Causality, Order, Information and Topology

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Outline

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Overview

- Causal structure - mathematically modelled as a partial order - can be taken to be the fundamental structure of spacetime.

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- The topology can be derived from this.
- Ordered topological spaces (domains) were used by Dana Scott to model computation as information processing.
- Spacetime carries a natural domain structure.

Background

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- A finite piece of information about the output should only require a finite piece of information about the input.
- This is just what the $\epsilon - \delta$ definition says.
- Data types are domains (ordered topological spaces) and computable functions are continuous.

Summary of Results

- The causal order alone determines the topology of globally hyperbolic spacetimes. [CMP Nov'06]

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- A (globally hyperbolic) spacetime can be given domain structure: approximate points. [CMP Nov'06]
- The space of causal curves in the Vietoris topology is compact (cf. Sorkin-Woolgar) [GRG '06]
- The geometry can be captured by a Martin "measurement." [AMS Symposia in Pure and Appliedd Math 2012]

The layers of spacetime structure

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- Parallel transport: affine structure.
- Lorentzian metric: gives a length scale.

The causal structure of spacetime

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- We assume that spacetime is *time-orientable*: there is a global notion of future and past.
- A *timelike* curve from x to y has a tangent vector that is everywhere timelike: we write $x \preceq y$. (We avoid $x \ll y$ for now.) A *causal* curve has a tangent that, at every point, is either timelike or null: we write $x \leq y$.

Kronheimer - Penrose axioms

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- A fundamental axiom is that $<$ is a partial order.
- Other axioms describe the interaction of $<$ and $\prec$.
- The $\leq$ and $\ll$ orders satisfy all the axioms of a causal space.

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- Chronology: $x \preceq y \Rightarrow y \not\preceq x$.
- Causality: $x \leq y$ and $y \leq x$ implies $x = y$.

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- In such a spacetime a future directed causal curve cannot get trapped in a compact set.
- Stable causality: perturbations of the metric do not cause violations of causality.
- Causal simplicity: for all $x \in M$, $J^{\pm}(x)$ are closed.

Global Hyperbolicity

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- Global hyperbolicity: M is strongly causal and for each p, q in M, $[p, q] := J^{+}(p) \cap J^{-}(q)$ is compact.

The Alexandrov Topology

Define

$$\langle x, y\rangle := I^{+}(x) \cap I^{-}(y).$$

The sets of the form $\langle x, y\rangle$ form a base for a topology on M called the Alexandrov topology.

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Theorem (Penrose): TFAE:

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3. The Alexandrov topology is Hausdorff.

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- For "directed set" think "chain."
- computable functions are viewed as *continuous* with respect to a suitable topology: the Scott topology.
- ideal (infinite) elements are limits of their (finite) approximations.

Examples of domains

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- Sequences of elements from $\{a, b\}$ ordered by prefix: the domain of streams.
- Compact non-empty intervals of real numbers ordered by *reverse* inclusion (with **R** thrown in).
- X a locally compact space with $K(X)$ the collection of compact subsets ordered by reverse inclusion.

The Way-below relation

- In addition to $\leq$ there is an additional, (often) irreflexive, transitive relation written $\ll$: $x \ll y$ means that x has a "finite" piece of information about y or x is a "finite approximation" to y. If $x \ll x$ we say that x is *finite*.

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- The relation $x \ll y$ - pronounced x is “way below” y - is directly defined from $\leq$.
- Official definition of $x \ll y$: If $X \subset D$ is directed and $y \leq (\bigvee^{\uparrow} X)$ then there exists $u \in X$ such that $x \leq u$. If a limit gets past y then some finite stage of the limiting process already got past x.

Domain theory continued

- A continuous domain D has a basis of elements $B \subset D$ such that for every x in D the set $x \Downarrow := \{u \in B | u \ll x\}$ is directed and $\bigvee^{\uparrow} (x \Downarrow) = x$.

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- A *continuous* function between domains is order monotone and preserves lubs (sups) of directed sets.
- Why are directed sets so important? They are collecting consistent pieces of information.
- Surely the words "continuous function" should have something to do with topology?

The dream

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- Find a topology so that Turing computability is precisely continuity.
- Scott's topology comes close.
- All computable functions are Scott continuous but one still needs some recursion theoretic machinery to pin down exactly what computable means.

Topologies of Domains 1: The Scott topology

- the open sets of D are upwards closed and if $\mathcal{O}$ is open, then if $X \subset D$ is directed and $\bigvee^{\uparrow} X \in \mathcal{O}$ it must be the case that some $x \in X$ is in $\mathcal{O}$.

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- The effectively checkable properties.
- This topology is T_0 but not T_1.

Topologies of Domains 2: The Lawson topology

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- This topology is metrizable.
- It has the same Borel algebra as the Scott topology.

Topologies of Domains 3: The interval topology

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- The domain theoretic analogue of the Alexandrov topology.
- Caveat: the "Alexandrov topology" means something else in the theory of topological lattices.

