Divergence-Least Semantics Of amb Is Hoare

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Abstract This note strengthens the hoary observation that McCarthy's amb is not monotone with respect to the Smyth and Plotkin powerdomains. It shows that there is no least fixpoint semantics for amb that is sensitive to divergence.

This paper is concerned with an *erratic choice* operator $M|M'$, and an *ambiguous choice* operator M amb M'. Recall that $M|M'$ means: either evaluate M or evaluate M'. And M amb M' means: evaluate both M and M' on an arbitrary fair scheduler, and return whatever answer you get first. We defer the study of ambiguous choice until Sect. 2.

1 Erratic Choice

Suppose we have a language $\mathfrak{L}$ containing the following:

- a boolean type bool, equipped with constants t and f, and a conditional operator if M then N else N' at every type
- a natural number type nat, equipped with a constant n for each $n \in \mathbb{N}$, and an equality operator $N = N'$
- a term d (short for diverge) at every type
- an erratic choice operator | at every type

The types bool and nat are called *ground types*. To describe operational semantics, suppose that we have a function $\mathsf{behs}[-]$

- from the set of closed terms of type bool to $\mathcal{P}\{\mathsf{true}, \mathsf{false}, \bot\}$
- from the set of closed terms of type nat to $\mathcal{P}(\mathbb{N} \cup \{\bot\})$

satisfying the following equations:

$$\begin{aligned}
\mathsf{behs}[\mathtt{t}] &= \{\mathsf{true}\} \\
\mathsf{behs}[\mathtt{f}] &= \{\mathsf{false}\} \\
\mathsf{behs}[n] &= \{n\} \\
\mathsf{behs}[\mathtt{d}] &= \{\bot\} \\
\mathsf{behs}[M|N] &= \mathsf{behs}[M] \cup \mathsf{behs}[N] \\
\mathsf{behs}[\mathtt{if}\ M\ \mathtt{then}\ N\ \mathtt{else}\ N'] &= \{x \in \mathsf{behs}[N] \mid \mathsf{true} \in \mathsf{behs}[M]\} \\
&\quad \cup \{x \in \mathsf{behs}[N'] \mid \mathsf{false} \in \mathsf{behs}[M']\} \\
&\quad \cup \{\bot \mid \bot \in \mathsf{behs}[M]\}
\end{aligned}$$

$$\begin{aligned}\mathsf{behs}[M=N] = &\{\mathsf{true} \mid \exists n \in \mathbb{N}.(n \in \mathsf{behs}[M] \wedge n \in \mathsf{behs}[N])\}\\ &\cup\{\mathsf{false} \mid \exists m, n \in \mathbb{N}.(m \neq n \wedge m \in \mathsf{behs}[M] \wedge n \in \mathsf{behs}[N])\}\\ &\cup\{\bot \mid \bot \in \mathsf{behs}[M] \vee \bot \in \mathsf{behs}[N]\}\end{aligned}$$

We write $\mathsf{vals}[N]$ for $\mathsf{behs}[N] \setminus \{\bot\}$, and write $M \Uparrow$ when $\bot \in \mathsf{behs}[M]$. We write $=_{\mathsf{beh}}$ for the kernel of $\mathsf{behs}[-]$.

Some reasonable laws for $\mathfrak{L}$ are shown in Fig. 1, and when we speak of a "denotational semantics", we mean one that validates all these laws. (It is not known whether these laws are complete in any sense.)

