Markov processes?

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- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

Motivation

Model and reason about systems with *continuous* state spaces or continuous time evolution or both.

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- Telecommunication systems with spatial variation; e.g. cell phones.
- Performance modelling.
- Continuous time systems.
- Probabilistic programming languages with recursion or iteration.

Markov Kernels

- A *Markov kernel* is a function $h : S \times \Sigma \longrightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \longrightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : S \longrightarrow [0, 1]$ a measurable function.

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- Though apparantly asymmetric, these are the stochastic analogues of binary relations
- and the uncountable generalization of a matrix.

Formal Definition of LMPs

- An LMP is a tuple $(S, \Sigma, \mathsf{L}, \forall \alpha \in \mathsf{L}.\tau_\alpha)$ where $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a *transition probability* function such that

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- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure
  and
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Larsen-Skou Bisimulation

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Let $\mathcal{S} = (S, i, \Sigma, \tau)$ be a labelled Markov process. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-closed **measurable** set $A \in \Sigma$, $\tau_a(s, A) = \tau_a(s', A)$.

Two states are bisimilar if they are related by a bisimulation relation.

Logical Characterization

The logic

$$\mathcal{L} ::== \mathsf{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$

We say $s \models \langle a \rangle_q \phi$ iff

$$\exists A \in \Sigma . (\forall s' \in A . s' \models \phi) \wedge (\tau_a(s, A) > q).$$

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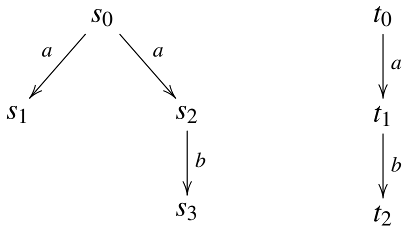
The main theorem

Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

That cannot be right?



Two processes that cannot be distinguished without negation.
The formula that distinguishes them is $\langle a \rangle (\neg \langle b \rangle \top)$.

But it is!

- $$\begin{tikzcd} & s_0 \arrow[dl, "{a[p]}"'] \arrow[dr, "{a[q]}"] & \\ s_1 & & s_2 \arrow[d, "b"] \\ & & s_3 \end{tikzcd} \qquad \begin{tikzcd} t_0 \arrow[d, "{a[r]}"] \\ t_1 \arrow[d, "b"] \\ t_2 \end{tikzcd}$$

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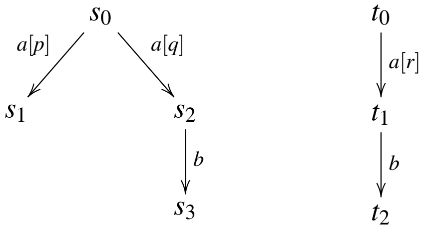
- 

  We add probabilities to the transitions.
- If $p+q<r$ or $p+q>r$ we can easily distinguish them.
- If $p+q=r$ and $p>0$ then $q<r$ so $\langle a\rangle_r\langle b\rangle_1\top$ distinguishes them.

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- Can easily show that $\tau_a(s, A) = \tau_a(s', A)$ for A of the form $[\![\phi]\!]$.
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- Use Dynkin's $\lambda - \pi$ theorem to show that we get a well defined measure on the σ-algebra generated by such sets and the above equality holds.
- Use special properties of analytic spaces to show that this σ-algebra is the same as the original σ-algebra.

The Easy Direction

- Let R be a bisimulation relation on an LMP (S, Σ, τ_a). We prove by induction on ϕ that $\forall \phi \in \mathcal{L}$

$$\forall s, s' \in S. sRs' \Rightarrow s \models \phi \Leftrightarrow s' \models \phi.$$

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- Base case trivial.
- $\wedge$ is obvious from Inductive Hypothesis.
- For $\phi = \langle a \rangle_q \psi$ we have that $[\![\psi]\!]$ is R-closed from inductive hypothesis. Thus

$$\tau_a(s, [\![\psi]\!]) = \tau_a(s', [\![\psi]\!])$$

  and thus $sRs' \Rightarrow s \models \phi \Leftrightarrow s' \models \phi$.

Digression on Analytic Spaces

- An analytic set A is the image of a Polish space X (or a Borel subset of X) under a continuous (or measurable) function $f : X \to Y$, where Y is Polish. If (S, Σ) is a measurable space where S is an analytic set in some ambient topological space and Σ is the Borel σ-algebra on S.

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- Analytic sets do not form a σ-algebra but they are in the completion of the Borel algebra under **any** measure. [Universally measurable.]
- Regular conditional probability densities (disintegrations) can be defined on analytic spaces.

