Quantum Field Theory for Leg Spinners 8/11/10

Lecture Notes 1: Geometric mechanics

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In the previous lecture we have rigorously defined smooth manifolds, and parallel transport of tangent vectors on them. So we are now able to speak of things like the flow of a Hamiltonian vector field as consisting of those tangent vectors that are unchanged when transported. The goal of this lecture is to apply all this mathematics to mechanics. Indeed, classical mechanics can be completely formulated in such terms, and hence we speak of "geometric mechanics". This formulation will be the launching pad for us to consider quantization.

1 The cotangent bundle

Geometric mechanics starts from a *configuration space*, taken to be a smooth manifold M. One should think of this as a "snapshot" of the system. Notice that M need not be $\mathbb{R}^n$; one could think of particles constrained to all kinds of configurations, such as pendula swinging along spheres.[1] Configuration space is not a complete description of a mechanical system, since there are no dynamics at all. In fact, the configuration space just contains information about position. In order to describe the dynamics completely we also need the momenta. If we also take momenta into account, we end up with *phase space*, which mathematically corresponds to the so-called cotangent bundle over M. We will now define this rigorously.

First of all, recall that a *vector bundle* over M is an open continuous map $\pi\colon B \to M$ from some topological space B (which is part of the data of a bundle), such that each $x \in M$ has a neighbourhood U for which

$$\pi^{-1}(U) \cong V \times U$$

for some fixed vector space V.

As we saw last lecture, an example of a bundle over M is its *tangent bundle* TM, given by $\pi^{-1}(p) = T_p$, *i.e.*the fibre over a point is the tangent space at that point. One can again define charts and transition functions for TM so that it becomes a smooth manifold in its own right whose dimension is twice that of M.

[1]Indeed, as long as the constraints are non-holonomic, every such physical collection of positions will be a configuration space.

The tangent bundle represents velocities, *i.e.*the $\dot{q}$ in the Lagrangian picture $L(q,\dot{q})$. When switching to Hamiltonian mechanics, one works with the dual bundle – the cotangent bundle – which uses the momenta $p = \frac{\partial L}{\partial \dot{q}}$ instead. To that end, the *cotangent bundle* T_*M is given at each point by the dual space of the tangent space: $\pi^{-1}(q) = T_q^*$. (Since T_q is finite-dimensional, T_q and T_q^* are of course isomorphic, but not naturally so.)

Again, the cotangent bundle T_*M can be made into a manifold in its own right by equipping it with smooth structure, which we now discuss in some detail. Let (U,ψ) be a chart for M. Define a chart (U',ψ') for T_*M as follows:

$$U' = \{(q,p_a) \mid q \in U, p_a \in T_q^*\}.$$

Hence U' is isomorphic to $U \times \mathbb{R}^n$, where n is the dimension of M. The idea for the definition of the chart ψ' is to use the local Euclidean structure provided by ψ and translate the result back to M. If $\psi(q) = (x^{(1)},\ldots,x^{(n)}) \in \mathbb{R}^n$, this gives n real-valued functions $x^{(i)}$ (that vary with the point q). By unwinding definitions, one sees that their gradients $\nabla_a x^{(i)}$ form a basis for T_q^*. Decompose p_q in this basis to get coefficients $\kappa_{(i)}$:

$$p_a = \sum_i \kappa_{(i)} \nabla_a x^{(i)}.$$

We define

$$\psi'(q,p_a) = (\underbrace{x^{(1)},\ldots,x^{(n)}}_{=\psi(q)}, \kappa_{(1)},\ldots,\kappa_{(n)}) \in \mathbb{R}^{2n}.$$

To verify that the transition functions are smooth, and hence that the above indeed defines the data for a ($2n$-dimensional) manifold, is left as an exercise for the conscientious reader.

To distinguish between tensors on the manifold M and tensors on the manifold T_*M, we use greek letters as indices on the cotangent bundle. Hence $\nabla_\alpha, \Omega_{\alpha\beta}, H^\alpha$ etc. denote derivative operators, tensor fields and vector field respectively on the cotangent bundle T_*M, whereas ∇_a, g_{ab} etc. denote derivative operators, tensor fields etc. on the base manifold M.

2 Symplectic forms

It turns out that the cotangent bundle T_*M of any manifold M comes equipped with a "canonical" 2-form. Via this so-called symplectic[2] form we will rediscover Poisson brackets

[2] The name "symplectic" has been introduced by Hermann Weyl in 1939 as the Greek adjective corresponding to the word "complex", which for him referred to the linear line complexes introduced by Plücker that satisfied condition (S1) below, because the word "complex" by that time had got strong connotations with complex numbers.

in this section.