Laws of Erratic Choice [Plo83]

$$\begin{aligned} M|M' &\simeq M'|M \\ (M|M')|M'' &\simeq M|(M'|M'') \\ M|M &\simeq M \end{aligned}$$

Laws of Conditionals [Lev04] (Fig. A.8, call-by-name equations)

$$\begin{aligned} \texttt{if t then } M \texttt{ else } M' &\simeq M \\ \texttt{if f then } M \texttt{ else } M' &\simeq M' \\ \texttt{if } N \texttt{ then t else f} &\simeq N \\ \texttt{if } (\texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & N' \\ \texttt{else} & N'' \end{array} \right\}) \texttt{ then } M \texttt{ else } M' &\simeq \texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N' \texttt{ then } M \texttt{ else } M') \\ \texttt{else} & (\texttt{if } N'' \texttt{ then } M \texttt{ else } M') \end{array} \right. \\ \texttt{if d then } M \texttt{ else } M' &\simeq \texttt{d} \\ \texttt{if } (N|N') \texttt{ then } M \texttt{ else } M' &\simeq (\texttt{if } N \texttt{ then } M \texttt{ else } M') \\ & \qquad |(\texttt{if } N' \texttt{ then } M \texttt{ else } M') \end{aligned}$$

Laws of Equality Testing

$$\begin{aligned} c = c &\simeq \texttt{t} && (c \text{ constant}) \\ c = c' &\simeq \texttt{f} && (c,\, c' \text{ distinct constants}) \\ \texttt{d} = N &\simeq \texttt{d} \\ (N|N') = N'' &\simeq (N = N'')|(N = N'') \\ (\texttt{if } N \texttt{ then } N' \texttt{ else } N'') = N''' &\simeq \texttt{if } N \texttt{ then } (N' = N''') \texttt{ else } (N'' = N''') \end{aligned}$$

Laws of Commutativity

$$\begin{aligned} \texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N' \texttt{ then } M \texttt{ else } M') \\ \texttt{else} & (\texttt{if } N' \texttt{ then } M'' \texttt{ else } M''') \end{array} \right. &\simeq \texttt{if } N' \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N \texttt{ then } M \texttt{ else } M'') \\ \texttt{else} & (\texttt{if } N \texttt{ then } M' \texttt{ else } M''') \end{array} \right. \\ N = N' &\simeq N' = N \end{aligned}$$

Fig. 1. Laws

Definition 1 If N, N' are of type `bool`, we define $N = N'$ to be

$$\texttt{if}\ N \left\{ \begin{array}{l} \texttt{then}\ \ (\texttt{if}\ N'\ \texttt{then t else f}) \\ \texttt{else}\ \ (\texttt{if}\ N'\ \texttt{then f else t}) \end{array} \right.$$

□

We call the seven closed terms of type `bool`

$$\{\texttt{t}, \texttt{f}, \texttt{t}|\texttt{f}, \texttt{d}, \texttt{t}|\texttt{d}, \texttt{f}|\texttt{d}, \texttt{t}|\texttt{f}|\texttt{d}\}$$

the *basic boolean terms.*

Proposition 1 Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\simeq$ satisfies all the laws of Fig. 1. Let $\Gamma \vdash M, M' : B$ be terms.

1. $M|M'$ is $\lesssim$ every upper bound of $\{M, M'\}$, and $\gtrsim$ every lower bound of $\{M, M'\}$.
2. If $M \lesssim M'$ then $M \lesssim M|M' \lesssim M'$.
3. If $M|M'$ is an upper bound of $\{M, M'\}$, then it is a least upper bound.
4. Dually, if $M|M'$ is a lower bound of $\{M, M'\}$, then it is a greatest lower bound.

□

Proof For (1), if P is an upper bound for $\{M, M'\}$, then $M|M' \lesssim P|P \simeq P$. The rest follows. □

Definition 2 We say that a congruence $\simeq$ on $\mathcal{L}$ is *ground-extensional* when $N{=_{\mathsf{beh}}}N'$ implies $N \simeq N'$ for closed terms N, N' of the same ground type. □

Proposition 2 Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\simeq$ satisfies all the laws of Fig. 1.