Amazing Facts about Analytic Spaces

- Given A an analytic space and $\sim$ an equivalence relation such that there is a *countable* family of real-valued measurable functions $f_i : S \longrightarrow \mathbf{R}$ such that

$$\forall s, s' \in S.s \sim s' \iff \forall f_i.f_i(s) = f_i(s')$$

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- If an analytic space (S, Σ) has a sub-σ-algebra Σ_0 of Σ which separates points and is countably generated then Σ_0 is Σ! The Unique Structure Theorem (UST).

The big picture

1. We have LMP $(S, \Sigma, \mathsf{L}, \tau_a)$ and we want to quotient by $\simeq$ where $s \simeq s'$ if they agree on all formulas of the logic.

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4. In lieu of an answer: maps between LMP's satisfying the above condition are called "zigzags" and bisimulation can be defined as the existence of a span of zigzags.

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- Easy to check that $q^{-1}(q([\![\phi]\!])) = [\![\phi]\!]$:

  $s \in q^{-1}(q([\![\phi]\!]))$ implies that $q(s) \in q([\![\phi]\!])$, i.e. $\exists s' \in [\![\phi]\!] . s \simeq s'$, so $s \models \phi$ so $s \in [\![\phi]\!]$.

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- Thus the σ-algebra generated -say, Λ - by $q([\![\phi]\!])$ is a sub-σ-algebra of Ω.
- Λ is countably generated and separates points so by UST it *is* Ω. Thus $q([\![\phi]\!])$ generates Ω.

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- Thus $\tau_a(s, q^{-1}(q(\llbracket\phi\rrbracket))) = \tau_a(s', q^{-1}(q(\llbracket\phi\rrbracket)))$ and hence ρ is well defined. We have $\rho_a(q(s), B) = \tau_a(s, q^{-1}(B))$.

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$$\tau_a(s,X) = \tau_a(s,q^{-1}(q(X))) = \rho_a(q(s),q(X)) =$$

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- Spoiler can only win if Duplicator is stuck. For example if C is all of S.
- s and t are bisimilar if and only if Duplicator has a winning strategy.

Simulation

Let $\mathcal{S} = (S, \Sigma, \tau)$ be a labelled Markov process. A preorder R on S is a **simulation** if whenever sRs', we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) \leq \tau_a(s', A)$. We say s is simulated by s' if sRs' for some simulation relation R.

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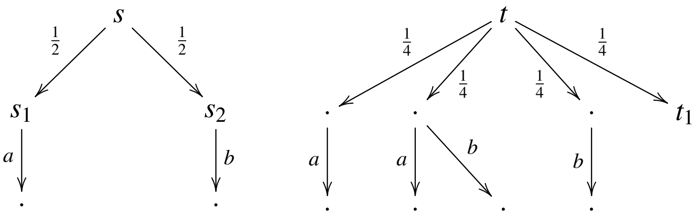
- One can show that if s simulates s' then s satisfies all the formulas of $\mathcal{L}$ that s' satisfies.
- What about the converse?

Counter example!

In the following picture, t satisfies all formulas of $\mathcal{L}$ that s satisfies but t does not simulate s.



All transitions from s and t are labelled by a.

Counter example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a \rangle_0 (\langle a \rangle_0 \mathsf{T} \wedge \langle b \rangle_0 \mathsf{T}).$$

Counter example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a\rangle_0(\langle a\rangle_0\mathsf{T}\wedge\langle b\rangle_0\mathsf{T}).$$

- A formula with disjunction that is satisfied by s but not by t:

$$\langle a\rangle_{\frac{3}{4}}(\langle a\rangle_0\mathsf{T}\vee\langle b\rangle_0\mathsf{T}).$$

A logical characterization for simulation

- The logic $\mathcal{L}$ does **not** characterize simulation. One needs disjunction.

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- New proof, with Nathanaël Fijalkow and Bartek Klin, works with countably many labels and uses topology.

Other Logics

$$\begin{array}{rcl}
\mathcal{L}_{\mathrm{Can}} & := & \mathcal{L}_0 \mid \mathrm{Can}(a) \\
\mathcal{L}_{\Delta} & := & \mathcal{L}_0 \mid \Delta_a \\
\mathcal{L}_{\neg} & := & \mathcal{L}_0 \mid \neg\phi \\
\mathcal{L}_{\vee} & := & \mathcal{L}_0 \mid \phi_1 \vee \phi_2 \\
\mathcal{L}_{\wedge} & := & \mathcal{L}_{\neg} \mid \bigwedge_{i \in \mathbf{N}} \phi_i
\end{array}$$

where

$s \models \mathrm{Can}(a)$ to mean that $\tau_a(s, S) > 0$;
$s \models \Delta_a$ to mean that $\tau_a(s, S) = 0$.

We need $\mathcal{L}_{\vee}$ to characterise simulation.

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- Recently, Fijalkow showed that if there are *uncountably many labels* then the logical characterization of bisimulation fails.
- However, if we introduce a topology on the space of labels and a continuity assumption, we can regain the logical characterization result.