Let (q, p_a) be a point on the cotangent bundle T_*M, and let f be a real-valued function on M such that $\nabla_a f = p_a$. Then the pullback $f \circ \pi \colon T_*M \to \mathbb{R}$ defines a scalar field on T_*M. Let ∇_α be a derivative operator on T_*M. Given all this data, we define a $2n$-dimensional gradient

$$A_\alpha = \nabla_\alpha (f \circ \pi),$$

in T_*M, and intuitively think of it as $p\,\mathrm{d}x$. This is a "natural" co-vector field on T_*M in the sense that it does not depend on the choice of f. In fact, we can eliminate more freedom by defining the 2-form

$$\Omega_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha,$$

which we intuitively think of as the "curl" $\mathrm{d}x \wedge \mathrm{d}p$ of A. It is independent of the choice of ∇_α and ∇_a.

The 2-form Ω satisfies the following three properties:

(S1) It is antisymmetric: $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$.

(S2) It is invertible, in the sense that there is an $\Omega^{\alpha\beta}$ satisfying $\Omega^{\alpha\gamma}\Omega^{\gamma\beta} = \delta^\alpha_\beta$, where the last symbol is the Kronecker delta.

(S3) $\Delta_{[\alpha}\Omega_{\beta\gamma]} = 0$, where the bracketed indices denote complete antisymmetrization. In general,

$$T_{[abc]} = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} T_{\sigma(a)\sigma(b)\sigma(c)},$$

and the antisymmetrization can be taken over indices of multiple tensors, as above.[3]

Any 2-form satisfying these three properties is called a *symplectic form*.

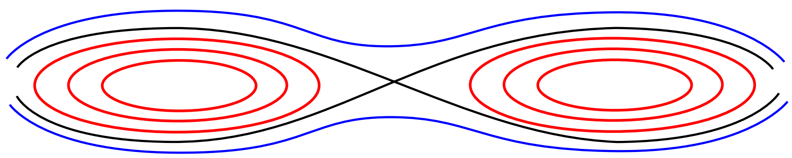
We can now finally define the arena in which dynamics can happen to be a *symplectic manifold*: a smooth manifold equipped with a symplectic form. It follows that such a manifold is automatically even-dimensional. In fact, a lot of mechanics could alternatively be called the geometry of symplectic manifold. A symplectic manifold is one of the two major pieces out of which Hilbert spaces are constructed; the other ingredient will follow in later lectures.

[3]Incidentally, parentheses around indices similarly denote complete symmetrization.

3 Dynamics

As phase space is supposed to give the "kinematics", we can trace every point along its tangent curve, thus extrapolating its "future" and "past". Hence the blank canvas of phase space comes "painted with curves", which are called *dynamical trajectories*. The whole picture is also called the *phase portrait*. The dynamical trajectories of different points cannot intersect, because behaviour is presupposed to be deterministic. Together, the dynamical trajectories must fill the whole of space, since every point lies on a tangent curve.

For example, the phase portrait for the harmonic oscillator looks as follows.



Think of the curves in the above picture as lines traced out by the weight swinging on a pendulum. The red ellipses correspond to states in which the pendulum doesn't go "over the top". The blue curves correspond to states in which the pendulum is swung with so much energy that it does go "over the top" (and hence keeps doing so). The two intermediate black curves corresponds to the state in which the pendulum just fails to reach the top, and then falls back. The apparent "point of intersection" is actually a single trajectory corresponding to a pendulum balanced on the top at rest. Notice that they do not intersect, although they do come infinitesimally close to each other.

The tangent vectors of the family of curves in the phase portrait form a vector field, called *dynamics*: there is a smooth scalar field on T_*M called the *hamiltonian* H, and there is an associated vector field, called the *hamiltonian vector field*, denoted by H^α, given by

$$H^\alpha = \Omega^{\alpha\beta}\nabla_\beta H.$$

The latter assertion follows from the theory of differential equations, that is lifted from $\mathbb{R}^n$ via local charts. The integral curves of H^α give the phase portrait discussed above.

Notice that $H^\alpha\nabla_\alpha H = \Omega^{\alpha\beta}\nabla_\beta H\nabla_\alpha H = 0$ by (S1), so that H is conserved along dynamical trajectories. This is, more rigorously, what we meant by "hamiltonian flow" in the introduction of this lecture.