1. On the basic boolean terms, it takes one of the 20 forms shown in Fig. 2–4.
2. In cases (1), (8), (8^{op}), (4), (11), (11^{op}) we have $M|\texttt{d} \simeq \texttt{d}$ for all $\Gamma \vdash M : B$.
3. In cases (1), (5), (5^{op}), (3), we have $M|\texttt{d} \simeq M$ for all $\Gamma \vdash M : B$.
4. In cases (1), (5), (5^{op}), (3), (6), (7), (8^{op}), (9^{op}), (11^{op}), (12), we have $M|\texttt{d} \gtrsim M$ for all $\Gamma \vdash M : B$.
5. Dually, in cases (1), (5), (5^{op}), (3), (6^{op}), (7^{op}), (8), (9), (11), (12^{op}), we have $M|\texttt{d} \lesssim M$ for all $\Gamma \vdash M : B$.
6. In cases (1),(5),(8), (9), (11), we have $\texttt{d} \lesssim M$ for all $\Gamma \vdash M : B$.
7. Dually, in cases (1), (5^{op}), (8^{op}), (9^{op}), (11^{op}), we have $\texttt{d} \lesssim M$ for all $\Gamma \vdash M : B$.
8. In case (1), we have $M \simeq M'$, for all $\Gamma \vdash M, M' : B$.
9. In cases (1), (5), (6), (8^{op}), the term $M|M'$ is a least upper bound of M and M', for all $\Gamma \vdash M, M' : B$.
10. Dually, in cases (1), (5^{op}), (6^{op}), (8), the term $M|M'$ is a greatest lower bound of M and M', for all $\Gamma \vdash M, M' : B$.

11. In cases (1), (5), (6), (8^{op}), (11^{op}), (12), the term $M|M'|\mathtt{d}$ is a least upper bound of M and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
12. Dually, in cases (1), (5^{op}), (6^{op}), (8), (11), (12^{op}), the term $M|M'|\mathtt{d}$ is a greatest lower bound of M and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
13. In cases (1), (5), (6), (8), (8^{op}), (11^{op}), (12), (4), (9), (10), the term $M|M'|\mathtt{d}$ is a least upper bound of $M|\mathtt{d}$ and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
14. Dually, in cases (1), (5^{op}), (6^{op}), (8), (8^{op}), (11), (12^{op}), (4), (9^{op}), (10^{op}) the term $M|M'|\mathtt{d}$ is a greatest lower bound of $M|\mathtt{d}$ and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
15. Suppose $\lesssim$ is ground-extensional. Let N and N' be closed terms of the same ground type. Then $N \lesssim N$ iff

	$N \Uparrow, N' \Uparrow$	$N \Uparrow, N' \not\Uparrow$	$N \not\Uparrow, N' \Uparrow$	$N \not\Uparrow, N' \not\Uparrow$
(1)	true	true	true	true
(2)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(3)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(4)	true	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(5)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$
(5^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(6)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$
(6^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(7)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(7^{op})	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(8)	true	true	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(8^{op})	true	false	true	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(9)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(9^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(10)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(10^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(11)	true	true	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(11^{op})	true	false	true	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(12)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(12^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$

$\square$

Proof

(1) Exhaustive analysis shows that these are the only preorders on this set for which $|$ and `if` are both monotone.

(2)–(7) Apply `if` $[\cdot]$ `then` M `else` M to the special case where M is `t`.

(8)–(14) We prove these results, using Prop. 1(3)–(4), by applying the context `if` $[\cdot]$ `then` M `else` M' to the special case where M is `t` and M' is `f`.

(15: $\Rightarrow$) We reason as follows.