The role of way below in spacetime structure

- **Theorem:** Let (M, g) be a spacetime with Lorentzian signature. Define $x \ll y$ as the way-below relation of the causal order. If (M, g) is globally hyperbolic then $x \ll y$ iff $y \in I^+(x)$.

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- One can recover I from J without knowing what smooth or timelike means.
- Intuition: any way of approaching y must involve getting into the timelike future of x.

We can stop being coy about notational clashes: henceforth $\ll$ is way-below *and* the timelike order.

Bicontinuity and Global Hyperbolicity

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- The definition of continuous domain - or poset - is biased towards approximation from below. If we symmetrize the definitions we get bicontinuity (details in the paper).
- Theorem: If (M, g) is globally hyperbolic then $(M, \leq)$ is a bicontinuous poset. In this case the interval topology is the manifold topology.

An “abstract” version of globally hyperbolic

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1. bicontinuous and,
2. all segments $[a, b] := \{x : a \leq x \leq b\}$ are compact in the interval topology on X.

An Important Example of a Domain: $\mathbf{I}\mathbb{R}$

- The collection of compact intervals of the real line

  $$\mathbf{I}\mathbb{R} = \{[a, b] : a, b \in \mathbb{R} \;\&\; a \leq b\}$$

  ordered under reverse inclusion

  $$[a, b] \sqsubseteq [c, d] \Leftrightarrow [c, d] \subseteq [a, b]$$

  is an ω-continuous dcpo.

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- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigvee^{\uparrow} S = \bigcap S$.

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$\mathbf{I}\mathbb{R}$ continued.

- $I \ll J \Leftrightarrow J \subseteq \mathrm{int}(I)$, and
- $\{[p, q] : p, q \in \mathbb{Q} \ \& \ p \leq q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$.

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- We have $\max(\mathbf{I}\mathbb{R}) \simeq \mathbb{R}$ in the Scott topology.
- The "classical" structure lives on top - ideal points,
- there is now a substrate of "approximate" elements.

Generalizing $\mathbf{I}\mathbb{R}$

- The closed segments of a globally hyperbolic poset X

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- X has a countable basis iff $\mathbf{I}X$ is ω-continuous.

$$\max(\mathbf{I}X) \simeq X$$

where the set of maximal elements has the relative Scott topology from $\mathbf{I}X$.

Spacetime from a discrete ordered set

If we have a countable dense subset $\mathcal{C}$ of $\mathcal{M}$, a globally hyperbolic spacetime, then we can view the induced causal order on C as defining a discrete poset. An ideal completion construction in domain theory, applied to a poset constructed from $\mathcal{C}$ yields a domain $\mathbf{I}C$ with

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements have the Scott topology. Thus from a countable subset of the manifold we can reconstruct the whole manifold.

Globally Hyperbolic Posets and Interval Domains

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- One can define categories of globally hyperbolic posets and an abstract notion of "interval domain": these can also be organized into a category.
- These two categories are equivalent.
- Thus globally hyperbolic spacetimes *are* domains - not just posets - but
- not with the causal order but, rather, with the order coming from the notion of intervals; i.e. from notions of approximation.

Spacetime as a domain

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- For globally hyperbolic spacetimes these are all compact.
- The order is inclusion.
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- The other elements are "approximate points."

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- metric structure as well.
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- Keye Martin defined a concept called a “measurement.” This is designed to capture quantitative notions on domains.
- Metric notions can be related to these measurements.

Keye's measurements

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- We write $ker(\mu)$ for $\{x | \mu(x) = 0\}$ and $\mu_\epsilon(x) = \{y | y \sqsubseteq x \text{ and } |\mu(x) - \mu(y)| \leq \epsilon\}$.
- For any Scott open set U and any $x \in ker(\mu)$

$$x \in U \Rightarrow (\exists \epsilon > 0) x \in \mu_\epsilon \subseteq U.$$

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- Maximal elements have zero uncertainty.
- On a suitable domain of probability distributions Shannon entropy is a measurement.

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- Does the volume of an interval or the length of the longest geodesic give a measurement on the domain of spacetime intervals?
- Unfortunately not! If *a* and *b* are null related then you get a nontrivial interval with zero volume.
- However, any globally hyperbolic spacetime (in fact any stably causal one) has a *global time function.* The difference in the global time function does give a measurement.

Measurements and Geometry

- Does the volume of an interval or the length of the longest geodesic give a measurement on the domain of spacetime intervals?
- Unfortunately not! If *a* and *b* are null related then you get a nontrivial interval with zero volume.
- However, any globally hyperbolic spacetime (in fact any stably causal one) has a *global time function.* The difference in the global time function does give a measurement.
- Knowing the global time function effectively gives the rest of the metric.

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- Can we think of spacetime geometry in terms of its capacity to convey information?