4 Poisson brackets

We can now define an *observable* as a scalar function on phase space T_*M. Additionally, for two observables A and B, we define their *Poisson bracket* $\{A, B\}$ to be

$$\{A, B\} = \Omega^{\alpha\beta}(\nabla_\alpha A)(\nabla_\beta B).$$

It follows that $\{A, B\}$ satisfies

$$\dot{A} = H^\alpha \nabla_\alpha A = \Omega^{\alpha\beta}\nabla_\beta H \nabla_\alpha A = \{A, H\}.$$

To justify the first equation, recall that $\dot{A}$ is interpreted as meaning how A changes over time. Hence we find that

$$\dot{H} = 0.$$

5 Canonical Transformations

This section was omitted in the actual lecture but was added later by the lecturer.

We need the notion of diffeomorphism of manifolds first. If M, N are smooth manifolds and $\psi : M \to N$ is a smooth bijection such that ψ^{-1} is also smooth, then ψ is called a *diffeomorphism*; it is isomorphism in the category of smooth manifolds and smooth maps. If f is a scalar field (real-valued function) on M we can define a scalar field $\psi(f)$ on N by $\psi(f) = f \circ \psi^{-1}$.[4] We now extend this transfer to any tensor field. First, note that for a vector field ξ^a on M we can get a vector field $\psi(\xi^a)$ on N by $\psi(\xi^a)\nabla_a\psi(f) = \psi(\xi^a\nabla_a f)$. The lhs of this equation defines the vector field $\psi(\xi^a)$ as a directional derivative and the rhs defines it as computing it in M and then transferring to N using ψ. Now we can transfer covector fields by the formula $\psi(\omega_a)\psi(\xi^a) = \psi(\omega_a\xi^a)$. Similarly, for higher-rank tensor fields.

Given a system with configuration space C, phase space Γ, symplectic form $\Omega_{\alpha\beta}$ and hamiltonian H we define a *canonical transformation* ψ to be a diffeomorphism from Γ to itself such that $\psi(\Omega) = \Omega$. Note, that we do not require the preservation of H. A canonical transformation that preserves H is called a *symmetry*. Physicists usually work

[4]Of course, for any smooth map $\phi : M \to N$ we can pull back functions from $N \to \mathbb{R}$ to obtain functions in $M \to \mathbb{R}$. This extends to any tensor fields in the same way as described in the main text

with infinitesimal canonical transformations; in order to define them we introduce the Lie derivative.

Suppose that we have a vector field ξ^a on a manifold M. Consider the family of integral curves defined by this vector field. These curves define a one-parameter family of diffeomorphisms as follows. Intuitively, the idea is that for a value of the parameter t we move every point a parameter distance t along the curve. This family of diffeomorphisms is written ψ_t. Now since we can push tensor fields along diffeomorphisms we can transform any tensor field T (indices omitted) to $\psi_t(T)$. We define the Lie derivative as

$$\pounds_\xi T = \lim_{t\to 0} \frac{\psi_{-t}(T) - T}{t}.$$

From this definition it follows that $\pounds$ is linear and obeys the Leibniz rule and that $\pounds_\xi(f) = \xi(f) = \xi^a \nabla_a f$. In the usual way we can extend the Lie derivative to vector fields, covector fields and higher tensor fields. For example, we have the formulas

$$\pounds_\xi \eta^a = \xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a \text{ and } \pounds_\xi \omega_a = \xi^b \nabla_b \omega_a + \omega_b \nabla_a \xi^b.$$

Now an *infinitesimal canonical transformation* is just a vector field ξ^a such that $\pounds_\xi \Omega_{\alpha\beta} = 0$ and an *infinitesimal symmetry* is an infinitesimal canonical transformation ξ^a such that $\pounds_\xi H = 0$.

6 Conclusions

Geometric mechanics has grown far beyond the setting of mechanics, and has become a beautiful piece of mathematics in its own right. There are many further topics of study, such as canonical transformations, Hamilton-Jacobi theory (leading to Schrödinger's equation), and symmetry (leading to Noether's theorem). For us, the plan for the subsequent lectures is to use the techniques discussed in this lecture as follows:

1. define Ω for the Klein-Gordon field;
2. use Ω (and another ingredient to be detailed later) to define H;
3. use Ω to define an algebra of observables (after Dirac);
4. give a representation of 3 on operators of 2.