– Suppose $\mathtt{t}|\mathtt{f} \not\lesssim \mathtt{t}$ and $N \leqslant N'$ and $N \not\Uparrow$, $N' \not\Uparrow$. Then $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$, because $c \in \mathsf{vals}[N] \setminus \mathsf{vals}[N']$ would imply

$$\begin{aligned} \mathtt{t}|\mathtt{f} &=_{\mathsf{beh}} (\mathtt{if}\ (N = c)\ \mathtt{then}\ \mathtt{f}\ \mathtt{else}\ \mathtt{t})|\mathtt{t} \\ &\lesssim (\mathtt{if}\ (N' = c)\ \mathtt{then}\ \mathtt{f}\ \mathtt{else}\ \mathtt{t})|\mathtt{t} =_{\mathsf{beh}} \mathtt{t} \end{aligned}$$

Dually, if $\mathtt{t}|\mathtt{f} \gtrsim \mathtt{t}$ and $N \leqslant N'$ and $N \not\Uparrow, N' \not\Uparrow$, then $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$.

– Suppose $\mathtt{t}|\mathtt{d} \not\lesssim \mathtt{t}$ and $N \leqslant N'$ and $N' \not\Uparrow$. Then $N \not\Uparrow$, because $N \Uparrow$ would imply

$$\begin{aligned}\mathtt{t}|\mathtt{d} &=_{\mathsf{beh}} (\mathtt{if}\ (N = N)\ \mathtt{then\ t\ else\ t})|\mathtt{t} \\ &\lesssim\ (\mathtt{if}\ (N' = N')\ \mathtt{then\ t\ else\ t})|\mathtt{t} =_{\mathsf{beh}} \mathtt{t}\end{aligned}$$

Dually, if $\mathtt{t} \not\gtrsim \mathtt{t}|\mathtt{d}$ and $N \leqslant N'$ and $N' \not\Uparrow$, then $N \not\Uparrow$.

– Suppose $\mathtt{t}|\mathtt{f}|\mathtt{d} \not\lesssim \mathtt{t}|\mathtt{d}$, and $N \leqslant N'$. Then $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$, because $c \in \mathsf{vals}[N] \setminus \mathsf{vals}[N']$ would imply

$$\begin{aligned}\mathtt{t}|\mathtt{f}|\mathtt{d} &=_{\mathsf{beh}} (\mathtt{if}\ N = c\ \mathtt{then\ f}|\mathtt{t})|\mathtt{t}|\mathtt{d} \\ &\lesssim\ (\mathtt{if}\ N' = c\ \mathtt{then\ f}|\mathtt{t})|\mathtt{t}|\mathtt{d} =_{\mathsf{beh}} \mathtt{f}|\mathtt{d}\end{aligned}$$

Dually, if $\mathtt{t}|\mathtt{f}|\mathtt{d} \not\gtrsim \mathtt{t}|\mathtt{d}$ and $N \leqslant N'$, then $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$.

(15: $\Leftarrow$) We reason as follows. Suppose $N \Uparrow, N' \Uparrow$.

– In the cases where Prop. 2(2) holds, we have

$$N =_{\mathsf{beh}} N|\mathtt{d} \simeq \mathtt{d} \simeq N'|\mathtt{d} =_{\mathsf{beh}} N'$$

– In the cases where Prop. 2(13) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|\mathtt{d} \lesssim N|N'|\mathtt{d} =_{\mathsf{beh}} N'$$

Dually, if $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$, then, in the cases where Prop. 2(14) holds, we have $N \gtrsim N'$.

– If $\mathsf{vals}[N] = \mathsf{vals}[N']$, then $N =_{\mathsf{beh}} N'$ so $N \lesssim N'$.

Suppose $N \Uparrow, N' \not\Uparrow$.

– In case (1), by Prop. 2(8), we have $N \lesssim N'$.

– In the cases where Prop. 2(13) and Prop. 2(5) both hold, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|\mathtt{d} \lesssim N|N'|\mathtt{d} =_{\mathsf{beh}} N'|\mathtt{d} \lesssim N'$$

– In the cases where Prop. 2(12) holds, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|N'|\mathtt{d} \lesssim N'$$

– In the cases where Prop. 2(5) holds, $\mathsf{vals}[N] = \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N'|\mathtt{d} \lesssim N'$$

Dually, suppose $N \not\Uparrow, N' \Uparrow$.

– In case (1), we have $N \lesssim N'$.

– In the cases where Prop. 2(14) and Prop. 2(4) both hold, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies $N \lesssim N'$

– In the cases where Prop. 2(11) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies $N \lesssim N'$.

– In the cases where Prop. 2(4) holds, $\mathsf{vals}[N] = \mathsf{vals}[N']$ implies $N \lesssim N'$.

Suppose $N \not\Uparrow, N' \not\Uparrow$.

– In case (1), by Prop. 2(8), we have $N \lesssim N'$.
– In cases where Prop. 2(9) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N \lesssim N|N' =_{\mathsf{beh}} N'$$

– Dually, in cases where Prop. 2(10) holds, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies $N \lesssim N'$.
– If $\mathsf{vals}[N] = \mathsf{vals}[N']$ then $N =_{\mathsf{beh}} N'$ so $N \lesssim N'$.

□ In the cases where Prop. 2(6) applies, we say that $\lesssim$ is *divergence-least*.

Since any congruence is a precongruence, we can specialize Prop. 2 as follows.

Proposition 3 Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying the laws of Fig. 1.

1. On the basic boolean terms, it takes one of the forms (1), (2), (3), (4).
2. In cases (1), (4), we have $M|\mathtt{d} \simeq \mathtt{d}$ for all $\Gamma \vdash M : B$.
3. In cases (1), (3), we have $M|\mathtt{d} \simeq M$ for all $\Gamma \vdash M : B$.
4. In case (1), we have $M \simeq M'$, for all $\Gamma \vdash M, M' : B$.
5. Suppose $\simeq$ is ground-extensional, and let N and N' be closed terms of the same ground type. Then $N \simeq N'$ iff

	$N \Uparrow, N' \Uparrow$	$N \Uparrow, N' \not\Uparrow$	$N \not\Uparrow, N' \Uparrow$	$N \not\Uparrow, N' \not\Uparrow$
case(1)	true	true	true	true
case(2)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
case(3)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
case(4)	true	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$

□

In the cases where Prop. 3(3) applies, we say that $\simeq$ is *divergence-insensitive*.

2 Ambiguous Choice

Suppose that $\mathfrak{L}$ contains an ambiguous choice operator `amb`—not necessarily at every type, but at least at type `bool`, and the function $\mathsf{behs}[-]$ has the property

$$\begin{aligned}\mathsf{behs}[M \ \mathtt{amb}\ N] = {} & ((\mathsf{behs}[M] \cup \mathsf{behs}[N]) \setminus \bot) \\ & \cup \{\bot \mid \bot \in \mathsf{behs}[M] \wedge \bot \in \mathsf{behs}[N]\}\end{aligned}$$

Laws pertaining to this operator are shown in Fig. 5. We can deduce from them the equation

$$(M|\mathtt{d}) \ \mathtt{amb}\ (N|\mathtt{d}) \simeq M|N|\mathtt{d} \tag{1}$$

as follows: the LHS $\simeq$

$$(M \ \mathtt{amb}\ N)|M|N|\mathtt{d} \tag{2}$$

while the RHS $\simeq$

$$(M|N|\mathtt{d}) \ \mathtt{amb}\ (M|N|\mathtt{d}) \simeq (2)$$

All the laws of Fig. 1 and 5 are satisfied by the congruence in [LM99] if the language treated there is extended with cost-free conditionals. All but the "laws of commutativity" are satisfied by the congruence in [Las05].

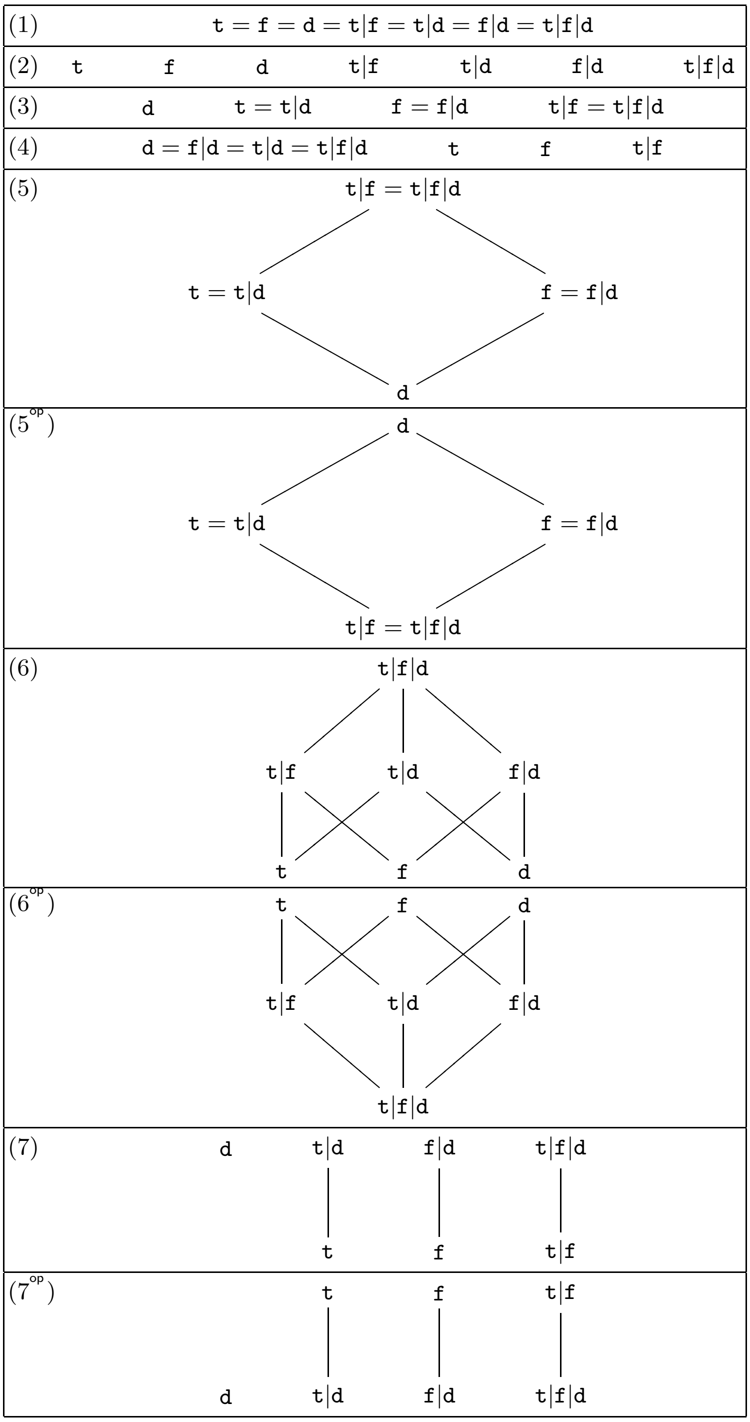

Fig. 2. The Twenty Precongruences

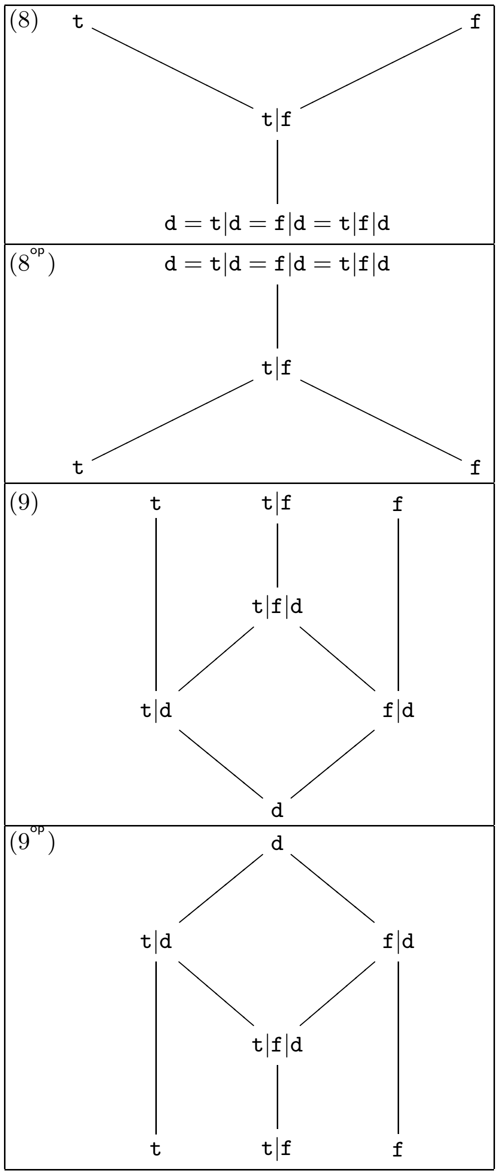

Fig. 3. The Twenty Precongruences (continued)

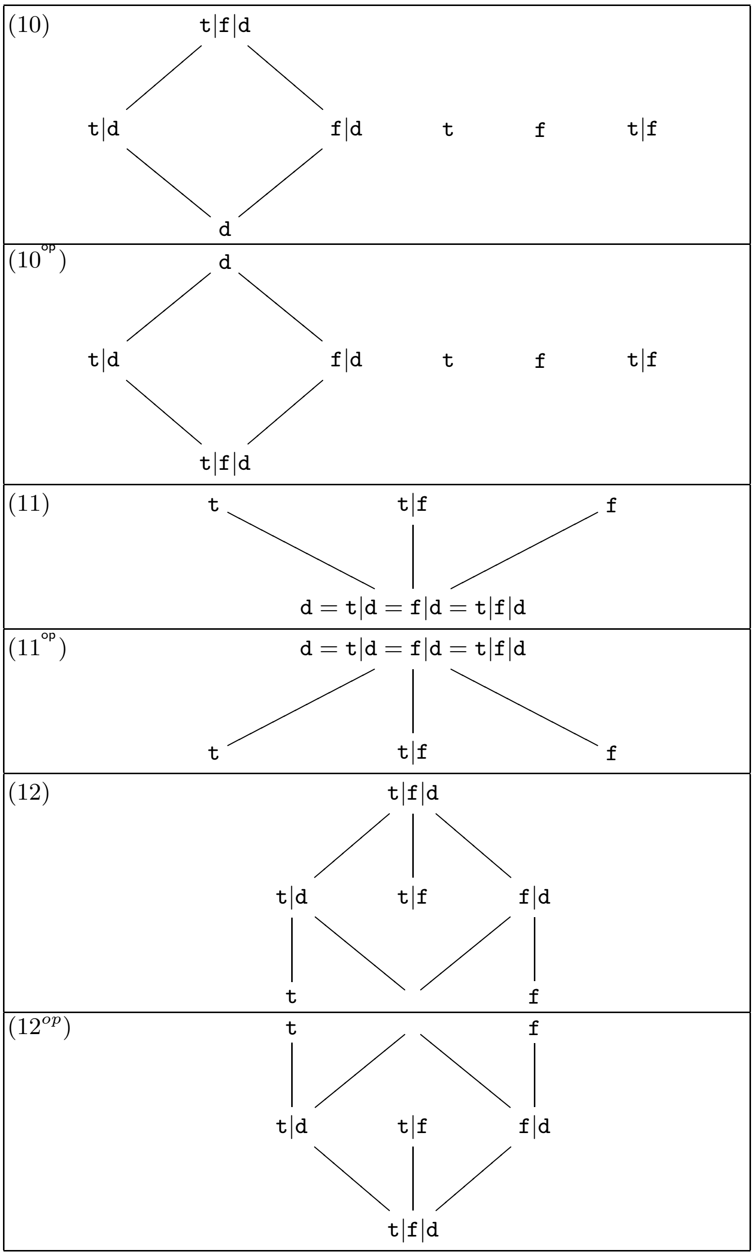

Fig. 4. The Twenty Precongruences (continued)

$$
\begin{aligned}
N \texttt{ amb } N' &\simeq N' \texttt{ amb } N \\
(N \texttt{ amb } N') \texttt{ amb } N'' &\simeq N \texttt{ amb } (N' \texttt{ amb } N'') \\
N \texttt{ amb } N &\simeq N \\
c \texttt{ amb } c' &\simeq c|c' \qquad\qquad (c,\ c' \text{ constants}) \\
\texttt{d amb } N &\simeq N \\
(N|N') \texttt{ amb } N'' &\simeq (N \texttt{ amb } N'')|(N' \texttt{ amb } N'') \\
(N \texttt{ amb } N')|N'' &\simeq (N|N'') \texttt{ amb } (N'|N'')
\end{aligned}
$$

Fig. 5. Laws of Ambiguous Choice

Proposition 4 Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying all the laws of Fig. 1–5. Then $\simeq$ is divergence-insensitive iff

$$N|N' \simeq N \texttt{ amb } N' \tag{3}$$

for all $\Gamma \vdash N, N' : B$ where B is an `amb` type. □

Proof If $\simeq$ is divergence-insensitive, then

$$
\begin{aligned}
M \texttt{ amb } N &\simeq (M|\texttt{d}) \texttt{ amb } (N|\texttt{d}) \\
&\simeq M|N|\texttt{d} \\
&\simeq M|N
\end{aligned}
$$

Conversely, (3) implies $\texttt{t}|\texttt{d} \simeq \texttt{t amb d} \simeq \texttt{t}$. □

Proposition 5 Any divergence-insensitive denotational semantics of the `amb`-free fragment of $\mathcal{L}$ has a unique extension to a denotational semantics of $\mathcal{L}$. It is obtained by setting $[\![N \texttt{ amb } N']\!]$ to be $[\![N|N']\!]$. □

Proof It is trivial to check the laws for ambiguous choice. Uniqueness follows from Prop. 4. □

Proposition 6 1. Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\lesssim$ satisfies all the laws of Fig. 1–Fig. 5. On the basic boolean terms, it takes one of the forms (1), (2), (3), (5), (5^{op}), (6), (6^{op}), (7), (7^{op}). Hence if $\lesssim$ is divergence-least, then it is divergence-insensitive.

2. Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying all the laws of Fig. 1–5. On the basic boolean terms, it takes one of the forms (1), (2), (3).

□

Proof These are the only cases for which `amb` is monotone. □

If $\mathcal{L}$ contains recursion, then, for any semantics that interprets recursion as a least fixpoint, the induced precongruence will be divergence-least. In a call-by-name language, for example, `diverge` can be expressed as $\mu\texttt{x.x}$, so it denotes the least fixpoint of the identity function. Therefore, Prop. 6(1) shows that there cannot be a least fixpoint semantics that is divergence-sensitive.

References

[Las05] S. B. Lassen. Normal form simulation for McCarthy's amb. In *Proceedings, 21st Annual Conference on Mathematical Foundations of Programming Semantics*, 2005. to appear in ENTCS.

[Lev04] P. B. Levy. *Call-By-Push-Value. A Functional/Imperative Synthesis*. Semantic Structures in Computation. Springer, 2004.

[LM99] Soren B. Lassen and Andrew K. Moran. Unique fixed point induction for McCarthy's amb. In *Proceedings of the 24th International Symposium on Mathematical Foundations of Computer Science*, volume 1672 of *"LNCS"*, pages 198–208. Springer, 1999.

[Plo83] G. Plotkin. Domains. prepared by Y. Kashiwagi, H. Kondoh and T. Hagino., 1983.